6 The Ho algorithm

An application of the matrix normal form discussed in Section 3.5 and of the Cayley-Hamilton theorem will be described. This important calculation solves the *inverse* of the problem of generating from $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ the Markov sequence

(5.43) $$\{\mathbf{H}_k\}_0^\infty = \left\{ \begin{array}{ll} \mathbf{D}, & k = 0 \\ \mathbf{C}\mathbf{A}^{k-1}\mathbf{B}, & k > 0 \end{array} \right\} .$$

That is, given $\{\mathbf{H}_k\}_0^\infty$ and the knowledge or assumption that this sequence can be generated by an LTI system, it is required to find at least one set of matrices $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ satisfying the above formula. Finding a state-space model from input-output information such as the Markov sequence is one kind of *system identification.*

Minimal order The system with matrices $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ produced by the Ho method has *minimal order* n, where $\mathbf{A} \in \mathbb{R}^{n\times n}$, in the class of LTI systems satisfying (5.43).

First some circumstances in which the $\{\mathbf{H}_k\}$ are obtained will be given, then the algorithm and its derivation, followed by examples.

6.1 The context

The sequence $\{\mathbf{H}_k\}$ is obtained in the following situations, and others:

1. $\{\mathbf{H}_k\}$ is the impulse-response sequence of an unknown discrete-time system, as in Section 1.5 of Chapter 2.
2. The rational proper discrete-time transfer matrix $\mathbf{H}(z)$ is known, and can be expanded (by long division!) as $\mathbf{H}(z) = \mathbf{H}_0 + \mathbf{H}_1 z^{-1} + \mathbf{H}_2 z^{-2} + \cdots$, as for continuous-time systems in Equation (3.25).
3. The transfer matrix $\mathbf{H}(s)$ of a continuous-time system is known and can be expanded as for the discrete-time system above into $\mathbf{H}(s) = \mathbf{H}_0 + \mathbf{H}_1 s^{-1} + \mathbf{H}_2 s^{-2} + \cdots$.
4. Given the continuous-time impulse response matrix $\mathbf{H}(t)$, the $\mathbf{H}_k$ can be obtained, using (2.33), as

$$\begin{aligned} \mathbf{H}_1 &= \frac{d^0}{dt^0}\mathbf{H}(t)|_{t=0+} \\ \mathbf{H}_2 &= \frac{d^1}{dt^1}\mathbf{H}(t)|_{t=0+} \\ &\vdots \\ \mathbf{H}_k &= \frac{d^{k-1}}{dt^{k-1}}\mathbf{H}(t)|_{t=0+}, \end{aligned}$$

with the zeroth term expressed consistently with the others, using the convention

$$\mathbf{H}_0 = \int_{0-}^{0+} \mathbf{H}(t)\,dt = \frac{d^{-1}}{dt^{-1}}\mathbf{H}(t)\;.$$

5. A realization $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ of $\{\mathbf{H}_k\}$ is known, but may not be of minimal order, for example, if the realization has been found by inspecting the transfer matrix, using the methods of Chapter 4. Then the $\mathbf{H}_k$ can be calculated directly, using (5.43).

6.2 Constructive solution

The solution to this problem is given by the following, known as the B. L. Ho algorithm:

Step 0 First, by definition,

(5.44) $$\mathbf{D} = \mathbf{H}_0.$$

Step 1 For r "large enough," construct the $pr \times mr$ matrix

(5.45) $$\mathbf{S}_r = \begin{bmatrix} \mathbf{H}_1 & \mathbf{H}_2 & \cdots & \mathbf{H}_r \\ \mathbf{H}_2 & \mathbf{H}_3 & \cdots & \mathbf{H}_{r+1} \\ \cdots & & & \\ \mathbf{H}_r & \mathbf{H}_{r+1} & \cdots & \mathbf{H}_{2r-1} \end{bmatrix}.$$

A matrix with the above structure is called a Hankel matrix. Find nonsingular $\mathbf{P}$, $\mathbf{Q}$ such that

(5.46) $$\mathbf{P}\mathbf{S}_r\mathbf{Q} = \begin{bmatrix} \mathbf{I}_n & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{N},$$

where $\mathbf{N}$ is the normal form of $\mathbf{S}_r$, and n is the rank of $\mathbf{S}_r$. The required value of r will become clear later in the discussion. As illustrated in Figure 5.6, partition $\mathbf{P}$, $\mathbf{Q}$ into

(5.47) $$\mathbf{P} = \begin{bmatrix} \mathbf{P}_1 \\ \mathbf{P}_2 \end{bmatrix}, \quad \mathbf{Q} = [\,\mathbf{Q}_1,\ \mathbf{Q}_2\,],$$

where $\mathbf{P}_1$ has n rows and $\mathbf{Q}_1$ has n columns.

Step 2 As illustrated in Figure 5.7, calculate the matrices

(5.48a) $$\mathbf{A} = \mathbf{P}_1 \begin{bmatrix} \mathbf{H}_2 & \mathbf{H}_3 & \cdots & \mathbf{H}_{r+1} \\ \mathbf{H}_3 & \mathbf{H}_4 & \cdots & \mathbf{H}_{r+2} \\ \cdots & & & \\ \mathbf{H}_{r+1} & \mathbf{H}_{r+2} & \cdots & \mathbf{H}_{2r} \end{bmatrix} \mathbf{Q}_1, \quad \mathbf{B} = \mathbf{P}_1 \begin{bmatrix} \mathbf{H}_1 \\ \mathbf{H}_2 \\ \vdots \\ \mathbf{H}_r \end{bmatrix},$$

(5.48b) $$\mathbf{C} = [\,\mathbf{H}_1,\ \mathbf{H}_2,\ \cdots \mathbf{H}_r\,]\,\mathbf{Q}_1.$$

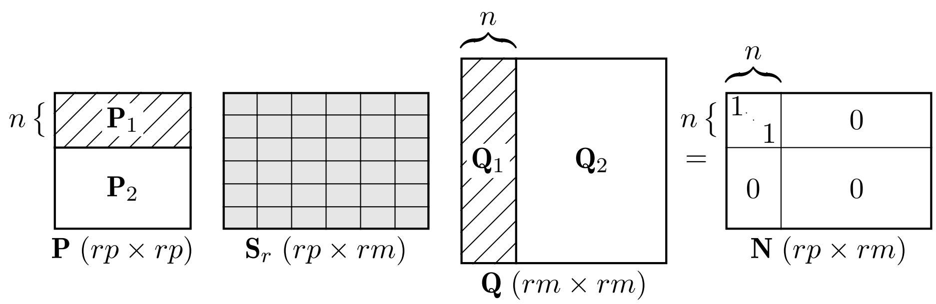

Fig. 5.6 Construction of $\mathbf{P}$, $\mathbf{Q}$, and $\mathbf{N}$, showing matrix dimensions.

6.3 Development of the algorithm

The proof that the previous construction produces a minimal system generating $\{\mathbf{H}_k\}_0^\infty$ rests on the following results.

Proposition 1 *If there is a realization of finite order* n, *then rank* $\mathbf{S}_r \le n$ *for all* $r = 1,\ 2, \cdots$.

Proof: Factor $\mathbf{S}_r$ as the product of matrices $\mathscr{O}\mathscr{C}$ as shown:

$$\mathbf{S}_r = \mathscr{O}\mathscr{C} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{r-1} \end{bmatrix} [\,\mathbf{B},\ \mathbf{AB}, \cdots \mathbf{A}^{r-1}\mathbf{B}\,], \tag{5.49}$$

where $\mathbf{C}$, $\mathbf{A}$, $\mathbf{B}$ are matrices of the finite-order realization. Because $\mathscr{O}$ has n columns and $\mathscr{C}$ has n rows, rank $\mathscr{O}\mathscr{C} \le \min\{\text{rank}\,\mathscr{O},\ \text{rank}\,\mathscr{C}\} \le n$. □

Proposition 2 *If there is a realization of finite order then there exist constants* $\alpha_1, \cdots \alpha_r$ *such that, for any* $k > 0$,

$$\mathbf{H}_{k+r} = \alpha_1 \mathbf{H}_{k+r-1} + \alpha_2 \mathbf{H}_{k+r-2} + \cdots \alpha_r \mathbf{H}_k\ . \tag{5.50}$$

Proof: Let $\mathbf{A}$ be the $n \times n$ state-vector coefficient matrix of a realization of order n, with characteristic polynomial

$$\det(\lambda \mathbf{I} - \mathbf{A}) = \lambda^n + a_1 \lambda^{n-1} + \cdots a_n\ . \tag{5.51}$$

Then by the Cayley-Hamilton theorem,

$$\mathbf{A}^n = -a_1 \mathbf{A}^{n-1} \cdots - a_n \mathbf{I}_n, \tag{5.52}$$

so that

$$\begin{aligned} \mathbf{H}_{k+n} &= \mathbf{CA}^{k+n-1}\mathbf{B} = \mathbf{CA}^{k-1}(\mathbf{A}^n)\mathbf{B} \\ &= \mathbf{CA}^{k-1}(-a_1\mathbf{A}^{n-1} - a_2\mathbf{A}^{n-2} - \cdots a_n\mathbf{I}_n)\mathbf{B} \\ &= -a_1\mathbf{H}_{k+n-1} - a_2\mathbf{H}_{k+n-2} - \cdots a_n\mathbf{H}_k. \end{aligned} \tag{5.53}$$