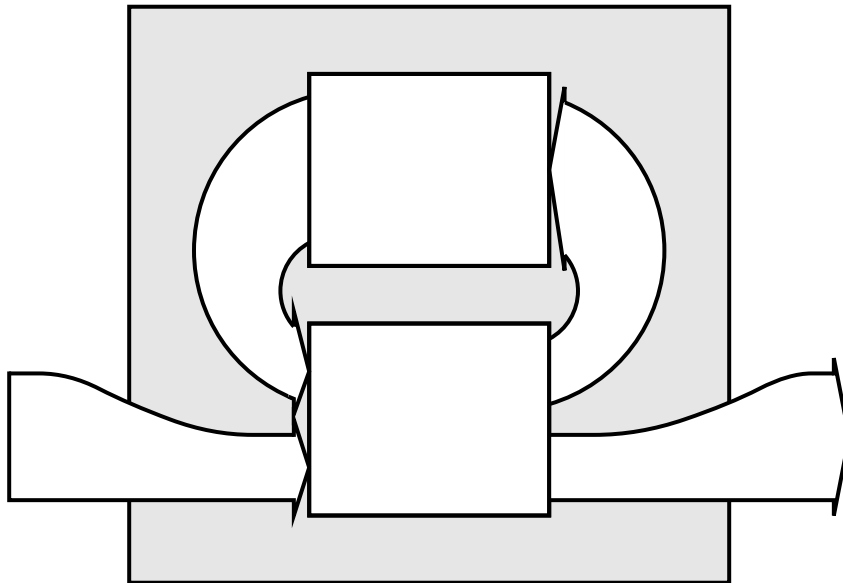


# The Essentials of Linear State-Space Systems

## *Supplementary chapters*

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# Contents

<b>Chapter 11</b>	<b>Time-varying systems</b>	<b>1</b>
	1 Homogeneous systems, <i>2</i>	
	1.1 Fundamental and state-transition matrices, <i>2</i>	
	2 Complete solution, <i>6</i>	
	3 Time-varying change of basis, <i>7</i>	
	4 Further study, <i>10</i>	
	5 Problems, <i>10</i>	
<b>Chapter 12</b>	<b>State feedback and pole placement</b>	<b>13</b>
	1 Single-input systems, <i>15</i>	
	2 Multi-input systems, <i>19</i>	
	2.1 Pole placement for multi-input systems, <i>21</i>	
	3 Feedback, controllability, and observability, <i>23</i>	
	4 Further study, <i>25</i>	
	5 Problems, <i>26</i>	
<b>Chapter 13</b>	<b>Optimal control</b>	<b>29</b>
	1 Formulation of the optimization problems, <i>30</i>	
	2 Continuous-time optimization, <i>33</i>	
	2.1 A generalized formulation, <i>36</i>	
	2.2 Particular cases, <i>37</i>	
	2.3 Constructing a state feedback solution, <i>40</i>	
	2.4 Properties of the optimal feedback solution, <i>42</i>	
	2.5 Infinite time interval and the Hamiltonian matrix, <i>43</i>	
	2.6 Solving for constant <b>P</b> : the continuous-time case, <i>46</i>	
	3 Discrete-time optimization, <i>52</i>	
	3.1 Feedback solutions, <i>55</i>	
	3.2 Computation of <b>P</b> : the discrete-time case, <i>56</i>	
	4 Further study, <i>59</i>	
	5 Problems, <i>59</i>	

<b>Chapter 14</b>	<b>Output feedback</b>	<b>63</b>
	1 Observers, 65	
	1.1 The full-order observer, 66	
	1.2 Reduced-order observers, 70	
	2 Optimal observers: the Kalman filter, 74	
	3 Further study, 75	
	4 Problems, 75	
<b>Chapter 15</b>	<b>High gain feedback</b>	<b>77</b>
	1 Root loci of multi-input, multi-output systems, 77	
	2 Integral control, 80	
	3 Derivative action, 82	
	4 Further study, 84	
	5 Problems, 84	
	<b>References</b>	<b>85</b>
<b>Appendix</b>	<b>Solutions</b>	<b>89</b>
	Chapter 11, 89	
	Chapter 12, 93	
	Chapter 13, 97	
	Chapter 14, 102	

# 11

## Time-varying systems

This chapter discusses the solution of time-varying linear systems, first encountered in Section 2 of Chapter 1 and described by Equations (1.11) for continuous-time systems, repeated here:

$$(1.11a) \quad \frac{d}{dt}\mathbf{X}(t) = \mathbf{A}(t)\mathbf{X}(t) + \mathbf{B}(t)\mathbf{U}(t)$$

$$(1.11b) \quad \mathbf{Y}(t) = \mathbf{C}(t)\mathbf{X}(t) + \mathbf{D}(t)\mathbf{U}(t).$$

Whether a system is considered to be linear and time-varying or nonlinear is sometimes a question of semantics and the designation of quantities as inputs, state variables, or time-varying parameters, as illustrated in Example 14 of Chapter 1. To focus the discussion, assume a model of the form of (1.11), where the external environment affects the system through  $\mathbf{U}(t)$ , and the system affects the environment through  $\mathbf{Y}(t)$  as usual, but the entries of the coefficient matrices  $\mathbf{A}(t)$ ,  $\mathbf{B}(t)$ ,  $\mathbf{C}(t)$ ,  $\mathbf{D}(t)$  are known functions of  $t$  that are independent of  $\mathbf{U}(t)$ . Solutions resulting from initial conditions  $\mathbf{X}(t_0)$  and input  $\mathbf{U}(t)$  will also depend on the initial time  $t_0$  and the interval  $[t_0, t)$  over which the solutions are obtained. One natural source of time-varying linear equations is the linearization of a time-invariant nonlinear system along a time-varying solution trajectory as discussed in Section 6 of Chapter 1, resulting in Equations (1.29), which are of the form of (1.11).

The discussion in this chapter concerns time-continuous systems because the recursion given in Section 1 of Chapter 2 is adequate for solving most discrete-time systems. However, the properties of linearity apply to both cases.

Provided that  $\mathbf{A}(t)$ ,  $\mathbf{B}(t)$ ,  $\mathbf{C}(t)$ ,  $\mathbf{D}(t)$  and  $\mathbf{U}(t)$  are bounded and piecewise continuous with respect to  $t$ , then by the discussion in Section 2.1 of Chapter 2, solutions exist, satisfy (1.11) except at points of discontinuity, and are continuous and unique. For simplicity in the following, the coefficient matrices will be assumed to be continuous functions of  $t$ , since discontinuities must be treated specially.

The transform methods of Chapter 3 are inapplicable to (1.11) because the coefficients in the equations are not constant.

The solution of the LTI case for initial state  $\mathbf{X}(t_0)$  obtained in Chapter 2 is repeated below for easy reference. The solution of the state for constant  $\mathbf{A}$ ,  $\mathbf{B}$ ,  $\mathbf{C}$ , and  $\mathbf{D}$  is

$$(2.25) \quad \mathbf{X}(t) = e^{(t-t_0)\mathbf{A}}\mathbf{X}(t_0) + \int_{t_0}^t e^{(t-\tau)\mathbf{A}}\mathbf{B}\mathbf{U}(\tau) d\tau,$$

and the output is

$$(2.29) \quad \mathbf{Y}(t) = \mathbf{C} \left( e^{(t-t_0)\mathbf{A}} \mathbf{X}(t_0) + \int_{t_0}^t e^{(t-\tau)\mathbf{A}} \mathbf{B} \mathbf{U}(\tau) d\tau \right) + \mathbf{D}\mathbf{U}(t).$$

The state-transition matrix  $\Phi(t, t_0) = e^{(t-t_0)\mathbf{A}}$  appears in the first term of (2.25) and is convolved with  $\mathbf{B}\mathbf{U}(t)$  in the second term. The output  $\mathbf{Y}(t)$  in (2.29) is obtained by multiplying the formula for  $\mathbf{X}(t)$  by  $\mathbf{C}$  and adding the term  $\mathbf{D}\mathbf{U}$ . Similarly for the time-varying equations, if  $\mathbf{X}(t)$  can be solved from (1.11a), it can be substituted into (1.11b) to obtain the output.

Solutions of (1.11) will be shown to exhibit the properties of linearity found for LTI systems in Section 2 of Chapter 2, but the state-transition matrix is not generally defined by a predetermined formula.

First, the solution for  $\mathbf{X}(t)$  when  $\mathbf{U}(t)$  is identically zero will be investigated, followed by the complete solution of (1.11).

## 1 Homogeneous systems

Setting  $\mathbf{U}(t) \equiv 0$  in (1.11a) results in

$$(11.1) \quad \frac{d}{dt} \mathbf{X}(t) = \mathbf{A}(t) \mathbf{X}(t).$$

Let an interval  $[t_0, t_f]$  over which solutions are to be found be given, together with the initial state  $\mathbf{X}(t_0)$ . In order to solve these equations, the entries of  $\mathbf{A}(t)$  must also be known over  $[t_0, t_f]$ . As usual, denote the dimension of  $\mathbf{X}(t)$  by  $n$ . There are  $n$  equations to be solved for the  $n$  unknown entries of  $\mathbf{X}(t)$  over the given interval.

In general, Equations (11.1) cannot be solved in closed form as for the LTI case but typically must be approximated by computer methods. For given  $t_0$  and  $\mathbf{A}(t)$ , there are infinitely many solutions corresponding to different initial states  $\mathbf{X}(t_0)$ . However, all of these solutions can be characterized by solving (11.1) exactly  $n$  times, as will be shown.

### 1.1 Fundamental and state-transition matrices

Let  $t_0$  be known and fixed in the following. Select a set of  $n$  linearly-independent initial-condition vectors  $\{\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \dots, \boldsymbol{\xi}_n\}$  and, for each  $i = 1, \dots, n$ , compute the solution  $\boldsymbol{\psi}_i(t)$  of (11.1) over  $[t_0, t_f]$  that corresponds to  $\boldsymbol{\xi}_i$ . The vectors  $\boldsymbol{\xi}_i$  may be chosen for convenience of solution provided they are independent. These solution vectors, written simply as  $\boldsymbol{\psi}_i$ , define a *fundamental matrix*  $\Psi(t) = [\boldsymbol{\psi}_1, \boldsymbol{\psi}_2, \dots, \boldsymbol{\psi}_n]$ , which obeys the differential equation

$$(11.2) \quad \frac{d}{dt} \Psi(t) = \mathbf{A}(t) \Psi(t),$$

with nonsingular initial-condition matrix  $\Psi(t_0) = [\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \dots, \boldsymbol{\xi}_n]$ . Post-multiply (11.2) by  $\Psi^{-1}(t_0)$  to get

$$(11.3) \quad \frac{d}{dt} (\Psi(t) \Psi^{-1}(t_0)) = \mathbf{A}(t) (\Psi(t) \Psi^{-1}(t_0)),$$

and let  $\Phi(t, t_0) = \Psi(t)\Psi^{-1}(t_0)$ . This fundamental matrix satisfies (11.2) as follows,

$$(11.4) \quad \frac{d}{dt}\Phi(t, t_0) = \mathbf{A}(t)\Phi(t, t_0)$$

with initial condition  $\Phi(t_0, t_0) = \Psi(t_0)\Psi^{-1}(t_0) = \mathbf{I}$ , and is unique because the properties of  $\mathbf{A}(t)$  guarantee unique solutions of (11.1). Postmultiplying (11.4) by any initial-condition vector  $\mathbf{X}(t_0)$  gives

$$(11.5) \quad \frac{d}{dt}(\Phi(t, t_0)\mathbf{X}(t_0)) = \mathbf{A}(t)(\Phi(t, t_0)\mathbf{X}(t_0)),$$

which shows that

$$(11.6) \quad \mathbf{X}(t) = \Phi(t, t_0)\mathbf{X}(t_0)$$

is a formula for the solution of (11.1) corresponding to  $\mathbf{X}(t_0)$ , since it satisfies (11.5) and the initial condition  $\Phi(t_0, t_0)\mathbf{X}(t_0) = \mathbf{X}(t_0)$ . Furthermore, any vector  $\mathbf{X}(t) \in \mathbb{R}^n$  corresponds to a unique initial condition  $\mathbf{X}(t_0)$  in (11.6) since  $\Phi(t, t_0)$  is nonsingular, as shown in the next result. The matrix  $\Phi(t, t_0)$  is therefore called the *state-transition matrix* corresponding to Equation (11.1).

**Example 1**  
State-transition  
matrix

Although simulation must typically be employed to solve Equation (11.1), some textbook examples can be solved in closed form. Suppose that solutions are to be found over  $[t_0, \infty)$  with  $\mathbf{A}(t) = \begin{bmatrix} 0 & e^t \\ 0 & -1 \end{bmatrix}$ . Choosing initial conditions  $\xi_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$  at  $t = 0$  gives solution  $\psi_1(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ , and choosing  $\xi_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$  gives solution  $\psi_2(t) = \begin{bmatrix} t \\ e^{-t} \end{bmatrix}$ . Therefore the state-transition matrix for initial time  $t_0$  is

$$\Phi(t, t_0) = \Psi(t)\Psi^{-1}(t_0) = \begin{bmatrix} 1 & t \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 1 & t_0 \\ 0 & e^{-t_0} \end{bmatrix}^{-1} = \begin{bmatrix} 1 & (t - t_0)e^{t_0} \\ 0 & e^{-(t-t_0)} \end{bmatrix}.$$

**Proposition 1** *The state-transition matrix  $\Phi(t, t_0)$  is nonsingular.*

**Proof:** The determinant of the state-transition matrix will be shown to be

$$(11.7) \quad \det \Phi(t, t_0) = e^{\int_{t_0}^t \text{trace}(\mathbf{A}(\tau)) d\tau}$$

where  $\text{trace } \mathbf{A} = \sum_{i=1}^n a_{ii}$ , and therefore, since the exponential function is nonzero for finite arguments,  $\det \Phi(t, t_0) \neq 0$ .

If (11.7) is true, differentiating both sides yields

$$(11.8) \quad \frac{d}{dt} \det \Phi(t, t_0) = e^{\int_{t_0}^t \text{trace}(\mathbf{A}(\tau)) d\tau} \text{trace}(\mathbf{A}(t)) = (\det \Phi(t, t_0)) \text{trace}(\mathbf{A}(t))$$

and conversely. Showing that (11.8) is true, therefore, shows that (11.7) holds. Using the notation  $\Phi(t, t_0) = [\phi_{ij}]$ , consider the derivative of the determinant computed as a sum of products by the first method in Section 2 of Chapter 5:

$$\begin{aligned}
 (11.9) \quad \frac{d}{dt}(\det [\phi_{ij}]) &= \frac{d}{dt} \sum \pm(\phi_{1i}\phi_{2j}\cdots) = \sum \pm \left( (\dot{\phi}_{1i}\phi_{2j}\cdots) + (\phi_{1i}\dot{\phi}_{2j}\cdots) + \cdots \right) \\
 &= \sum \pm(\dot{\phi}_{1i}\phi_{2j}\cdots) + \sum \pm(\phi_{1i}\dot{\phi}_{2j}\cdots) + \cdots \\
 &= \det \begin{bmatrix} \dot{\phi}_{11} & \dot{\phi}_{12} & \cdots \\ \phi_{21} & \phi_{22} & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix} + \det \begin{bmatrix} \phi_{11} & \phi_{12} & \cdots \\ \dot{\phi}_{21} & \dot{\phi}_{22} & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix} + \cdots,
 \end{aligned}$$

which is a sum of  $n$  determinants, each determinant containing a differentiated row. In the first determinant, substitute  $\dot{\phi}_{ij} = \sum_{k=1}^n a_{ik}\phi_{kj}$  from Equation (11.4) to obtain

$$(11.10) \quad \det \begin{bmatrix} \dot{\phi}_{11} & \dot{\phi}_{12} & \cdots \\ \phi_{21} & \phi_{22} & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix} = \det \begin{bmatrix} a_{11}\phi_{11}+a_{12}\phi_{21}+\cdots & a_{11}\phi_{12}+a_{12}\phi_{22}+\cdots & \cdots \\ \phi_{21} & \phi_{22} & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}.$$

Now add multiples of lower rows to the first using operations  $H_{12}(-a_{12})$ ,  $H_{13}(-a_{13})$ ,  $\dots$ ,  $H_{1n}(-a_{1n})$  as defined in Section 3.1 of Chapter 5 to get

$$(11.11) \quad \det \begin{bmatrix} \dot{\phi}_{11} & \dot{\phi}_{12} & \cdots \\ \phi_{21} & \phi_{22} & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix} = \det \begin{bmatrix} a_{11}\phi_{11} & a_{11}\phi_{12} & \cdots \\ \phi_{21} & \phi_{22} & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix} = a_{11} \det \Phi(t, t_0).$$

Similarly, the other determinants in (11.9) are  $a_{ii} \det \Phi(t, t_0)$ ,  $i = 2, \dots, n$ , and the sum of the  $n$  determinants is  $\text{trace}(\mathbf{A}(t)) \det \Phi(t, t_0)$ , as required to demonstrate Equation (11.8).  $\square$

**Inverse** A formula for the inverse of  $\Phi(t, t_0)$  can be found easily. Let  $t_1, t_2$ , and  $t_3$  be in  $[t_0, t_f]$ . Then from the fundamental properties of  $\Phi(t, t_0)$ ,

$$(11.12) \quad \mathbf{X}(t_3) = \Phi(t_3, t_2)\mathbf{X}(t_2) = \Phi(t_3, t_2)\Phi(t_2, t_1)\mathbf{X}(t_1) = \Phi(t_3, t_1)\mathbf{X}(t_1),$$

which must be true for any  $\mathbf{X}(t_1)$ . Therefore,

$$(11.13) \quad \Phi(t_3, t_1) = \Phi(t_3, t_2)\Phi(t_2, t_1)$$

for any three points  $t_1, t_2$ , and  $t_3$ . Let  $t_3 = t_1$ , so that

$$(11.14) \quad \Phi(t_1, t_1) = \mathbf{I} = \Phi(t_1, t_2)\Phi(t_2, t_1);$$

that is,

$$(11.15) \quad \Phi^{-1}(t_1, t_2) = \Phi(t_2, t_1).$$

Although formulas for  $\Phi(t, t_0)$  are not available in general, some special cases can be solved in closed form, as in the following.

**Piecewise constant systems** If  $\mathbf{A}(t)$  is piecewise constant but has step changes at times  $t_1, t_2, \dots$  in  $[t_0, t_f]$ , then let  $\mathbf{A}_i$  be the value of  $\mathbf{A}(t)$  between  $t_{i-1}$  and  $t_i$ . By continuity of the solution  $\mathbf{X}(t)$ , the



composition property (11.13), and the formula for the state-transition matrix for time-invariant systems,  $\Phi(t, t_0)$  over  $t_k < t < t_{k+1}$  is the product

$$(11.16) \quad \Phi(t, t_0) = e^{(t-t_k)\mathbf{A}_k} \dots e^{(t_2-t_1)\mathbf{A}_1} e^{(t_1-t_0)\mathbf{A}_0}.$$

### Triangular systems

If  $\mathbf{A}(t)$  can be written in upper triangular form, so that in each row  $i$ , the only nonzero entries are in columns  $i, \dots, n$ , choose initial conditions  $\xi_1 = \mathbf{e}_1, \dots, \xi_n = \mathbf{e}_n$ . Then in each solution  $\psi_i$ , entries  $i+1, \dots, n$  are zero, entry  $i$  satisfies a scalar differential equation with initial condition 1, and superior entries can be solved as functions of known quantities.

### Commutation

In the special case that

$$(11.17) \quad \mathbf{A}(t)\mathbf{A}(\tau) = \mathbf{A}(\tau)\mathbf{A}(t)$$

for every  $t, \tau$ , the state-transition matrix  $\Phi(t, \tau)$  is

$$(11.18) \quad \Phi(t, \tau) = e^{\int_{\tau}^t \mathbf{A}(s) ds},$$

which will be shown to satisfy Equation (11.4). The series for the right-hand side is

$$(11.19) \quad e^{\int_{\tau}^t \mathbf{A}(s) ds} = \mathbf{I} + \int_{\tau}^t \mathbf{A}(s) ds + \frac{1}{2} \left( \int_{\tau}^t \mathbf{A}(s) ds \right) \left( \int_{\tau}^t \mathbf{A}(s) ds \right) + \dots$$

and its derivative with respect to  $t$  is

$$(11.20) \quad \frac{d}{dt} e^{\int_{\tau}^t \mathbf{A}(s) ds} = \mathbf{A}(t) + \frac{1}{2} \mathbf{A}(t) \left( \int_{\tau}^t \mathbf{A}(s) ds \right) + \frac{1}{2} \left( \int_{\tau}^t \mathbf{A}(s) ds \right) \mathbf{A}(t) + \dots,$$

whereas

$$(11.21) \quad \mathbf{A}(t) e^{\int_{\tau}^t \mathbf{A}(s) ds} = \mathbf{A}(t) + \mathbf{A}(t) \left( \int_{\tau}^t \mathbf{A}(s) ds \right) + \dots,$$

and the latter two series are equal if and only if

$$(11.22) \quad \mathbf{A}(t) \int_{\tau}^t \mathbf{A}(s) ds = \int_{\tau}^t \mathbf{A}(s) ds \mathbf{A}(t).$$

Assume that (11.17) is true, and take  $\mathbf{A}(t)$  inside the integrals in (11.22), obtaining identical integrands. Conversely, if (11.22) is true, differentiating by  $\tau$  and multiplying by  $-1$  produces (11.17). Therefore, (11.22) is true if and only if (11.17) is true.

---

### Example 2

Verifying nonsingularity and

Equation (11.7)

The nonsingularity of the state-transition matrix  $\Phi(t, t_0)$  of Example 1 will be checked. In this example, the determinant of  $\Phi(t, t_0)$  is  $e^{-(t-t_0)} \neq 0$ , whereas  $\text{trace } \mathbf{A}(t) = -1$ , and

$$e^{\int_{t_0}^t \text{trace}(\mathbf{A}(\tau)) d\tau} = e^{\int_{t_0}^t (-1) d\tau} = e^{-(t-t_0)} = \det \Phi(t, t_0),$$

which also verifies Equation (11.7).

**Example 3**  
Inverse of  
 $\Phi(t, t_0)$

Calculating the inverse of  $\Phi(t, t_0)$  from Example 1, we get

$$\begin{aligned}\Phi^{-1}(t, t_0) &= \frac{1}{\det \Phi(t, t_0)} \text{adj}(\Phi(t, t_0)) = \frac{1}{e^{-(t-t_0)}} \begin{bmatrix} e^{-(t-t_0)} & -(t-t_0)e^{t_0} \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & (t_0-t)e^t \\ 0 & e^{-(t_0-t)} \end{bmatrix} = \Phi(t_0, t).\end{aligned}$$

**Example 4**  
Piecewise  
constant  $\mathbf{A}(t)$

Let  $\mathbf{A}(t) = \begin{bmatrix} 0 & 0 \\ -\text{step}(t-1) & 0 \end{bmatrix}$ . Then  $\Phi(t, 0) = \mathbf{I}$  for  $t < 1$ , and, for  $t \geq 1$ ,  $\Phi(t, 0) = e^{\left( (t-1) \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} \right)} \times \mathbf{I} = \begin{bmatrix} 1 & 0 \\ -t+1 & 1 \end{bmatrix}$ .

**Example 5**  
Triangular  $\mathbf{A}(t)$

Suppose that  $\mathbf{A}(t) = \begin{bmatrix} 1 & 1 \\ 0 & t \end{bmatrix}$ . Then initial condition  $\boldsymbol{\xi}_1 = \mathbf{e}_1$  gives  $\boldsymbol{\psi}_1 = \begin{bmatrix} e^t \\ 0 \end{bmatrix}$  and  $\boldsymbol{\xi}_2 = \mathbf{e}_2$  results in  $\boldsymbol{\psi}_2 = \begin{bmatrix} e^t 0 + \int_0^t e^{t-\tau} e^{\tau^2/2} d\tau \\ e^{t^2/2} \end{bmatrix} = \begin{bmatrix} e^t \int_0^t e^{-\tau^2/2} d\tau \\ e^{t^2/2} \end{bmatrix}$ .

**Example 6**  
Diagonal  
time-varying  
coefficients

Let  $\frac{d}{dt}\mathbf{X} = \mathbf{A}(t)\mathbf{X}$  with  $\mathbf{A}(t) = \begin{bmatrix} -\alpha + \beta \cos t & -3 \\ 3 & -\alpha + \beta \cos t \end{bmatrix} = \mathbf{R}(t) + \mathbf{Q}$ , where  $\mathbf{R} = \text{diag}[\beta \cos t]$  and  $\mathbf{Q} = \begin{bmatrix} -\alpha & -3 \\ 3 & -\alpha \end{bmatrix}$ . In this special case,  $\mathbf{A}(t)$  satisfies Equation (11.17) and, therefore,

$$\Phi(t, 0) = e^{\int_0^t (\mathbf{R}(\tau) + \mathbf{Q}) d\tau} = e^{\int_0^t \mathbf{R}(\tau) d\tau} e^{t\mathbf{Q}} = e^{\int_0^t \mathbf{R}(\tau) d\tau} e^{t\mathbf{Q}}$$

where the rightmost equality is true because  $\mathbf{Q} \int_0^t \mathbf{R}(\tau) d\tau = \int_0^t \mathbf{R}(\tau) d\tau \mathbf{Q}$  (see Equation (2.21)) for this example.

## 2

### Complete solution

It will be shown below that over an interval not containing discontinuities, the solution for  $\mathbf{X}(t)$  is given by the formula

$$(11.23) \quad \mathbf{X}(t) = \Phi(t, t_0)\mathbf{X}(t_0) + \int_{t_0}^t \Phi(t, \tau)\mathbf{B}(\tau)\mathbf{U}(\tau) d\tau.$$

Substituting this formula into Equation (1.11b) gives the complete solution of (1.11) for  $\mathbf{Y}(t)$  as

$$(11.24) \quad \mathbf{Y}(t) = \mathbf{C}(t)\Phi(t, t_0)\mathbf{X}(t_0) + \mathbf{C}(t) \int_{t_0}^t \Phi(t, \tau)\mathbf{B}(\tau)\mathbf{U}(\tau) d\tau + \mathbf{D}(t)\mathbf{U}(t).$$

Equation (11.23) is the equivalent, for time-varying systems, of (2.25) for LTI systems, and Equation (11.24) is the time-varying equivalent to (2.29).

To show that (11.23) solves (1.11a), first note that at  $t = t_0$ , the first right-hand term of (11.23) is  $\mathbf{X}(t_0)$  and the integral is zero, so the formula (11.23) satisfies the initial condition. Now (11.23) will be shown to satisfy (1.11a), as follows:

$$\begin{aligned}
 (11.25) \quad \frac{d}{dt}\mathbf{X}(t) &= \frac{d}{dt}\Phi(t, t_0)\mathbf{X}(t_0) + \frac{d}{dt}\left(\Psi(t) \int_{t_0}^t \Psi^{-1}(\tau)\mathbf{B}(\tau)\mathbf{U}(\tau) d\tau\right) \\
 &= \mathbf{A}(t)\Phi(t, t_0)\mathbf{X}(t_0) + \mathbf{A}(t)\Psi(t) \int_{t_0}^t \Psi^{-1}(\tau)\mathbf{B}(\tau)\mathbf{U}(\tau) d\tau \\
 &\quad + \Psi(t)\Psi^{-1}(t)\mathbf{B}(t)\mathbf{U}(t) \\
 &= \mathbf{A}(t)\left(\Phi(t, t_0)\mathbf{X}(t_0) + \int_{t_0}^t \Phi(t, \tau)\mathbf{B}(\tau)\mathbf{U}(\tau) d\tau\right) + \mathbf{B}(t)\mathbf{U}(t) \\
 &= \mathbf{A}(t)\mathbf{X}(t) + \mathbf{B}(t)\mathbf{U}(t).
 \end{aligned}$$

**Linearity** The form of (11.23) shows that the solution of the state  $\mathbf{X}(t)$  is the sum of two parts: the free response, which is the initial vector multiplied by the state-transition matrix, and the forced response, which is an integral. The properties of linearity analyzed in Chapter 2 are preserved. For fixed  $t_0$  and  $t$ , the free response is a linear function of the initial state, and the forced response is a linear function of the input  $\mathbf{U}(t)$  over the interval  $[t_0, t)$ . Similarly, the first term on the right of (11.24) is the free response, and the forced response is the two right-most terms. Therefore, although time-varying systems do not have a predefined formula for the state-transition matrix  $\Phi(t, t_0)$  as LTI systems do, solutions can be constructed by superposition in the same way.

**Example 7**  
Complete  
solution

The complete solution of Equation (1.11a) will be found for the system with  $\mathbf{A}(t)$  and  $\Phi(t, t_0)$  given in Example 1,  $\mathbf{U}(t) = \text{step}(t - t_0)$ ,  $\mathbf{C}(t) = [t, 0]$ ,  $\mathbf{D}(t) = 1$ ,  $\mathbf{B}(t) = \begin{bmatrix} 0 \\ e^{-t} \end{bmatrix}$ , and initial state  $\mathbf{X}(t_0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ . Applying Equation (11.23),

$$\begin{aligned}
 \mathbf{X}(t) &= \begin{bmatrix} 1 & (t - t_0)e^{t_0} \\ 0 & e^{-(t-t_0)} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \int_{t_0}^t \begin{bmatrix} 1 & (t - \tau)e^\tau \\ 0 & e^{-(t-\tau)} \end{bmatrix} \begin{bmatrix} 0 \\ e^{-\tau} \end{bmatrix} \text{step}(\tau - t_0) d\tau \\
 &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \int_{t_0}^t \begin{bmatrix} t - \tau \\ e^{-t} \end{bmatrix} d\tau = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} (t - t_0)^2/2 \\ -(t - t_0)e^{-t} \end{bmatrix},
 \end{aligned}$$

which is the complete solution for the state  $\mathbf{X}(t)$ . Substituting this into Equation (1.11b) gives the formula for the output for  $t \geq t_0$ :

$$\mathbf{Y}(t) = [t, 0] \begin{bmatrix} 1 + (t - t_0)^2/2 \\ -(t - t_0)e^{-t} \end{bmatrix} + 1 \text{step}(t - t_0) = t + t(t - t_0)^2/2 + 1.$$

## 3

**Time-varying change of basis**

The inputs  $\mathbf{U}(t)$  and outputs  $\mathbf{Y}(t)$  of a system are normally predefined, but the state variables may be chosen to give the desired form of state-space equations. Chapter 7 explores the effect of changing the basis for the state space by making the change of state variables

$$(7.1) \quad \mathbf{X} = \mathbf{S}\mathbf{X}',$$

where  $\mathbf{S}$  is nonsingular and constant. Here, the effect of allowing  $\mathbf{S}$  to vary with  $t$  while remaining nonsingular will be investigated.

To reduce clutter in the equations,  $\mathbf{X}(t)$  will be written simply as  $\mathbf{X}$  in the following, and similarly for other variables where ambiguity does not result. Differentiating (7.1) and equating the result to (1.11a) while assuming that  $\mathbf{S}$  is *not* constant gives

$$(11.26) \quad \frac{d}{dt}\mathbf{X} = \dot{\mathbf{S}}\mathbf{X}' + \mathbf{S}\dot{\mathbf{X}}' = \mathbf{A}\mathbf{S}\mathbf{X}' + \mathbf{B}\mathbf{U},$$

and solving for  $\dot{\mathbf{X}}'(t)$  gives the new state model

$$(11.27a) \quad \frac{d}{dt}\mathbf{X}' = \mathbf{S}^{-1}(\mathbf{A}\mathbf{S} - \dot{\mathbf{S}})\mathbf{X}' + (\mathbf{S}^{-1}\mathbf{B})\mathbf{U} = \mathbf{A}'\mathbf{X}' + \mathbf{B}'\mathbf{U}$$

$$(11.27b) \quad \mathbf{Y} = (\mathbf{C}\mathbf{S})\mathbf{X}' + \mathbf{D}\mathbf{U} = \mathbf{C}'\mathbf{X}' + \mathbf{D}'\mathbf{U}.$$

where the matrices defining the new system are  $\mathbf{A}' = \mathbf{S}^{-1}(\mathbf{A}\mathbf{S} - \dot{\mathbf{S}})$ ,  $\mathbf{B}' = \mathbf{S}^{-1}\mathbf{B}$ ,  $\mathbf{C}' = \mathbf{C}\mathbf{S}$ ,  $\mathbf{D}' = \mathbf{D}$ . Some useful properties of these transformations will be investigated in the following.

**Equivalent  
fundamental  
matrices**

If  $\Psi'$  is a fundamental matrix for (11.27a), then  $\Psi = \mathbf{S}\Psi'$  is a fundamental matrix for (11.1). To show this, observe that if  $\Psi'$  is a fundamental matrix of (11.27a), then

$$(11.28) \quad \frac{d}{dt}\Psi' = \mathbf{S}^{-1}(\mathbf{A}\mathbf{S} - \dot{\mathbf{S}})\Psi',$$

with  $\Psi'(t_0)$  nonsingular. By differentiation,

$$(11.29) \quad \frac{d}{dt}\Psi = \dot{\mathbf{S}}\Psi' + \mathbf{S}\dot{\Psi}' = \dot{\mathbf{S}}\Psi' + \mathbf{S}\mathbf{S}^{-1}(\mathbf{A}\mathbf{S} - \dot{\mathbf{S}})\Psi' = \dot{\mathbf{S}}\Psi' + \mathbf{A}\mathbf{S}\Psi' - \dot{\mathbf{S}}\Psi' = \mathbf{A}\Psi,$$

with  $\Psi(t_0) = \mathbf{S}(t_0)\Psi'(t_0)$  nonsingular, as required. As a consequence, the state-transition matrices of the original and transformed systems are related as follows:

$$(11.30) \quad \Phi(t, t_0) = \Psi(t)\Psi^{-1}(t_0) = (\mathbf{S}(t)\Psi'(t))(\mathbf{S}(t_0)\Psi'(t_0))^{-1} = \mathbf{S}(t)\Phi'(t, t_0)\mathbf{S}^{-1}(t_0).$$

**Equivalent  
response**

The output of the transformed system (11.27) will be written and related to the original system using (11.30) and (7.1):

$$(11.31) \quad \begin{aligned} \mathbf{Y}(t) &= \mathbf{C}'(t) \left( \Phi'(t, t_0)\mathbf{X}'(t_0) + \int_{t_0}^t \Phi'(t, \tau)\mathbf{B}'(\tau)\mathbf{U}(\tau) d\tau \right) + \mathbf{D}(t)\mathbf{U}(t) \\ &= \mathbf{C}(t) \left( \mathbf{S}(t)\Phi'(t, t_0)\mathbf{S}^{-1}(t_0) \right) \mathbf{S}(t_0)\mathbf{X}'(t_0) \\ &\quad + \mathbf{C}(t) \int_{t_0}^t \left( \mathbf{S}(t)\Phi'(t, \tau)\mathbf{S}^{-1}(\tau) \right) \mathbf{B}(\tau)\mathbf{U}(\tau) d\tau + \mathbf{D}(t)\mathbf{U}(t) \\ &= \mathbf{C}(t)\Phi(t, t_0)\mathbf{X}(t_0) + \mathbf{C}(t) \int_{t_0}^t \Phi(t, \tau)\mathbf{B}(\tau)\mathbf{U}(\tau) d\tau + \mathbf{D}(t)\mathbf{U}(t), \end{aligned}$$

which is (11.24), showing that the original and transformed systems have identical outputs for identical inputs if the initial states satisfy  $\mathbf{X}(t_0) = \mathbf{S}(t_0)\mathbf{X}'(t_0)$ . The forced responses, in particular, are identical.

**Transforming  
A to desired  
form**

From the form of Equation (11.27a), a change of variables  $\mathbf{X} = \mathbf{S}\mathbf{X}'$  with time-varying transformation matrix  $\mathbf{S}$  will replace state matrix  $\mathbf{A}$  by  $\mathbf{A}' = \mathbf{S}^{-1}(\mathbf{A}\mathbf{S} - \dot{\mathbf{S}})$ . Several

resulting forms may be desirable, depending on the context, and two such cases will be discussed. However, the lack of a closed-form solution for  $\Psi(t)$  may limit their usefulness.

The transformation matrix  $\mathbf{S}$  will be chosen as the product of two matrices  $\mathbf{S} = \Psi\Psi_1$  where  $\Psi$  is a fundamental matrix for Equation (11.1), giving  $\mathbf{A}'$  as shown:

$$(11.32) \quad \begin{aligned} \mathbf{A}' &= \Psi_1^{-1}\Psi^{-1}(\mathbf{A}\Psi\Psi_1 - \dot{\Psi}\Psi_1 - \Psi\dot{\Psi}_1) = \Psi_1^{-1}\Psi^{-1}(\mathbf{A}\Psi\Psi_1 - \mathbf{A}\Psi\Psi_1 - \Psi\dot{\Psi}_1) \\ &= -\Psi_1^{-1}\dot{\Psi}_1. \end{aligned}$$

**Constant  $\mathbf{A}'$**  First, if  $\Psi_1 = e^{-t\mathbf{Q}}$  with constant  $\mathbf{Q}$ , then the transformed matrix is

$$(11.33) \quad \mathbf{A}' = -e^{t\mathbf{Q}}e^{-t\mathbf{Q}}(-\mathbf{Q}) = \mathbf{Q},$$

which applies, for example, when  $\mathbf{Q}$  is chosen to be 0 to give  $\mathbf{A}' = 0$ .

**Example 8**  
Transforming  
Example 1

Applying Equation (7.1) with  $\mathbf{S} = \Psi(t)$  and both  $\mathbf{A}(t)$  and  $\Psi(t)$  given by Example 1 results in

$$\mathbf{A}' = \mathbf{S}^{-1}(\mathbf{A}\mathbf{S} - \dot{\mathbf{S}}) = \frac{1}{e^{-t}} \begin{bmatrix} e^{-t} & -t \\ 0 & 1 \end{bmatrix} \left( \begin{bmatrix} 0 & 1 \\ 0 & -e^{-t} \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & -e^{-t} \end{bmatrix} \right),$$

which reduces to 0, as expected from Equation (11.33). With  $\mathbf{B}(t)$  and  $\mathbf{C}(t)$  given in Example 7, the new matrices are

$$\begin{aligned} \mathbf{B}' &= \mathbf{S}^{-1}\mathbf{B} = \frac{1}{e^{-t}} \begin{bmatrix} e^{-t} & -t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ e^{-t} \end{bmatrix} = \begin{bmatrix} -t \\ 1 \end{bmatrix}, \\ \mathbf{C}' &= \mathbf{C}\mathbf{S} = [t, 0] \begin{bmatrix} 1 & t \\ 0 & e^{-1} \end{bmatrix} = [t, t^2]. \end{aligned}$$

**Periodic systems**

Suppose that  $\mathbf{A}(t)$  is periodic with period  $T$ , so that  $\mathbf{A}(t) = \mathbf{A}(t + T)$ . It will be shown that the fundamental matrix  $\Phi(t, 0)$  can be factored as

$$(11.34) \quad \Phi(t, 0) = \mathbf{S}(t) e^{t\mathbf{Q}},$$

where  $\mathbf{S}(t)$  is periodic with period  $T$  and where time-varying similarity transformation (7.1) applied as in Section 11 using this  $\mathbf{S}(t)$  results in  $\mathbf{A}' = \mathbf{Q}$ . Then the stability properties of the system are determined by  $\mathbf{Q}$  since, for example, if all eigenvalues of  $\mathbf{Q}$  are in the left half-plane,  $e^{t\mathbf{Q}}$  is decreasing and  $\Phi(t, 0)$  is decreasing.

To show the required properties of  $\Phi(t, 0)$ , let  $\mathbf{Q}$  satisfy

$$(11.35) \quad e^{T\mathbf{Q}} = \Phi(T, 0);$$

that is,  $T\mathbf{Q}$  is the logarithm of  $\Phi(T, 0)$ , computed as described in Section 5 of Chapter 7. Let the transformation matrix be  $\mathbf{S}(t) = \Phi(t, 0) e^{-t\mathbf{Q}}$ . Then since  $\Phi(t, 0)$  is a fundamental matrix,  $\mathbf{A}' = \mathbf{Q}$  as in (11.33). Furthermore,  $\mathbf{S}(t)$  is periodic of period  $T$ , which is shown by computing  $\mathbf{S}(t + T)$  as

$$(11.36) \quad \begin{aligned} \mathbf{S}(t + T) &= \Phi(t + T, 0) e^{-(t+T)\mathbf{Q}} = \Phi(t + T, 0) e^{-T\mathbf{Q}} e^{-t\mathbf{Q}} \\ &= \Phi(t + T, 0)\Phi(0, T) e^{-t\mathbf{Q}} = \Phi(t + T, T) e^{-t\mathbf{Q}}, \end{aligned}$$

where  $e^{-T\mathbf{Q}}$  has been replaced using (11.35). Differentiating the left factor of the right-hand side gives

$$(11.37) \quad \frac{d}{dt}\Phi(t+T, T) = \mathbf{A}(t+T)\Phi(t+T, T) = \mathbf{A}(t)\Phi(t+T, T)$$

which shows that  $\Phi(t+T, T)$  is a fundamental matrix, with  $\Phi(T, T) = \mathbf{I}$ . Consequently,  $\Phi(t+T, T) = \Phi(t, 0)$  so that in (11.36),  $\mathbf{S}(t+T) = \Phi(t, 0)e^{-t\mathbf{Q}} = \mathbf{S}(t)$  as required.

**Example 9**  
Periodic  $\mathbf{A}(t)$

Consider the matrix  $\mathbf{A}(t)$  given in Example 6. This matrix is periodic with period  $2\pi$  for  $\beta \neq 0$ . The state-transition matrix was found in Example 6 to have factored form  $\Phi(t, 0) = \mathbf{S}(t)e^{t\mathbf{Q}}$ , where  $\mathbf{S}(t) = e^{\int_0^t \mathbf{R}(\tau) d\tau}$ .

Writing  $\Phi(t, 0)$  in more detail gives

$$\Phi(t, 0) = e^{\int_0^t \beta \cos \tau d\tau} \mathbf{I} \times e^{t \begin{bmatrix} -\alpha & -3 \\ 3 & -\alpha \end{bmatrix}} = (e^{\beta \sin t} e^{-\alpha t}) \begin{bmatrix} \cos 3t & -\sin 3t \\ \sin 3t & \cos 3t \end{bmatrix},$$

where the rightmost matrix has been obtained as in Example 19 of Chapter 2. At  $t = T = 2\pi$ , the value of  $e^{\beta \sin t}$  is 1 and  $\Phi(T, 0) = e^{T\mathbf{Q}}$ , so (11.35) is true as expected. From the form of  $e^{t\mathbf{Q}}$ , solutions will be asymptotic to 0 provided  $\alpha > 0$ ; that is, provided the eigenvalues of  $\mathbf{Q}$  are in the left half-plane.

## 4

**Further study**

Methods for obtaining  $\Phi(t, \tau)$  as an infinite series can be found in textbooks such as [3]. Perturbation methods for systems in which the time-varying components are functions of a small parameter can also be found. However, the expressions rapidly become unmanageable for design purposes.

Some time-varying equations have received extensive study and are found in classical books on differential equations. Examples are the Bessel, Legendre, Hermite, Laguerre, and Mathieu equations, and the hypergeometric equations [1].

## 5

**Problems**

- 1 Find the state-transition matrix  $\Phi(t, t_0)$  valid for  $t$  and  $t_0$  in  $[0, \infty)$ , for the following matrices:

$$(a) \mathbf{A}(t) = \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix}, \quad (b) \mathbf{A}(t) = \begin{bmatrix} t & 1 \\ 0 & t \end{bmatrix}.$$

- 2 Find the state-transition matrix  $\Phi(t, t_0)$  valid for  $t$  and  $t_0$  in  $[0, \infty)$ , for the following matrices:

$$(a) \mathbf{A}(t) = t, \quad (b) \mathbf{A}(t) = \begin{bmatrix} 0 & 0 \\ -\text{step}(t-1) & \text{step}(t-1) - 1 \end{bmatrix}.$$

---

**3** Find the complete solution over  $[0, \infty)$  (a) for  $\mathbf{X}(t)$  and (b) for  $\mathbf{Y}(t)$  for  $\mathbf{A}(t)$  given in Problem 1(a),  $\mathbf{B} = \begin{bmatrix} 0 \\ t+1 \end{bmatrix}$ ,  $\mathbf{C}(t) = [1, -t^2/4]$ ,  $\mathbf{D} = 0$ ,  $\mathbf{X}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ , and  $\mathbf{U}(t) = \text{step}(t)$ .

---

**4** Find the complete solution over  $[0, \infty)$  (a) for  $\mathbf{X}(t)$  and (b) for  $\mathbf{Y}(t)$  for  $\mathbf{A}(t)$  given in Problem 1(a),  $\mathbf{B} = \begin{bmatrix} t \\ 0 \end{bmatrix}$ ,  $\mathbf{C}(t) = [t, 0]$ ,  $\mathbf{D} = 0$ ,  $\mathbf{X}(0) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ , and  $\mathbf{U}(t) = \text{step}(t)$ .

---

**5** For  $\mathbf{A}(t)$  given in Problem 1(a) and  $\mathbf{B}(t)$ ,  $\mathbf{C}(t)$ ,  $\mathbf{D}(t)$  given in Problem 3, find a nonsingular state transformation  $\mathbf{X}(t) = \mathbf{S}(t)\mathbf{X}'(t)$  such that the resulting matrix  $\mathbf{A}'(t)$  is zero, and find the resulting matrices  $\mathbf{B}'(t)$  and  $\mathbf{C}'(t)$  for the transformed system.

---

**6** For  $\mathbf{A}(t)$  given in Problem 1(b) and  $\mathbf{B}(t)$ ,  $\mathbf{C}(t)$ ,  $\mathbf{D}(t)$  given in Problem 4, find a nonsingular state transformation  $\mathbf{X}(t) = \mathbf{S}(t)\mathbf{X}'(t)$  such that the resulting matrix  $\mathbf{A}'(t)$  is zero, and find the resulting matrices  $\mathbf{B}'(t)$  and  $\mathbf{C}'(t)$  for the transformed system.

---

**7** For the system for which  $\mathbf{A}(t) = \begin{bmatrix} \sin t & 1 \\ 0 & 0 \end{bmatrix}$ , find the state-transition matrix in the form  $\Phi(t, 0) = \mathbf{S}(t)e^{t\mathbf{Q}}$  where  $\mathbf{S}(t)$  is periodic and  $\mathbf{Q}$  is constant. Do free solutions approach 0 as  $t \rightarrow \infty$ ?

---

**8** Find a state transformation  $\mathbf{X} = \mathbf{S}(t)\mathbf{X}'$  that transforms the system with the matrices shown to an input-output equivalent system for which  $\mathbf{A}(t) = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$ :

$$\mathbf{A}(t) = \begin{bmatrix} -1 & 1 + e^{-t} \\ 0 & -1 \end{bmatrix}, \quad \mathbf{B}(t) = \begin{bmatrix} -e^{-t} \\ 1 \end{bmatrix}, \quad \mathbf{C}(t) = [1, e^{-t}], \quad \mathbf{D}(t) = 0.$$


---

**9** From Equation (11.24), the forced response of the output can be rewritten as

$$\mathbf{Y}(t) = \int_{t_0}^t \mathbf{H}(t, \tau) \mathbf{U}(\tau) d\tau,$$

provided that  $\mathbf{B}(t)\mathbf{U}(t)$  is a continuous function of  $t$ . Find an expression for  $\mathbf{H}(t, \tau)$  in terms of  $\mathbf{B}$ ,  $\mathbf{C}$ , and  $\Phi(t, \tau)$ . The  $i$ -th column of  $\mathbf{H}(t, t_0)$  is the forced response to an impulse applied to the  $i$ -th input at time  $t_0$ . For the system matrices given in Problem 3, find the impulse-response matrix  $\mathbf{H}(t, t_0)$ .

---

**10** Solve Problem 9 for the system defined by the matrices shown:

$$\mathbf{A}(t) = \begin{bmatrix} t & 1 \\ 0 & t \end{bmatrix}, \quad \mathbf{B}(t) = \begin{bmatrix} t \\ 0 \end{bmatrix}, \quad \mathbf{C}(t) = [1, -t^2/4], \quad \mathbf{D}(t) = 0.$$

**11** The matrix  $\mathbf{A}(t)$  for a system is given by  $\mathbf{A}(t) = \begin{bmatrix} a & e^{-t} \\ -e^{-t} & a \end{bmatrix}$ , with  $a > 0$ . Find the state-transition matrix for this matrix and show that it satisfies Equations (11.7), (11.13), and (11.15).

---

**12** Find the state-transition matrix  $\Phi(t, \tau)$  for the system with matrix  $\mathbf{A}(t) = t^2\mathbf{M}$  where  $\mathbf{M} \in \mathbb{R}^{n \times n}$  is constant.



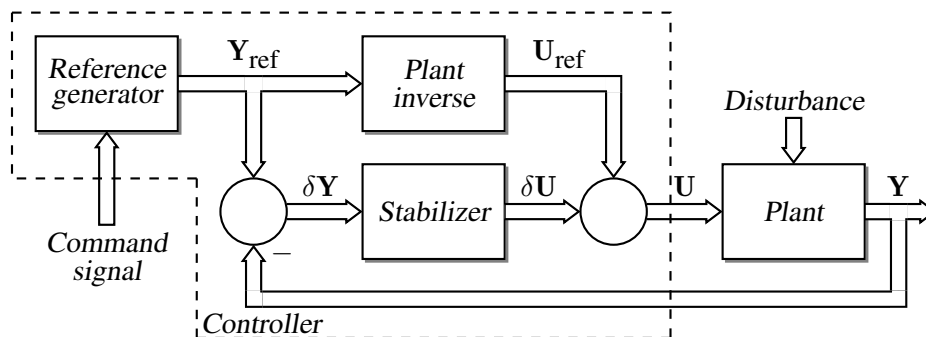
# 12

## State feedback and pole placement

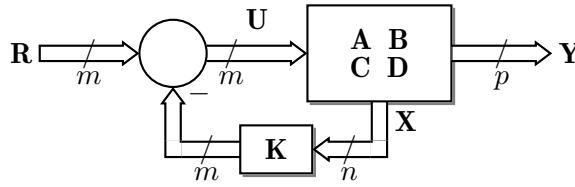
Methods for altering and controlling the free and forced responses of dynamical systems are introduced in this chapter. The three-purpose controller shown in Figure 12.1, for example, generates a reference signal for the plant output, a feed-forward signal for good transient performance, and a feedback signal for stability and disturbance cancellation. On receipt of a command signal, the reference generator produces the desired plant output  $\mathbf{Y}_{\text{ref}}(t)$ . The controller is assumed to contain an inverse model of the plant, which generates the control signal  $\mathbf{U}_{\text{ref}}(t)$  from  $\mathbf{Y}_{\text{ref}}(t)$ , so that the plant output is then ideally equal to the reference output. Disturbances, measurement errors, plant instability, or modelling errors may cause the actual plant output to differ from the reference value, and the stabilizer must be designed to keep the resulting error signal  $\delta\mathbf{Y}(t)$  small. Assuming that both  $\delta\mathbf{Y}(t)$  and the required stabilizing signal  $\delta\mathbf{U}(t)$  are small, then the stabilizer can be designed for a small-signal model (linearization) of the plant, which, in principle, may be nonlinear, provided the required inverse exists.

The fundamentals of stabilizer design will be investigated, assuming for simplicity that the reference signals  $\mathbf{Y}_{\text{ref}}$  and  $\mathbf{U}_{\text{ref}}(t)$  are zero. The plant output is measured and fed back to the controller, with a negative sign by convention.

Before considering the output-feedback stabilization shown in Figure 12.1, a simpler configuration will be considered. Figure 12.2 shows an LTI plant, for which the state vector  $\mathbf{X}$  is measured, rather than the output  $\mathbf{Y}$ . The measured vector is weighted by a constant feedback gain matrix  $\mathbf{K}$ , and the result subtracted from a reference signal  $\mathbf{R}$ .



**Fig. 12.1** The controller in this prototypical control system contains a reference-signal generator, an inverse of the possibly nonlinear plant, and a feedback stabilizer.



**Fig. 12.2** The state feedback configuration. The plant is described by matrices  $\mathbf{A}$ ,  $\mathbf{B}$ ,  $\mathbf{C}$ , and  $\mathbf{D}$ . The state is fed back to the input through matrix  $\mathbf{K}$ . The vector dimensions are marked on the figure.

The closed-loop equations for a time-continuous system are then as follows:

$$(12.1a) \quad \frac{d}{dt} \mathbf{X}(t) = \mathbf{A} \mathbf{X}(t) + \mathbf{B} (\mathbf{R} - \mathbf{K} \mathbf{X}) = (\mathbf{A} - \mathbf{B} \mathbf{K}) \mathbf{X}(t) + \mathbf{B} \mathbf{R}(t)$$

$$(12.1b) \quad \mathbf{Y}(t) = \mathbf{C} \mathbf{X}(t) + \mathbf{D} (\mathbf{R} - \mathbf{K} \mathbf{X}) = (\mathbf{C} - \mathbf{D} \mathbf{K}) \mathbf{X}(t) + \mathbf{D} \mathbf{R}(t).$$

The equations for a discrete-time configuration are identical except that the left side of (12.1a) is replaced by  $\mathbf{X}(t+1)$ .

The closed-loop system will have dynamics and stability determined by the matrix  $\mathbf{A}_{CL} = \mathbf{A} - \mathbf{B} \mathbf{K}$ . In particular, closed-loop stability is a typical design criterion, according to which the eigenvalues of  $\mathbf{A}_{CL}$  must be in the left half of the complex plane for time-continuous systems and in the unit disk for discrete-time systems. This criterion leads to the fundamental question of whether the closed-loop eigenvalues can be placed at arbitrary predefined locations by choosing the constant state-feedback gain matrix  $\mathbf{K}$ . The eigenvalues of  $\mathbf{A}_{CL}$  are the transfer-function poles if the system is minimal, and pole placement for LTI models is based on the following result:

**Proposition 1** *If  $(\mathbf{A}, \mathbf{B})$  is a controllable pair, then for any set of real numbers  $\beta_1, \beta_2, \dots, \beta_n$ , there exists a constant matrix  $\mathbf{K}$  such that the closed-loop matrix  $\mathbf{A}_{CL} = (\mathbf{A} - \mathbf{B} \mathbf{K})$  has characteristic polynomial  $\det(s\mathbf{I} - \mathbf{A} + \mathbf{B} \mathbf{K}) = s^n + \beta_1 s^{n-1} + \dots + \beta_n$ .*

A formal proof of this result can be assembled using the constructions given in the following sections.

**Pole assignment** Because the characteristic polynomial has real coefficients, its roots are either real or symmetrically located with respect to the real axis of the complex plane and, within this limitation, the closed-loop poles can be assigned arbitrarily.

**Transient design** Pole assignment alone does not completely determine transient response to initial conditions or inputs because the zeros of the corresponding transfer functions have not been considered. For multi-input systems, the situation is further complicated by the non-uniqueness of the matrix  $\mathbf{K}$  that produces a given set of closed-loop poles. Thus, when the shapes of responses are important, pole placement is only part of a complete design procedure.

**Example 1**  
Degree of  
stability

Sometimes it is desirable to ensure that all closed-loop time-constants are smaller than a maximum value  $1/\alpha$ ; that is for time-continuous systems, that the real parts of all closed-loop eigenvalues are more negative than a given negative number  $-\alpha$ . This is

referred to as *degree-of-stability control* and is accomplished by constant state feedback as follows. Define

$$(12.2) \quad \mathbf{Z} = e^{\alpha t} \mathbf{X}, \quad \mathbf{V} = e^{\alpha t} \mathbf{U},$$

and employ the equation  $\dot{\mathbf{X}} = \mathbf{A}\mathbf{X} + \mathbf{B}\mathbf{U}$  to obtain

$$(12.3) \quad \begin{aligned} \dot{\mathbf{Z}} &= \alpha e^{\alpha t} \mathbf{X} + e^{\alpha t} (\mathbf{A}\mathbf{X} + \mathbf{B}\mathbf{U}) = \alpha e^{\alpha t} e^{-\alpha t} \mathbf{Z} + e^{\alpha t} (\mathbf{A}e^{-\alpha t} \mathbf{Z} + \mathbf{B}e^{-\alpha t} \mathbf{V}) \\ &= (\mathbf{A} + \alpha \mathbf{I}) \mathbf{Z} + \mathbf{B}\mathbf{V}. \end{aligned}$$

For any stabilizing constant state-feedback  $\mathbf{V} = -\mathbf{K}\mathbf{Z}$  for this system, the closed-loop eigenvalues satisfy  $\text{Re } \lambda_i(\mathbf{A} + \alpha \mathbf{I} - \mathbf{B}\mathbf{K}) < 0$ . From Problem 13 of Chapter 7, the eigenvalues of this matrix are  $\lambda_i(\mathbf{A} + \alpha \mathbf{I} - \mathbf{B}\mathbf{K}) = \alpha + \lambda_i(\mathbf{A} - \mathbf{B}\mathbf{K})$ , which implies that  $\text{Re } \lambda_i(\mathbf{A} - \mathbf{B}\mathbf{K}) < -\alpha$ . Therefore, the required input  $\mathbf{U}$  is given by calculating  $\mathbf{K}$  so that (12.3) is asymptotically stable and obtaining the input as

$$(12.4) \quad \mathbf{U} = e^{-\alpha t} \mathbf{V} = -e^{-\alpha t} \mathbf{K} e^{\alpha t} \mathbf{X} = -\mathbf{K}\mathbf{X}.$$

## 1

## Single-input systems

Pole placement can be highly effective for stabilizing a single-input linear system. Let  $\hat{\mathbf{a}} = [a_n, \dots, a_1]$  be the row vector of coefficients in the characteristic equation  $\det(s\mathbf{I} - \mathbf{A}) = s^n + s^{n-1}a_1 + \dots + a_n$  and let  $\hat{\mathbf{b}} = [\beta_n, \dots, \beta_1]$  be the desired coefficients. Assume that a change of basis  $\mathbf{X} = \mathbf{S}\mathbf{X}'$  can be found to transform the original system matrices  $\mathbf{A}, \mathbf{B}$  to  $\mathbf{A}', \mathbf{B}'$ , where

$$(12.5) \quad \mathbf{A}' = \mathbf{S}^{-1} \mathbf{A} \mathbf{S} = \begin{bmatrix} 0 & 1 & & \\ & & \ddots & \\ & & & 1 \\ -a_n & \dots & & -a_1 \end{bmatrix}, \quad \mathbf{B}' = \mathbf{S}^{-1} \mathbf{B} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Let the input vector be calculated as  $\mathbf{U} = \mathbf{K}'\mathbf{X}'$  to weight the entries of  $\mathbf{X}'$  by the entries of state feedback matrix  $\mathbf{K}' = [k'_n, \dots, k'_1]$ . Written in detail, the product  $\mathbf{B}'\mathbf{K}'$  is

$$(12.6) \quad \mathbf{B}'\mathbf{K}' = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} [k'_n, \dots, k'_1] = \begin{bmatrix} 0 & \dots & 0 \\ & \dots & \\ 0 & \dots & 0 \\ k'_n & \dots & k'_1 \end{bmatrix},$$

so the closed-loop transformed system matrix is

$$(12.7) \quad \mathbf{A}' - \mathbf{B}'\mathbf{K}' = \begin{bmatrix} 0 & 1 & & \\ & & \ddots & \\ & & & 1 \\ -a_n - k'_n & \dots & & -a_1 - k'_1 \end{bmatrix},$$

which will be given the desired characteristic equation provided that the entries of  $\mathbf{K}'$  satisfy  $-a_i - k'_i = -\beta_i$ , for  $i = 1, \dots, n$ ; that is,

$$(12.8) \quad \mathbf{K}' = \hat{\mathbf{b}} - \hat{\mathbf{a}}.$$

The feedback signal  $\mathbf{K}'\mathbf{X}'$  gives the transformed system the desired closed-loop poles. Expressed in terms of the measured untransformed state  $\mathbf{X} = \mathbf{S}\mathbf{X}'$ , this signal is  $\mathbf{K}'\mathbf{S}^{-1}\mathbf{X}$ . Therefore, let

$$(12.9) \quad \mathbf{K} = \mathbf{K}'\mathbf{S}^{-1} = (\hat{\mathbf{b}} - \hat{\mathbf{a}})\mathbf{S}^{-1},$$

for which, since a similarity transformation does not affect the eigenvalues of a matrix, the eigenvalues of  $\mathbf{A}' - \mathbf{B}'\mathbf{K}'$  are those of

$$(12.10) \quad \mathbf{S}(\mathbf{A}' - \mathbf{B}'\mathbf{K}')\mathbf{S}^{-1} = \mathbf{S}(\mathbf{S}^{-1}\mathbf{A}\mathbf{S} - \mathbf{S}^{-1}\mathbf{B}\mathbf{K}\mathbf{S})\mathbf{S}^{-1} = \mathbf{A} - \mathbf{B}\mathbf{K} = \mathbf{A}_{\text{CL}}.$$

Thus, the closed-loop untransformed system matrix  $\mathbf{A}_{\text{CL}}$  has the required eigenvalues.

It remains to find the similarity transformation  $\mathbf{S}$  that produces (12.5). A formula for this matrix is given by the product

$$(12.11) \quad \mathbf{S} = \mathbf{S}_0\mathbf{S}_1,$$

where  $\mathbf{S}_0$  is the column-interchanged controllability matrix

$$(12.12) \quad \mathbf{S}_0 = [\mathbf{A}^{n-1}\mathbf{B}, \mathbf{A}^{n-2}\mathbf{B}, \dots, \mathbf{B}],$$

and

$$(12.13) \quad \mathbf{S}_1 = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ a_1 & 1 & 0 & \cdots & 0 \\ a_2 & a_1 & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ a_{n-1} & \cdots & a_2 & a_1 & 1 \end{bmatrix}.$$

The matrix  $\mathbf{S}_0$  is invertible as required if and only if  $(\mathbf{A}, \mathbf{B})$  is a controllable pair, and brings  $[\mathbf{A}, \mathbf{B}]$  to lower-left reduced row echelon form in a manner to be illustrated in Example 3 and similar to the process described in Problem 6 of Chapter 9. The second matrix  $\mathbf{S}_1$ , which is nonsingular by its structure, transforms the result into *control canonical form*, that is, (12.5). Thus, the required transformation  $\mathbf{S}$  exists provided  $(\mathbf{A}, \mathbf{B})$  is a controllable pair.

The feedback matrix  $\mathbf{K}$  can be found by first calculating  $\hat{\mathbf{a}}$  by any method, constructing the product  $\mathbf{S} = \mathbf{S}_0\mathbf{S}_1$ , and then by applying (12.9). The coefficients in  $\hat{\mathbf{a}}$  are available from the left column of the transformed  $\mathbf{A}$  matrix after application of  $\mathbf{S}_0$ , but other numerical techniques may be preferable in practice, such as computing the eigenvalues of  $\mathbf{A}$  and computing  $\hat{\mathbf{a}}$  from them.

### High-gain feedback

Equation (12.9) shows that large differences between the desired coefficients and those of the original characteristic polynomial produce large entries in  $\mathbf{K}$ . Additionally, large entries in  $\mathbf{K}$  result from the near-singularity of the controllability matrix  $\mathbf{S}_0$ , since  $\mathbf{S}_1$  in the product  $\mathbf{S} = \mathbf{S}_0\mathbf{S}_1$  is identically nonsingular. In other words, near-uncontrollability requires high feedback gain. High feedback gain implies large plant input signals in response to disturbances or reference-input changes.

**Example 2**  
Single-input  
system

It is desired to place the poles of the system with matrices shown below at locations  $-1, -1, -1$ , for example:

$$\mathbf{A} = \begin{bmatrix} -9 & 7 & -1 \\ -11 & 8 & -1 \\ -7 & 4 & -1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}.$$

The desired characteristic polynomial is

$$(s + 1)(s + 1)(s + 1) = s^3 + 3s^2 + 3s + 1,$$

so that  $\hat{\mathbf{b}} = [1, 3, 3]$ . The matrix  $\mathbf{K}$  may be computed as follows. First,  $\mathbf{S}_0$  and  $\mathbf{S}_0^{-1}\mathbf{A}\mathbf{S}_0$  are

$$\mathbf{S}_0 = [\mathbf{A}^2\mathbf{B}, \mathbf{A}\mathbf{B}, \mathbf{B}] = \begin{bmatrix} 0 & 2 & -1 \\ -1 & 3 & -1 \\ -5 & 3 & 0 \end{bmatrix}, \quad \mathbf{S}_0^{-1}\mathbf{A}\mathbf{S}_0 = \begin{bmatrix} -2 & 1 & 0 \\ -3 & 0 & 1 \\ -4 & 0 & 0 \end{bmatrix}.$$

Thus by inspection,  $\hat{\mathbf{a}}$  and  $\mathbf{S}_1$  are

$$\hat{\mathbf{a}} = [4, 3, 2], \quad \mathbf{S}_1 = \begin{bmatrix} 1 & 0 & 0 \\ a_1 & 1 & 0 \\ a_2 & a_1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix},$$

and the required feedback matrix is

$$\mathbf{K} = (\hat{\mathbf{b}} - \hat{\mathbf{a}})\mathbf{S}^{-1} = ([1, 3, 3] - [4, 3, 2])\mathbf{S}_1^{-1}\mathbf{S}_0^{-1} = [2, -3, 1].$$

**Example 3**  
Pole placement  
by elementary  
similarities

As an aid to understanding the multivariable case to follow, the transformations required in Problem 2 will be performed using a sequence of elementary similarity transformations, which are defined as follows (see Problem 6 of Chapter 9):

1. Operation  $O_{ij}$  interchanges rows  $i, j$  in  $\mathbf{A}$  and  $\mathbf{B}$ , then columns  $i, j$  in the resulting  $\mathbf{A}$  and in  $\mathbf{C}$ .
2. Operation  $O_i(\alpha)$ , for  $\alpha \neq 0$ , multiplies row  $i$  of  $\mathbf{A}$  and  $\mathbf{B}$  by  $\alpha$ , then divides column  $i$  of  $\mathbf{A}$  and  $\mathbf{C}$  by  $\alpha$ .
3. Operation  $O_{ij}(\alpha)$  adds the product of (row  $j$ )  $\times \alpha$  to row  $i$  in  $\mathbf{A}$  and  $\mathbf{B}$ , then subtracts the product of (column  $i$ )  $\times \alpha$  from column  $j$  in  $\mathbf{A}$  and  $\mathbf{C}$ .

Each of these operation pairs corresponds to a similarity transformation using an elementary matrix. If a matrix  $\mathbf{S}$  is initially set equal to  $\mathbf{I}$  and each column operation is performed on this matrix as it is performed on  $\mathbf{A}$ , then the resulting  $\mathbf{S}$  is the similarity transformation matrix that has an effect on  $\mathbf{A}$  identical to that of the sequence of operation pairs. Performing Gaussian elimination using a sequence of these operation pairs to transform  $[\mathbf{A}, \mathbf{B}]$  to lower-left row echelon form is equivalent to testing the columns of  $\mathbf{S}_0$  for linear independence from the right.

Starting with the matrix  $[\mathbf{A}, \mathbf{B}]$  from Problem 2 and transforming to lower left reduced row echelon form as shown produces the result of transformation  $\mathbf{S}_0$ :

$$[\mathbf{A}, \mathbf{B}] = \left[ \begin{array}{ccc|c} -9 & 7 & -1 & -1 \\ -11 & 8 & -1 & -1 \\ -7 & 4 & -1 & 0 \end{array} \right] \xrightarrow{\begin{array}{l} O_{23} \\ O_3(-1) \\ O_{13}(1) \end{array}} \left[ \begin{array}{ccc|c} 2 & 0 & -1 & 0 \\ -7 & -1 & 3 & 0 \\ 11 & 1 & -3 & 1 \end{array} \right]$$

$$\begin{array}{c}
 O_{12} \\
 O_2(-1) \rightarrow \left[ \begin{array}{ccc|c} -1 & 7 & 3 & 0 \\ 0 & 2 & 1 & 0 \\ 1 & -11 & -3 & 1 \end{array} \right] \xrightarrow{O_{12}(-3)} \left[ \begin{array}{ccc|c} -1 & -2 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & -2 & 0 & 1 \end{array} \right] \\
 O_1(-1/2) \\
 O_{21}(1) \\
 O_{31}(2) \rightarrow \left[ \begin{array}{ccc|c} -2 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ -4 & 0 & 0 & 1 \end{array} \right].
 \end{array}$$

The effect of transformation  $\mathbf{S}_1$  can be obtained by further elementary similarities applied to the above result as follows:

$$\left[ \begin{array}{ccc|c} -2 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ -4 & 0 & 0 & 1 \end{array} \right] \xrightarrow{\begin{array}{c} O_{21}(-2) \\ O_{31}(-3) \\ O_{32}(-2) \end{array}} \left[ \begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -4 & -3 & -2 & 1 \end{array} \right].$$

Setting  $\mathbf{T}$  to  $\mathbf{I}$  and performing each elementary row operation on  $\mathbf{T}$  as it is performed on  $\mathbf{A}$  results in  $\mathbf{T} = \mathbf{S}_1^{-1}\mathbf{S}_0^{-1}$  at the termination of the above two sequences. Premultiplying this matrix by  $\hat{\mathbf{b}} - \hat{\mathbf{a}}$  produces  $\mathbf{K}$ .

**Example 4**  
Inverse  
pendulum

A linearization of an inverse pendulum at the vertical position was obtained in Example 19 of Chapter 1 and is repeated here without assuming  $M \gg m$ :

$$(12.14) \quad \frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -mg/M & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & g(m+M)/(M\ell) & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/M \\ 0 \\ -1/(M\ell) \end{bmatrix} u.$$

From (12.12) the matrix  $\mathbf{S}_0$ , which is column-equivalent to controllability matrix  $\mathcal{C}$ , is

$$(12.15) \quad \mathbf{S}_0 = \begin{bmatrix} mg/(M^2\ell) & 0 & 1/M & 0 \\ 0 & mg/(M^2\ell) & 0 & 1/M \\ -g(m+M)/(M^2\ell^2) & 0 & -1/(M\ell) & 0 \\ 0 & -g(m+M)/(M^2\ell^2) & 0 & -1/(M\ell) \end{bmatrix} \\
 \approx \begin{bmatrix} 0 & 0 & 1/M & 0 \\ 0 & 0 & 0 & 1/M \\ -g/(M\ell^2) & 0 & -1/(M\ell) & 0 \\ 0 & -g/(M\ell^2) & 0 & -1/(M\ell) \end{bmatrix},$$

which has full rank by inspection of the column-equivalent matrix shown. Therefore, the system is completely controllable and its poles can be placed arbitrarily using constant state-feedback matrix  $\mathbf{K}$ .

The open-loop characteristic polynomial is

$$(12.16) \quad \det(s\mathbf{I} - \mathbf{A}) = s^2 \left( s^2 - \frac{g(m+M)}{M\ell} \right) = s^2(s^2 - \gamma^2),$$

where the fraction has been replaced by  $\gamma^2$  for convenience. The system has a double pole at the origin and poles at  $\pm\gamma = \pm\sqrt{g(m+M)/(M\ell)} \simeq \pm\sqrt{g/\ell}$ , where the approximation applies for  $M \gg m$ . The system is unstable and the shorter the pendulum length  $\ell$ , the faster its transient behavior.

The transformation matrix is  $\mathbf{S} = \mathbf{S}_0\mathbf{S}_1$  where  $\mathbf{S}_0$  is given by (12.15); the matrix  $\mathbf{S}_1$  in (12.13) and  $\mathbf{S}$  are

$$(12.17) \quad \mathbf{S}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\gamma^2 & 0 & 1 & 0 \\ 0 & -\gamma^2 & 0 & 1 \end{bmatrix}, \quad \mathbf{S} = \frac{1}{M\ell} \begin{bmatrix} -g & 0 & \ell & 0 \\ 0 & -g & 0 & \ell \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

One pole-placement strategy might be as follows. Stabilize the system by changing the sign of the eigenvalue at  $+\gamma$  and replacing the eigenvalues at the origin by a complex pair with damping ratio equal to  $1/\sqrt{2}$  for good transient performance. The desired closed-loop characteristic polynomial is then

$$(12.18) \quad (s + \gamma)^2(s^2 + 2\zeta\omega s + \omega^2) = s^4 + 2(\zeta\omega + \gamma)s^3 + (\omega^2 + 4\gamma\zeta\omega + \gamma^2)s^2 + 2\gamma\omega(\omega + \zeta\gamma)s + \gamma^2\omega^2,$$

where  $\zeta$  is the damping ratio and  $\omega$  the undamped natural frequency of the complex eigenvalues. The entries in  $\hat{\mathbf{a}}$  are available from (12.16) and those of  $\hat{\mathbf{b}}$  are obtained from (12.18), from which the required state-feedback matrix  $\mathbf{K}$  given by (12.9) is

$$(12.19) \quad \mathbf{K} = (\hat{\mathbf{b}} - \hat{\mathbf{a}})\mathbf{S}^{-1} = [\gamma^2\omega^2, 2\gamma\omega(\omega + \zeta\gamma), \omega^2 + 4\gamma\zeta\omega + 2\gamma^2, 2(\zeta\omega + \gamma)] \frac{M\ell}{-g} \begin{bmatrix} 1 & 0 & \ell & 0 \\ 0 & 1 & 0 & \ell \\ 0 & 0 & g & 0 \\ 0 & 0 & 0 & g \end{bmatrix}.$$

## 2

### Multi-input systems

The multivariable method to be described is a generalization of the technique used for the single-input case: first transform the system to a form for which the required feedback is simple to calculate, then account for the transformation. In general, the resulting feedback matrix  $\mathbf{K}$  is not unique and contains more parameters than are required to place the poles. These additional parameters may be employed to improve system robustness with respect to parameter variations, for example.

**Comment** The absence of definitive criteria for deciding where to place the poles or for defining the redundant parameters in  $\mathbf{K}$  is a major reason why pole placement is typically not a preferred design method for multi-input, multi-output systems. However, an understanding of pole placement is essential for understanding the consequences of other methods of state feedback, and the remainder of this section and Section 2.1 should be read with this in mind.

**Solving for  $\mathbf{K}$**  A state feedback matrix  $\mathbf{K}$  can be obtained as follows.

1. Put  $[\mathbf{A}, \mathbf{B}]$  into lower-left reduced row echelon form with pivots above the main diagonal by means of a similarity transformation. If any pivots were on or below the main diagonal, the system would be of the form of (9.3a) and, by inspection, not completely controllable. A conceptual method for performing the transformation is given in Example 3, and this method corresponds to defining the transformation matrix  $\mathbf{S}_0$  to contain the rightmost  $n$  linearly independent vectors found by scanning

the matrix  $[\mathbf{A}^{n-1}\mathbf{B}, \dots, \mathbf{AB}, \mathbf{B}]$  from right to left. Otherwise, the unreduced echelon form can be obtained using a sequence of QR decompositions for numerical robustness, after which the pivots are transformed to units and the nonpivot entries in the pivot columns are zeroed.

2. Put the result into control form:

$$i \leftarrow 1$$

while (in row  $i$  of  $\mathbf{A}$  there is a pivot from Step 1 in some column  $j$ ):

zero row  $i$  except for the pivot by adding multiples of column  $j$  to other columns using operations of the type  $O_{ij}(\alpha)$

$$i \leftarrow i + 1$$

The result for a controllable system is

$$(12.20) \quad [\mathbf{S}^{-1}\mathbf{AS}, \mathbf{S}^{-1}\mathbf{B}] = [\mathbf{A}', \mathbf{B}'] = \left[ \begin{array}{c|c} \mathbf{J}_1 & \mathbf{0} \\ \mathbf{L} & \mathbf{J}_0 \end{array} \right]$$

which is in lower-left row echelon form with unit pivots. Submatrix  $\mathbf{J}_1$  is zero except for the unit pivots contained in it, and is illustrated below for a three-input system with  $\text{rank}(\mathbf{B}) = m = 3$ :

$$(12.21) \quad \left[ \begin{array}{c|c} \mathbf{J}_1 & \mathbf{0} \\ \mathbf{L} & \mathbf{J}_0 \end{array} \right] = \left[ \begin{array}{cccc|ccc} 0 & \dots & 0 & 1 & & & \\ & & & & \ddots & & \\ & & & & & & 1 \\ \hline \times & & \dots & \times & & 1 & \\ \times & & \dots & \times & \times & 1 & \\ \times & & \dots & \times & \times & \times & 1 \end{array} \right].$$

The pivots in the transformed  $\mathbf{A}$  are above the main diagonal, but not necessarily in a straight line. Parameters not identically zero are shown as  $\times$ .

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**Example 5**  
Discrete time  
deadbeat control

The free component of the response of a discrete-time system is of the form given in Equation (2.4). Therefore, the state feedback  $\mathbf{U}(t) = -\mathbf{KX}(t) + \mathbf{R}(t)$  produces a closed-loop free response which is

$$(12.22) \quad \mathbf{Y}_{\text{free}}(\ell) = (\mathbf{C} - \mathbf{DK})(\mathbf{A} - \mathbf{BK})^{\ell-t_0} \mathbf{X}(t_0).$$

If all poles of  $\mathbf{A}_{\text{CL}} = \mathbf{A} - \mathbf{BK}$  are placed at the origin, then  $\mathbf{A}_{\text{CL}}^\gamma = \mathbf{0}$ , where  $\gamma \leq n$  is the size of the largest block in the Jordan form (see Section 3 of Chapter 7) of  $\mathbf{A}_{\text{CL}}$ . Therefore, transients caused by initial conditions or disturbances become zero after  $\gamma$  time steps, and this strategy is called deadbeat control.

From (12.20), solve  $\mathbf{L} - \mathbf{J}_0\mathbf{K}' = \mathbf{0}$  and let  $\mathbf{K} = \mathbf{K}'\mathbf{S}^{-1}$ . The matrix  $\mathbf{K}$  is not unique in general since  $\mathbf{J}_0$  has full row rank but may not be square, but this method gives at least one  $\mathbf{K}$  which places the poles at the origin. Since transients decay to zero in at most  $n$  time steps, this method can result in large control signals for sampled systems with small sampling period.



**2.1 Pole placement for multi-input systems**

Performing interchange operations of type  $O_{ij}$  only, transform the result of Step 2 to the block companion form illustrated below for example (12.21):

$$(12.23) \quad \mathbf{A}' = \mathbf{S}^{-1}\mathbf{AS} = \begin{bmatrix} 0 & 1 & & & & \\ & & \cdot & & & \\ & & & 1 & & \\ \times & \cdot & \cdot & \times & \times & \cdot & \cdot & \times & \times & \cdot & \cdot & \times \\ \hline & & & 0 & 1 & & & & & & & \\ & & & & & \cdot & & & & & & \\ & & & & & & 1 & & & & & \\ \times & \cdot & \cdot & \times & \times & \cdot & \cdot & \times & \times & \cdot & \cdot & \times \\ \hline & & & & & & & 0 & 1 & & & \\ & & & & & & & & & \cdot & & \\ \times & \cdot & \cdot & \times & \times & \cdot & \cdot & \times & \times & \cdot & \cdot & \times \\ \hline & & & & & & & & & & & 1 \\ \times & \cdot & \cdot & \times & \times & \cdot & \cdot & \times & \times & \cdot & \cdot & \times \end{bmatrix}, \quad \mathbf{B}' = \mathbf{S}^{-1}\mathbf{B} = \begin{bmatrix} & & & \\ & & & \\ & & & \\ 1 & & & \\ \hline & & & \\ & & & \\ \times & 1 & & \\ \hline & & & \\ & & & \\ \times & \times & 1 & \end{bmatrix}.$$

On the diagonal of  $\mathbf{A}'$  there are ( $k = \text{rank } \mathbf{B}$ ) square blocks of dimension  $n_1, \dots, n_k$ , and in the example,  $k = 3$ . Not all of the off-diagonal nonzeros ( $\times$ 's) shown occur in every case; some elements may be identically zero, depending on the  $n_i$ . The above matrices  $\mathbf{A}'$ ,  $\mathbf{B}'$  will be written

$$(12.24) \quad \mathbf{A}' = \begin{bmatrix} \mathbf{J}_1 & 0 & 0 \\ -\mathbf{A}_{11} & -\mathbf{A}_{12} & -\mathbf{A}_{13} \\ 0 & \mathbf{J}_2 & 0 \\ -\mathbf{A}_{21} & -\mathbf{A}_{22} & -\mathbf{A}_{23} \\ 0 & 0 & \mathbf{J}_3 \\ -\mathbf{A}_{31} & -\mathbf{A}_{32} & -\mathbf{A}_{33} \end{bmatrix}, \quad \mathbf{B}' = \begin{bmatrix} 0 \\ \mathbf{B}_1 \\ 0 \\ \mathbf{B}_2 \\ 0 \\ \mathbf{B}_3 \end{bmatrix}$$

where the  $\mathbf{A}_{ij}$  have dimension  $1 \times n_j$  and the  $\mathbf{B}_i$  have dimension  $1 \times m$ . Let the  $m \times n$  feedback matrix for the transformed system be  $\mathbf{K}'$ . The effect of closing the feedback loop, that is, subtracting  $\mathbf{B}'\mathbf{K}'$  from  $\mathbf{A}'$ , is to subtract linear combinations of the rows of  $\mathbf{K}'$  from the bottom rows of the blocks of  $\mathbf{A}'$ .

Using the bottom row of each block, solve for  $\mathbf{K}'$  in the equation

$$(12.25) \quad \mathbf{A}' - \mathbf{B}'\mathbf{K}' = \begin{bmatrix} \mathbf{J}_1 & & \\ -\hat{\mathbf{b}}_1 & & \\ \hline 0 & \mathbf{J}_2 & \\ -\mathbf{A}_{21} & -\hat{\mathbf{b}}_2 & \\ \hline 0 & 0 & \mathbf{J}_3 \\ -\mathbf{A}_{31} & -\mathbf{A}_{32} & -\hat{\mathbf{b}}_3 \end{bmatrix},$$

where the  $\hat{\mathbf{b}}_i$  contain coefficients of the desired characteristic equation; that is, in the example, solve

$$(12.26) \quad \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \\ \mathbf{B}_3 \end{bmatrix} \mathbf{K}' = \begin{bmatrix} \hat{\mathbf{b}}_1 & 0 & 0 \\ \mathbf{A}_{21} & \hat{\mathbf{b}}_2 & 0 \\ \mathbf{A}_{31} & \mathbf{A}_{32} & \hat{\mathbf{b}}_3 \end{bmatrix} - \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \mathbf{A}_{13} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \mathbf{A}_{23} \\ \mathbf{A}_{31} & \mathbf{A}_{32} & \mathbf{A}_{33} \end{bmatrix} \\ = \begin{bmatrix} \hat{\mathbf{b}}_1 - \mathbf{A}_{11} & -\mathbf{A}_{12} & -\mathbf{A}_{13} \\ 0 & \hat{\mathbf{b}}_2 - \mathbf{A}_{22} & -\mathbf{A}_{23} \\ 0 & 0 & \hat{\mathbf{b}}_3 - \mathbf{A}_{33} \end{bmatrix}.$$

The closed-loop characteristic polynomial is the product of the polynomials for the diagonal blocks of (12.25):

$$(12.27) \quad (s^{n_1} + \beta_{11}s^{n_1-1} + \cdots + \beta_{1n_1})(s^{n_2} + \beta_{21}s^{n_2-1} + \cdots + \beta_{2n_2})(s^{n_3} + \beta_{31}s^{n_3-1} + \cdots + \beta_{3n_3}),$$

where the coefficients are contained in the row vectors  $\hat{\mathbf{b}}_i = [\beta_{in_i} \cdots \beta_{i1}]$ ,  $i = 1, 2, 3$ .

**Notes**

1. In the closed loop system, the lower-left off-diagonal blocks in (12.25) have no effect on pole locations, and the bottom rows of these blocks can be varied, for example, to reduce system sensitivity to parameter variations, or to minimize a norm of  $\mathbf{K}$ .
2. The desired closed-loop characteristic polynomial does not always admit of all complex pole locations. For example, if  $n_1 = n_2 = n_3 = 1$  all three closed-loop poles must be real. This method can be modified to account for this mild constraint by not zeroing all upper-right blocks in (12.25) with feedback. With the above block sizes, if block 23 is not zeroed, then the system has diagonal blocks of size 1 and 2, instead of 1, 1, 1, so complex poles can be obtained.
3. Equation (12.24) contains exactly the same coefficients as are available in (12.21); the transformation from (12.21) to (12.24) serves only to identify those parameters of (12.21) that are to be changed to zero, to values  $\beta_{..}$ , or to arbitrary values.
4. The feedback matrix can be chosen to make the closed-loop matrix upper triangular rather than lower triangular, giving a possibly different number of arbitrary parameters.

**Example 6**  
Multiple-input  
system

To illustrate the above with a very small system, let  $\mathbf{A}$ ,  $\mathbf{B}$  be

$$[\mathbf{A}, \mathbf{B}] = \left[ \begin{array}{ccc|cc} 1.4 & 1.6 & 1.4 & -1 & 2 \\ 3.6 & 4.4 & 5.6 & 0 & -2 \\ -4.8 & -5.2 & -7.8 & 1 & 6 \end{array} \right].$$

This system in reduced row echelon form is

$$\left[ \begin{array}{ccc|cc} -2.2222 & 0.0556 & 1 & 0 & 0 \\ 1.1111 & 0.2222 & 0 & 1 & 0 \\ 8.4444 & -0.1111 & 0 & 0 & 1 \end{array} \right],$$

from which, zeroing all rows to the left of the single pivot in (the current)  $\mathbf{A}$  gives,

$$\left[ \begin{array}{ccc|cc} 0 & 0 & 1 & 0 & 0 \\ 1.1111 & 0.2222 & 0 & 1 & 0 \\ 8.5062 & -0.0988 & -2.2222 & 0.0556 & 1 \end{array} \right],$$

and, after re-arranging to exhibit the  $1 \times 1$  and  $2 \times 2$  controllable blocks the result is

$$\left[ \begin{array}{ccc|cc} 0.2222 & 1.1111 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -0.0988 & 8.5062 & -2.2222 & 0.0556 & 1 \end{array} \right],$$

which is obtained from the original system via the similarity transformation matrix

$$\mathbf{S} = \left[ \begin{array}{ccc} -1.1111 & 12.4444 & 2 \\ 0.1111 & 27.5556 & -2 \\ 0.6667 & -32.6667 & 6 \end{array} \right].$$

The  $2 \times 3$  feedback matrix  $\mathbf{K}'$  in (12.25) is obtained by equating the closed-loop matrix  $\mathbf{A}' - \mathbf{B}'\mathbf{K}'$  to a desired matrix, as shown,

$$\left[ \begin{array}{c|cc} 0.2222 & 1.1111 & 0 \\ \hline 0 & 0 & 1 \\ -0.0988 & 8.5062 & -2.2222 \end{array} \right] - \left[ \begin{array}{c|c} 1 & 0 \\ \hline 0 & 0 \\ 0.0556 & 1 \end{array} \right] \mathbf{K}' = \left[ \begin{array}{c|cc} -\beta_{11} & 0 & 0 \\ \hline 0 & 0 & 1 \\ \times & -\beta_{22} & -\beta_{21} \end{array} \right]$$

where each  $\beta_{ij}$  appearing above is determined by the desired closed-loop poles, and  $\times$  is an arbitrary parameter. That is,  $\mathbf{K}'$  is the solution of

$$\left[ \begin{array}{c|c} 1 & 0 \\ \hline 0.0556 & 1 \end{array} \right] \mathbf{K}' = \left[ \begin{array}{ccc} 0.2222 & 1.1111 & 0 \\ -0.0988 & 8.5062 & -2.2222 \end{array} \right] - \left[ \begin{array}{ccc} -\beta_{11} & 0 & 0 \\ \times & -\beta_{22} & -\beta_{21} \end{array} \right],$$

where the rightmost matrix can also be replaced by other choices, such as

$$\left[ \begin{array}{ccc} -\beta_{11} & \times & \times \\ 0 & -\beta_{22} & -\beta_{21} \end{array} \right]$$

to give an upper-right closed-loop triangular form, with two arbitrary parameters. Which result might be more desirable would depend on the actual plant and the relative locations of the closed-loop block eigenvalues.

Finally, with  $\mathbf{K}'$  obtained from (12.28), the state-feedback matrix for the original system is  $\mathbf{K} = \mathbf{K}'\mathbf{S}^{-1}$ .

## 3

## Feedback, controllability, and observability

State feedback does not affect the controllability of a system, as will be shown. From the BPH test described on Page 226, the original system is completely controllable if and only if  $\text{rank}[\mathbf{A} - \lambda\mathbf{I}, \mathbf{B}] = n$  for all  $\lambda \in \mathbb{C}$ , where  $\mathbf{A} \in \mathbb{R}^{n \times n}$ . But for any  $\lambda$ , the rank of this matrix is

$$(12.28) \quad \text{rank}[\mathbf{A} - \lambda\mathbf{I}, \mathbf{B}] = \text{rank}\left([\mathbf{A} - \lambda\mathbf{I}, \mathbf{B}] \begin{bmatrix} \mathbf{I} & 0 \\ -\mathbf{K} & \mathbf{I} \end{bmatrix}\right) = \text{rank}[\mathbf{A} - \mathbf{BK} - \lambda\mathbf{I}, \mathbf{B}],$$

which shows that the closed-loop system is completely controllable if and only if the original system has this property.

Observability may be changed by feedback, as in the following example.

**Example 7**  
Feedback affects  
observability

The system described by matrices  $\mathbf{A} = 1$ ,  $\mathbf{B} = 1$ ,  $\mathbf{C} = 1$ ,  $\mathbf{D} = 1$  is minimal (completely controllable and observable). Applying state feedback  $\mathbf{U} = \mathbf{R} - \mathbf{K}\mathbf{X}$  with  $\mathbf{K} = 1$  produces the closed-loop matrices

$$\mathbf{A}_{\text{CL}} = \mathbf{A} - \mathbf{BK} = 0, \quad \mathbf{B}_{\text{CL}} = \mathbf{B} = 1, \quad \mathbf{C}_{\text{CL}} = \mathbf{C} - \mathbf{DK} = 0, \quad \mathbf{D}_{\text{CL}} = \mathbf{D} = 1$$

which is controllable but not observable.

**Stabilizable  
systems**

If the first step of the transformation described in Step 1 on Page 19 succeeds, then the closed-loop poles can be placed as described in Section 2. This is the multivariable generalization of the transformation affected by  $\mathbf{S}_0$  for single-input systems, as

illustrated in Example 3. If the step fails, then the system has the form shown in Equation (9.3a) and is not completely controllable. Given any state feedback matrix, partition it as  $\mathbf{K} = [\mathbf{K}_1, \mathbf{K}_2]$  to produce

$$(12.29) \quad \begin{bmatrix} \mathbf{A}_{11} & 0 \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} - \begin{bmatrix} 0 \\ \mathbf{B}_2 \end{bmatrix} [\mathbf{K}_1, \mathbf{K}_2] = \begin{bmatrix} \mathbf{A}_{11} & 0 \\ \mathbf{A}_{21} - \mathbf{B}_2 \mathbf{K}_1 & \mathbf{A}_{22} - \mathbf{B}_2 \mathbf{K}_2 \end{bmatrix}.$$

The form of the closed-loop matrix shows that the eigenvalues  $\mathbf{A}_{11}$ , which represents the uncontrollable part of the system, are unaffected by state feedback. Consequently, state feedback can place poles to be the roots of an arbitrary characteristic polynomial if and only if the system is completely controllable. If the eigenvalues of  $\mathbf{A}_{11}$  are in the stable region of the complex plane, then state feedback can be used to place the other eigenvalues, and the system is described as *stabilizable*.

**Zeros are unaffected**

State feedback changes the system poles, but has no effect on system zeros. This conclusion can be seen by writing the closed-loop equations in the form of Equation (10.16):

$$(12.30) \quad \begin{bmatrix} \mathbf{A} - \mathbf{BK} - s\mathbf{I} & 0 & \mathbf{B} \\ \mathbf{C} - \mathbf{DK} & -\mathbf{I} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{X}(s) \\ \mathbf{Y}(s) \\ \mathbf{R}(s) \end{bmatrix} = 0,$$

where  $\mathbf{X}$  is an  $n$ -vector,  $\mathbf{Y}$  is a  $p$ -vector, and  $\mathbf{R}$  is an  $m$ -vector. Then, setting the input  $\mathbf{R}(s)$  to zero, the system poles are the values of  $s$  for which nonzero solutions exist; that is, for which

$$(12.31) \quad \text{rank} \begin{bmatrix} \mathbf{A} - \mathbf{BK} - s\mathbf{I} & 0 \\ \mathbf{C} - \mathbf{DK} & -\mathbf{I} \end{bmatrix} < n + p.$$

These values of  $s$  are the eigenvalues of  $\mathbf{A} - \mathbf{BK}$ , as previously seen. Similarly, the system zeros are the values of  $s$  for which nonzero solutions exist for zero output  $\mathbf{Y}(s)$ ; that is, the values for which the following condition holds:

$$(12.32) \quad n + m > \text{rank} \begin{bmatrix} \mathbf{A} - \mathbf{BK} - s\mathbf{I} & \mathbf{B} \\ \mathbf{C} - \mathbf{DK} & \mathbf{D} \end{bmatrix} \\ = \text{rank} \left( \begin{bmatrix} \mathbf{A} - \mathbf{BK} - s\mathbf{I} & \mathbf{B} \\ \mathbf{C} - \mathbf{DK} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{I} & 0 \\ \mathbf{K} & \mathbf{I} \end{bmatrix} \right) = \text{rank} \begin{bmatrix} \mathbf{A} - s\mathbf{I} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}.$$

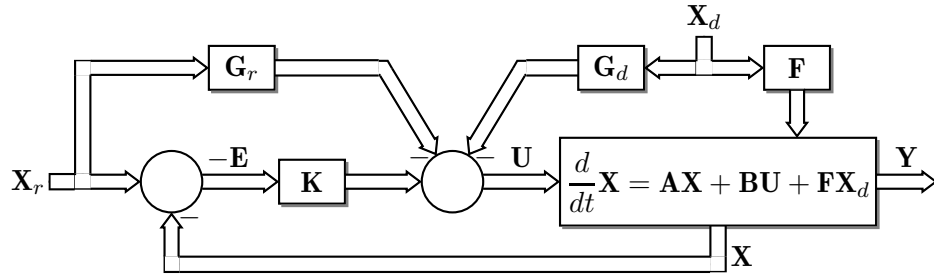
This shows that the zeros of the closed-loop system are the zeros of the original system.

**Example 8**  
Reference inputs  
and disturbance  
rejection

The basic state-feedback design that sends the closed-loop state asymptotically to the origin is modified in Figure 12.3 to include a reference input  $\mathbf{X}_r$  and disturbance vector  $\mathbf{X}_d$  that is assumed to affect the plant through a matrix  $\mathbf{F}$  as shown in the figure. so that the plant model is given by

$$(12.33) \quad \frac{d}{dt} \mathbf{X} = \mathbf{A}\mathbf{X} + \mathbf{B}\mathbf{U} + \mathbf{F}\mathbf{X}_d, \quad \text{with } \mathbf{U} = \mathbf{K}(-\mathbf{E}) + [\mathbf{G}_r, \mathbf{G}_d] \begin{bmatrix} \mathbf{X}_r \\ \mathbf{X}_d \end{bmatrix}.$$

The design requirement is to cancel the effect of  $\mathbf{X}_d$  if possible and to find a state-feedback control that sends the quantity  $\mathbf{E} = \mathbf{X} - \mathbf{X}_r$  to zero in steady state or, more generally, that sends a given linear combination  $\mathbf{C}_r \mathbf{E}$  of this quantity to zero in steady state.



**Fig. 12.3** The matrices  $\mathbf{G}_r$ ,  $\mathbf{G}_d$ , and  $\mathbf{K}$  are to be designed to cancel the effect of disturbance  $\mathbf{X}_d$ , which is assumed to be measurable, and to control the quantity  $\mathbf{C}_r \mathbf{E}$  to zero.

Assume that the plant is controllable and  $\mathbf{K}$  has been designed so that  $\mathbf{A} - \mathbf{BK}$  defines an asymptotically stable system. Since no eigenvalues of this matrix can be zero and  $\det(\mathbf{A} - \mathbf{BK})$  is the product of the eigenvalues, this matrix is nonsingular. Assume also that  $\mathbf{X}_r$  is the free response of a linear system given by  $\dot{\mathbf{X}}_r = \mathbf{A}_r \mathbf{X}_r$  with  $\mathbf{A}_r$  known, so that  $\mathbf{X}_r$  can contain steps, ramps, sinusoids, and other signals generated by a linear system. From the diagram and Equations (12.33), the evolution of  $\mathbf{E}$  is given by

$$\begin{aligned}
 (12.34) \quad \frac{d}{dt} \mathbf{E} &= \frac{d}{dt} \mathbf{X} - \frac{d}{dt} \mathbf{X}_r = \mathbf{A} \mathbf{X} + \mathbf{B} \mathbf{U} + \mathbf{F} \mathbf{X}_d - \mathbf{A}_r \mathbf{X}_r \\
 &= \mathbf{A} (\mathbf{X} - \mathbf{X}_r) + \mathbf{B} (-\mathbf{K} \mathbf{E}) + [\mathbf{G}_r, \mathbf{G}_d] \begin{bmatrix} \mathbf{X}_r \\ \mathbf{X}_d \end{bmatrix} + (\mathbf{A} - \mathbf{A}_r) \mathbf{X}_r + \mathbf{F} \mathbf{X}_d \\
 &= (\mathbf{A} - \mathbf{BK}) \mathbf{E} + ([\mathbf{A} - \mathbf{A}_r, \mathbf{F}] - \mathbf{B} [\mathbf{G}_r, \mathbf{G}_d]) \begin{bmatrix} \mathbf{X}_r \\ \mathbf{X}_d \end{bmatrix}.
 \end{aligned}$$

The left side of this equation is zero in steady state; therefore,  $\mathbf{E}$  can be solved and  $\mathbf{C}_r \mathbf{E}$  set to zero to give

$$(12.35) \quad \mathbf{C}_r (\mathbf{A} - \mathbf{BK})^{-1} \mathbf{B} [\mathbf{G}_r, \mathbf{G}_d] = \mathbf{C}_r (\mathbf{A} - \mathbf{BK})^{-1} [\mathbf{A} - \mathbf{A}_r, \mathbf{F}].$$

The solution and uniqueness of the unknowns  $\mathbf{G}_r$  and  $\mathbf{G}_d$  depend on the details of the known matrices (see Section 7 of Chapter 5), but certain assumptions are typical. If the plant type is 1 for example, so that there is a pole at the origin, and if  $\mathbf{X}_r$  represents a step reference, then  $\mathbf{A}_r$  can often be assumed equal to  $\mathbf{A}$  and the feed-forward matrix  $\mathbf{G}_r$  set to zero. Typically also, the matrix  $\mathbf{C}_r (\mathbf{A} - \mathbf{BK})^{-1} \mathbf{B}$  has full rank but is not necessarily square.

The solution sketched above depends on the disturbance vector  $\mathbf{X}_d$  containing measurable quantities (such as ambient temperature, for example). When  $\mathbf{X}_d$  is not directly measurable or when it represents plant-model errors, then it must be estimated as described in Chapter 14.

The constructive demonstration of pole placement given here has the advantage of simplicity and of explicitly showing the free parameters in the feedback matrix  $\mathbf{K}$  after the poles have been placed. Other methods, such as given by Brogan [9], are possible. Except for single-input systems however, eigenvalue placement in the pure form

discussed in this chapter is not a preferred method of designing state feedback for the reasons previously mentioned.

## 5 Problems

- 1 The matrices shown correspond to a continuous-time LTI system. In each case, find a state-feedback matrix, if possible, that places all system poles at  $-5 + j0$  on the complex plane.

$$(a) \quad \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}; \quad (b) \quad \mathbf{A} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

- 2 Let the desired closed-loop system pole locations for Problem 1(a) be all at  $-\alpha + j0$  on the complex plane. Determine, for (a)  $\alpha = 10$ , and (b)  $\alpha = 1$ , whether the entries of  $\mathbf{K}$  will be larger or smaller in magnitude than for  $\alpha = 5$  as in the solution of Problem 1.

- 3 An LTI system is defined by matrices  $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ ,  $\mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ . Let the poles all be placed at  $-\alpha + j0$  using state feedback  $u = r - \mathbf{K}\mathbf{X}(t)$ . Treating  $\alpha$  as a fixed parameter, find the value of the closed-loop control signal  $u(0+)$  at  $t = 0+$  for the free response to initial condition  $\mathbf{X}(0) = [0, 1]^T$ . How does this initial value of  $u$  vary as a function of  $\alpha$ ? How does this calculation relate to the discussion of high-gain feedback in Section 1?

- 4 Repeat Problem 3 for the matrices of Problem 1(a) and for matrices of the same form but of dimension  $n \times n$  for  $\mathbf{A}$  and  $n \times 1$  for  $\mathbf{B}$ . In each case, the initial state  $\mathbf{X}(0)$  is zero except for the last entry, which is 1.

- 5 State feedback  $\mathbf{U} = \mathbf{R} - \mathbf{K}\mathbf{X}$  is to be designed so that the system defined by the matrices shown will have closed-loop poles at  $-1 + j0$  and  $-2 + j0$ .

$$\mathbf{A} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{C} = [1, 1], \quad \mathbf{D} = 0.$$

Find the required state-feedback matrix for  $\mathbf{K}$  in the form of each of the three cases shown below. Case (a) provides a nonzero value only to the first input of the plant, case (b) only to the second input, and case (c) provides nonzero values to both inputs.

$$(a) \quad \mathbf{K} = \begin{bmatrix} k_{11} & k_{12} \\ 0 & 0 \end{bmatrix}, \quad (b) \quad \mathbf{K} = \begin{bmatrix} 0 & 0 \\ k_{21} & k_{22} \end{bmatrix}, \quad (c) \quad \mathbf{K} = \begin{bmatrix} k_{11} & 0 \\ 0 & k_{22} \end{bmatrix}$$

For each case, calculate the transfer matrix  $\mathbf{H}(s)$  and the forced response to the input  $\mathbf{R}(t) = \begin{bmatrix} \text{step}(t) \\ 0 \end{bmatrix}$ . Sketch the responses or use a computer to graph and compare them.

This problem illustrates that for a multi-input system, the pole locations do not uniquely define either the state feedback or the system response.

- 
- 6** Sometimes state-feedback results can be extended to the case of output feedback. Suppose that the state  $\mathbf{X}(t)$  of the linear system given in Problem 5 cannot be measured, but that  $\mathbf{Y}(t)$  is available. Design a constant output feedback matrix  $\mathbf{K}$  so that if  $\mathbf{U}(t) = \mathbf{R}(t) - \mathbf{K}\mathbf{Y}(t)$  the closed-loop poles are placed as required as in Problem 5. How would this result change if, instead of as given in the problem, the matrix  $\mathbf{B}$  were changed to  $\mathbf{B} = [0, 1]^T$ ?
- 

- 7** A plant with disturbance and reference inputs as shown in Figure 12.3 has the following matrices:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad \mathbf{D} = 0.$$

The reference input  $\mathbf{X}_r$  contains step functions and is generated by the  $2 \times 2$  matrix  $\mathbf{A}_r = 0$ . Design  $\mathbf{K}$ ,  $\mathbf{G}_r$ , and  $\mathbf{G}_d$  so that the closed-loop poles are at  $-3, -3$  and  $\mathbf{C}(\mathbf{X} - \mathbf{X}_r) = 0$  in steady state.

---

- 8** A system is described by the matrices shown:

$$\mathbf{A} = \begin{bmatrix} -28 & 10 & 9 \\ -29 & 10 & 10 \\ -36 & 13 & 11 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix}, \quad \mathbf{C} = [15, -6, -7], \quad \mathbf{D} = 0.$$

Find constant state-feedback matrix  $\mathbf{K}$  to place the closed-loop system eigenvalues at  $-3, -4$ , and  $-5$ .

---

- 9** An LTI system with size parameters  $n = 7, m = 3$  has been transformed to the form of  $\mathbf{A}'$  and  $\mathbf{B}'$  in Equation (12.23), where the diagonal blocks are of size  $n_1 = 2, n_2 = 2$ , and  $n_3 = 3$ . A state-feedback matrix  $\mathbf{K}'$  is to be found to preserve the diagonal-block sizes and place the closed-loop eigenvalues at predetermined locations. How many arbitrary parameters are available in the computation of  $\mathbf{K}'$ ?
- 

- 10** A system is described by the matrices given below:

$$\mathbf{A} = \begin{bmatrix} 5 & 1 & -6 \\ 4 & 1 & -5 \\ 8 & 1 & -9 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix}, \quad \mathbf{C} = [34, -14, -15], \quad \mathbf{D} = 0.$$

Calculate the open-loop poles and zeros, and then the state-feedback matrix  $\mathbf{K}$  that places the closed-loop eigenvalues at  $-3, -4$ , and  $-5$ . Is the resulting closed-loop system a minimal realization of its transfer function?

- 11** The plant dimension  $n$  can have a significant effect on the size of the entries of the state-feedback matrix. Consider a single-input plant that has been transformed so that  $\mathbf{A}'$  and  $\mathbf{B}'$  have the form shown in Equation (12.5). For simplicity, assume that the bottom row of  $\mathbf{A}'$  contains only zeros and that a state-feedback matrix  $\mathbf{K}'$  is to be found to place all closed-loop poles at  $-10$ . Determine the magnitude of the largest entry of  $\mathbf{K}'$  for arbitrary  $n$ . What conclusion can you draw about the feasibility of controlling systems of high order  $n$  using one input?



# 13

## Optimal control

This chapter introduces a design method that automatically calculates the gain matrix  $\mathbf{K}$  in the state-feedback controller for which the plant input is  $\mathbf{U}(t) = \mathbf{R}(t) - \mathbf{K}\mathbf{X}(t)$ . The matrices describing the plant must be known and the plant must be stabilizable as described in Chapter 12. An optimization problem is solved to produce  $\mathbf{K}$ ; consequently, the parameters in the function to be optimized must be chosen according to design specifications such as transient performance, within constraints imposed by the plant structure.

Some design specifications inherently require optimization of a function of the input or other variables, without reference to transient performance. However, the feedback matrix  $\mathbf{K}$  can also be related to dynamic response. Loosely speaking, the more it is desired to change the free response of a system, the more input is required and the larger the entries of  $\mathbf{K}$ . Therefore, there is a trade-off in the optimization between good transient response and the size of the inputs.

The controller obtained from optimization may be a relatively complex when  $n$  is large, since there are  $mn$  parameters in  $\mathbf{K}$  when  $\mathbf{X} \in \mathbb{R}^n$  and  $\mathbf{U} \in \mathbb{R}^m$ . Normally, no additional derivative action is required in the controller to further improve transient performance, but the controller does not automatically incorporate integral or other dynamic action as may be required to eliminate the effect of step or other disturbances. Alternative controller structures are required for disturbance cancellation.

**LQ design** The class of feedback optimization problems discussed here is known as LQ, since the dynamical system is linear and the optimization function is quadratic.

The optimization problems for both continuous and discrete time are formulated in the next section. Basic continuous-time solutions are then derived. This optimization problem has many variations, and it is shown how to reduce essentially all of them to the basic form. The solution provides an open-loop control, but because the system is linear, it is possible to obtain the same input by means of a state-feedback matrix, resulting in closed-loop control. An important case, that of infinite time interval, results in a constant state-feedback matrix, which is derived. The essential computation required to produce this matrix is the solution of an algebraic Riccati equation (ARE), the solution of which is discussed.

The LQ optimal-control solution for discrete-time systems is also analyzed. It turns out that the continuous-time optimization can be transformed into an equivalent discrete-time optimization, and vice-versa.

**Continuous time** In basic continuous-time form, the function to be minimized is derived as follows. Let  $\mathbf{X}(t_0)$  be a fixed but arbitrary initial state of a system that obeys the state equation

$$(13.1) \quad \frac{d}{dt}\mathbf{X} = \mathbf{A}\mathbf{X} + \mathbf{B}\mathbf{U},$$

where, as usual,  $\mathbf{X} \in \mathbb{R}^n$ , and  $\mathbf{U} \in \mathbb{R}^m$ . The matrices  $\mathbf{A}$  and  $\mathbf{B}$  will be assumed to be constant although much of the development can be extended to time-varying systems with little difficulty. It is desired to minimize the scalar

$$(13.2) \quad \phi = \frac{1}{2}\mathbf{X}(t)^T\mathbf{S}(t)\mathbf{X}(t)\Big|_{t_0}^{t_f} + \frac{1}{2}\int_{t_0}^{t_f} \left( \mathbf{X}(t)^T\mathbf{Q}\mathbf{X}(t) + \mathbf{U}(t)^T\mathbf{R}\mathbf{U}(t) \right) dt$$

over a specified interval  $t \in [t_0, t_f)$  with  $t_f = \infty$  allowed. The scalar  $\phi$  is called a cost function. In this expression, the integral of the term  $\mathbf{X}(t)^T\mathbf{Q}\mathbf{X}(t)$  is the cost of nonzero state, and the smaller that  $\mathbf{X}(t)$  is or the faster it approaches zero, the less will be the integral of this term. The integral of the term  $\mathbf{U}(t)^T\mathbf{R}\mathbf{U}(t)$  is the cost of the control signal. The matrix  $\mathbf{R}$  is symmetric and positive definite (see Section 3.1 of Chapter 8) and  $\mathbf{Q}$  is symmetric but only positive semi-definite in general since not all entries of  $\mathbf{X}$  may be of interest.

The notation of the first term on the right-hand side of (13.2) simply means that

$$(13.3) \quad \mathbf{X}(t)^T\mathbf{S}(t)\mathbf{X}(t)\Big|_{t_0}^{t_f} = \mathbf{X}(t_f)^T\mathbf{S}(t_f)\mathbf{X}(t_f) - \mathbf{X}(t_0)^T\mathbf{S}(t_0)\mathbf{X}(t_0)$$

so that  $\mathbf{S}(t)$  need only be defined at  $t_0$  and  $t_f$ , such that  $\mathbf{S}(t_f)$  and  $-\mathbf{S}(t_0)$  are symmetric and positive semi-definite. This term explicitly weights the initial and the final state vectors. In most control problems, it suffices to specify that  $\mathbf{S}(t_0) = 0$  because  $\mathbf{X}(t_0)$  is fixed, and when  $t_f = \infty$ , the matrix  $\mathbf{S}(t_f)$  is also zero.

A controller that generates  $\mathbf{U}(t)$  over  $[t_0, t_f)$  such that  $\phi$  is a minimum is required. A closed-loop control law will be found for which the input vector  $\mathbf{U}$  is a linear function of the state vector  $\mathbf{X}$  at any time.

**Discrete time** The discrete-time problem is analogous: minimize

$$(13.4) \quad \phi = \frac{1}{2}\mathbf{X}(t)^T\mathbf{S}(t)\mathbf{X}(t)\Big|_{t_0}^{t_f} + \frac{1}{2}\sum_{t=t_0}^{t_f-1} \left( \mathbf{X}(t)^T\mathbf{Q}\mathbf{X}(t) + \mathbf{U}(t)^T\mathbf{R}\mathbf{U}(t) \right)$$

subject to

$$(13.5) \quad \mathbf{X}(t+1) = \mathbf{A}\mathbf{X}(t) + \mathbf{B}\mathbf{U}(t), \quad t = t_0, \dots, t_f-1.$$

**Choosing  $\mathbf{Q}$  and  $\mathbf{R}$**  In both the continuous-time and discrete-time optimization problems, the matrices  $\mathbf{Q}$  and  $\mathbf{R}$  weight the entries of  $\mathbf{X}(t)$  and  $\mathbf{U}(t)$  respectively and are chosen during the design process. Increasing the entries of  $\mathbf{R}$  relative to those in  $\mathbf{Q}$  increases the weight associated with  $\mathbf{U}(t)$ , so the entries of  $\mathbf{U}(t)$  will become smaller at optimum, resulting in longer-lasting transients. Conversely, smaller values of  $\mathbf{R}$  will result in larger entries of  $\mathbf{U}(t)$  at optimum, corresponding to faster transients. Changing the entries of  $\mathbf{Q}$  has the opposite

effect. Typically both  $\mathbf{Q}$  and  $\mathbf{R}$  are chosen to be diagonal to provide a simple means of weighting the entries of  $\mathbf{X}(t)$  and  $\mathbf{U}(t)$  individually, but off-diagonal terms are allowed when cross-products of entries of  $\mathbf{X}$  or  $\mathbf{U}$  are to be part of the cost function. Thus, the transients of the controlled system can be shaped indirectly by choice of  $\mathbf{Q}$  and  $\mathbf{R}$ .

Variations in the cost functions are obtained by including cross-products of the entries in  $\mathbf{X}$  and  $\mathbf{U}$  in the scalar  $\phi$ , by including quadratic terms containing entries of  $\mathbf{Y} \in \mathbb{R}^p$ , where

$$(13.6) \quad \mathbf{Y}(t) = \mathbf{C}\mathbf{X}(t) + \mathbf{D}\mathbf{U}(t)$$

with  $\mathbf{C}$ ,  $\mathbf{D}$  known constant matrices, and by considering the two cases in which  $\mathbf{D}$  is zero or not.

### Choosing diagonal weights

The entries of  $\mathbf{Q}$  and  $\mathbf{R}$  must be chosen according to the problem at hand, but one rule of thumb that is sometimes applied when the maximum desired magnitudes are known is to weight the variables according to these bounds. Then the cost becomes

$$(13.7) \quad \phi = \frac{1}{2} \sum_{i=1}^n \left( \frac{x_i(t)}{x_{i(\max)}} \right)^2 \Big|_{t_0}^{t_f} + \frac{1}{2} \int_{t_0}^{t_f} \left( \sum_{i=1}^n \left( \frac{x_i(t)}{x_{i(\max)}} \right)^2 + \sum_{i=1}^m \left( \frac{u_i}{u_{i(\max)}} \right)^2 \right) dt,$$

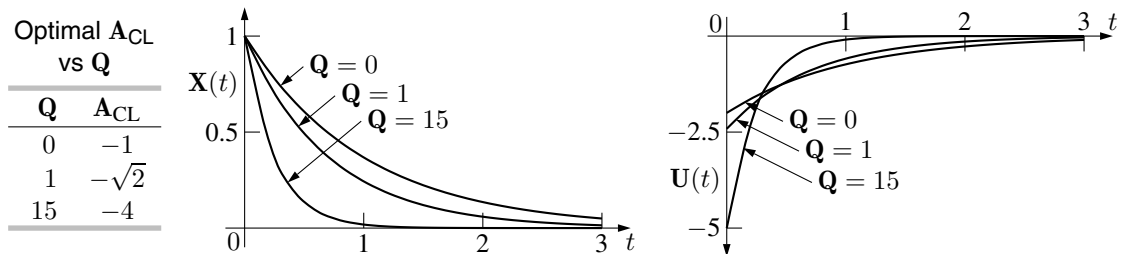
in the time-continuous case, for example, where the max-subscripted values are the desired maximum values. This rule of thumb gives a starting point for some designs; the parameters may have to be adjusted to produce acceptable responses and there is no guarantee that the desired bounds will not be exceeded.

Another method of similar simplicity is to weight variables according to the steady-state magnitude of their response to step inputs, for example.

### Example 1

The effect of varying weights

The effect of varying weights can be illustrated using a simple system described by  $\mathbf{A} = 1$ ,  $\mathbf{B} = 1$ ,  $\mathbf{C} = 1$ ,  $\mathbf{D} = 0$ ,  $\mathbf{X}(0) = 1$ ,  $t_0 = 0$ ,  $t_f = \infty$ . Choosing a fixed weight  $\mathbf{R} = 1$  for the input  $\mathbf{U}$  in the cost and varying the weight  $\mathbf{Q}$  on  $\mathbf{X}$  produces the transients shown in Figure 13.1. The results of Section 2.5 have been applied to obtain the optimal closed-loop  $\mathbf{A}_{CL}$ . Weighting  $\mathbf{X}$  more heavily with respect to  $\mathbf{U}$  produces faster transient response but larger initial inputs.



**Fig. 13.1** Table of optimal matrix  $\mathbf{A}_{CL}$  and the transients that result when  $\mathbf{Q}$  is varied. The larger the weight  $\mathbf{Q}$  on  $\mathbf{X}$ , the faster the resulting optimal transient response and the larger the initial input.

**Example 2**  
Scalar  
continuous-time  
system

For the the system described by matrices  $\mathbf{A} = \alpha$  and  $\mathbf{B} = 1$ , for which  $m = n = 1$ , let the cost function be

$$\phi = \int_{t_0}^{\infty} (qx(t)^2 + ru(t)^2) dt,$$

where  $r > 0$  and  $q \geq 0$ , so that  $\phi$  is non-negative. If the size of  $x(t)$  is of no interest, then setting  $q = 0$  and  $u(t) = 0$  will trivially minimize  $\phi$ . Otherwise, this integral is required to converge for  $\phi$  to be a minimum. For example, if the input is required to be  $u(t) = -kx(t)$  for nonzero feedback gain  $k$ , then minimizing  $\phi$  implies that  $x(t)$  and  $u(t)$  approach zero as  $t \rightarrow \infty$ .

**Example 3**  
Scalar  
discrete-time  
system and  
Lagrange  
multipliers

Let  $m = n = 1$  and let the discrete-time interval be  $t \in [0, 1]$  so that the cost function is

$$\phi = \frac{1}{2}(s(1)x(1)^2 - s(0)x(0)^2) + \frac{1}{2}(qx(0)^2 + ru(0)^2)$$

which is to be minimized subject to the constraint

$$ax(0) + bu(0) - x(1) = 0,$$

where the system matrices have been written  $\mathbf{A} = a$  and  $\mathbf{B} = b$ . Because  $x(0)$  is arbitrarily fixed, the factor  $s(0)$  is 0 with no loss of generality and  $s(1)$  can be written simply as  $s$ . The unknowns in this minimization are  $u(0)$  and  $x(1)$ . The method of Lagrange multipliers of elementary calculus can be used to obtain the minimum. Introduce a new variable  $\gamma_i$  for each constraint, add the products of the constraints and their multipliers  $\gamma_i$  to the function to be minimized, and equate the derivatives with respect to all unknown variables to zero. For the case at hand there is one constraint; let  $\gamma_1$  be the Lagrange multiplier, and differentiate the function

$$\frac{sx(1)^2}{2} + \frac{qx(0)^2 + ru(0)^2}{2} + \gamma_1(ax(0) + bu(0) - x(1))$$

with respect to unknowns  $\gamma_1$ ,  $u(0)$ , and  $x(1)$ . Equating the derivatives to zero results in three algebraic equations in the three unknowns. The derivative with respect to  $\gamma_1$  simply produces the constraint equation. Generalising this method to more general LTI systems and arbitrary discrete-time intervals solves the discrete-time optimal control problem and provides clues to the formulation of the continuous-time problem.

**Symmetric  
weighting  
matrices**

The matrices  $\mathbf{S}(t_f)$ ,  $\mathbf{S}(t_0)$ ,  $\mathbf{Q}$ , and  $\mathbf{R}$  in (13.2) and (13.4) can be assumed to be symmetric. To show that this assumption implies no loss of generality, consider an arbitrary real square matrix  $\mathbf{Q}$  and the quadratic term

$$(13.8) \quad \begin{aligned} \mathbf{X}^T \mathbf{Q} \mathbf{X} &= \mathbf{X}^T \left( \frac{1}{2}(\mathbf{Q} + \mathbf{Q}^T) + \frac{1}{2}(\mathbf{Q} - \mathbf{Q}^T) \right) \mathbf{X} \\ &= \mathbf{X}^T \frac{1}{2}(\mathbf{Q} + \mathbf{Q}^T) \mathbf{X} + \mathbf{X}^T \frac{1}{2}(\mathbf{Q} - \mathbf{Q}^T) \mathbf{X} = \mathbf{X}^T \mathbf{Q}_s \mathbf{X} + \mathbf{X}^T \mathbf{Q}_k \mathbf{X} \end{aligned}$$

where  $\mathbf{Q}$  has been written as the sum of two parts  $\mathbf{Q}_s$  and  $\mathbf{Q}_k$  as shown. By construction,  $\mathbf{Q}_s = \mathbf{Q}_s^T$  and this matrix is called the *symmetric part* of  $\mathbf{Q}$ . The matrix  $\mathbf{Q}_k$  satisfies

$\mathbf{Q}_k = -\mathbf{Q}_k^T$  and is said to be skew-symmetric; that is, the entries  $q_{ij}$  of  $\mathbf{Q}_k$  satisfy  $q_{ij} = -q_{ji}$ . Consequently, the term  $\mathbf{X}^T \mathbf{Q}_k \mathbf{X} = \sum_{i,j=1}^n q_{ij} x_i x_j$  is zero, and the value of the quadratic term is determined entirely by the symmetric part of  $\mathbf{Q}$ . Similarly, the other matrices in the cost function can be assumed to be symmetric.

### Tests for positive definiteness

From the above, the symmetric part of a real matrix determines whether it is positive-definite. From the discussion in Section 3 of Chapter 8, a symmetric real matrix has real eigenvalues and can be diagonalized by an orthogonal similarity transformation. For the scalar  $\mathbf{X}^T \mathbf{Q} \mathbf{X}$  and symmetric  $\mathbf{Q}$ , let  $\mathbf{M}$  be such a transformation matrix, so that if  $\mathbf{X} = \mathbf{M} \mathbf{Y}$ , the scalar can be written

$$(13.9) \quad \mathbf{X}^T \mathbf{Q} \mathbf{X} = \mathbf{Y}^T \mathbf{M}^T \mathbf{Q} \mathbf{M} \mathbf{Y} = \sum_{i=1}^n \lambda_i y_i^2.$$

This shows that a symmetric matrix  $\mathbf{Q}$  is positive definite if and only its eigenvalues are all positive, and  $\mathbf{Q}$  is positive semidefinite if its eigenvalues are all non-negative.

A further test of symmetric real matrices that is useful for hand calculation is to compute the *leading principal minors* of  $\mathbf{Q} = [q_{ij}]$ :

$$\det q_{11}, \quad \det \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix}, \quad \det \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{bmatrix}, \quad \dots \det \mathbf{Q}.$$

The symmetric matrix  $\mathbf{Q}$  is positive-definite if and only if all of these minors are positive, and positive semidefinite if and only if *all* principal minors are non-negative. The principal minors include the leading principal minors above and the determinants of all submatrices of  $\mathbf{Q}$  obtained by successively removing an  $i$ th row and  $i$ th column; that is, of all square submatrices with diagonals on the diagonal of  $\mathbf{Q}$ .

## 2

### Continuous-time optimization

It will be shown that the solutions of the continuous-time and discrete-time optimization problems are closely related, but first the continuous-time case will be discussed in detail.

The minimisation of (13.2) subject to (13.1) is accomplished as follows. Let  $\mathbf{\Gamma}(t) \in \mathbb{R}^n$  be an arbitrary vector, and add a product, equal to zero when (13.1) holds, to  $\phi$  in (13.2) as follows:

$$(13.10) \quad \phi = \frac{1}{2} \mathbf{X}(t)^T \mathbf{S}(t) \mathbf{X}(t) \Big|_{t_0}^{t_f} + \int_{t_0}^{t_f} \left( \frac{1}{2} \mathbf{X}(t)^T \mathbf{Q} \mathbf{X}(t) + \frac{1}{2} \mathbf{U}(t)^T \mathbf{R} \mathbf{U}(t) + \mathbf{\Gamma}^T (\mathbf{A} \mathbf{X}(t) + \mathbf{B} \mathbf{U}(t) - \dot{\mathbf{X}}) \right) dt.$$

Integrating the term containing  $\dot{\mathbf{X}}$  by parts results in

$$(13.11) \quad \int_{t_0}^{t_f} -\mathbf{\Gamma}^T \dot{\mathbf{X}} dt = -\mathbf{\Gamma}(t)^T \mathbf{X}(t) \Big|_{t_0}^{t_f} + \int_{t_0}^{t_f} \dot{\mathbf{\Gamma}}^T \mathbf{X} dt,$$

so that (13.10) becomes

$$(13.12) \quad \phi = \left( \frac{1}{2} \mathbf{X}^T \mathbf{S} \mathbf{X} - \mathbf{X}^T \mathbf{\Gamma} \right) \Big|_{t_0}^{t_f} + \int_{t_0}^{t_f} \left( \frac{1}{2} \mathbf{X}^T \mathbf{Q} \mathbf{X} + \frac{1}{2} \mathbf{U}^T \mathbf{R} \mathbf{U} + \mathbf{X}^T \mathbf{A}^T \mathbf{\Gamma} + \mathbf{U}^T \mathbf{B}^T \mathbf{\Gamma} + \mathbf{X}^T \dot{\mathbf{\Gamma}} \right) dt.$$

By the calculus of variations, if  $\mathbf{X}(\cdot)$ ,  $\mathbf{U}(\cdot)$ , and  $\mathbf{\Gamma}(\cdot)$  minimize  $\phi$ , then the sensitivity of  $\phi$  to arbitrary small changes  $\delta\mathbf{X}(\cdot)$ ,  $\delta\mathbf{U}(\cdot)$ , and  $\delta\mathbf{\Gamma}(\cdot)$  in these functions will be zero. Equating the differential with respect to  $\mathbf{X}$  of  $\phi$  in (13.12) to zero gives

$$(13.13) \quad 0 = \delta\mathbf{X}^T (\mathbf{S} \mathbf{X} - \mathbf{\Gamma}) \Big|_{t_0}^{t_f} + \int_{t_0}^{t_f} \delta\mathbf{X}^T (\mathbf{Q} \mathbf{X} + \mathbf{A}^T \mathbf{\Gamma} + \dot{\mathbf{\Gamma}}) dt$$

in which the left term applies only at  $t = t_0$  and  $t = t_f$ , whereas the right term implies that

$$(13.14) \quad \dot{\mathbf{\Gamma}} = -\mathbf{Q} \mathbf{X} - \mathbf{A}^T \mathbf{\Gamma}$$

over  $(t_0, t_f)$ . In typical optimal control problems,  $\mathbf{X}(t_0)$  is an arbitrary but fixed initial value, implying that  $\delta\mathbf{X}(t_0) = 0$  and  $\mathbf{S}(t_0)$  can be set to 0. Then  $\mathbf{S}(t_f)$  is simply written as  $\mathbf{S}$  and the first term of (13.13) reduces to

$$(13.15) \quad \mathbf{\Gamma}(t_f) = \mathbf{S} \mathbf{X}(t_f),$$

which is a final-value constraint on the solution of the differential equation given in (13.14).

Equating the differential of  $\phi$  with respect to  $\mathbf{\Gamma}$  in (13.10) to zero gives

$$(13.16) \quad 0 = \int_{t_0}^{t_f} \delta\mathbf{\Gamma}^T (\mathbf{A} \mathbf{X} + \mathbf{B} \mathbf{U} - \dot{\mathbf{X}}) dt,$$

and since  $\delta\mathbf{\Gamma}$  is arbitrary, this equation reduces to the original constraint given by (13.1).

Finally, taking the differential with respect to  $\mathbf{U}$  yields

$$(13.17) \quad 0 = \int_{t_0}^{t_f} \delta\mathbf{U}^T (\mathbf{R} \mathbf{U} + \mathbf{B}^T \mathbf{\Gamma}) dt,$$

which must be true for arbitrary changes  $\delta\mathbf{U}$ , implying that

$$(13.18) \quad \mathbf{U} = -\mathbf{R}^{-1} \mathbf{B}^T \mathbf{\Gamma},$$

where  $\mathbf{R}$  is invertible since it is positive-definite.

Combining (13.1), (13.14), and (13.18) results in

$$(13.19) \quad \begin{bmatrix} \mathbf{A} - \mathbf{I} \frac{d}{dt} & \mathbf{B} & 0 \\ \mathbf{Q} & 0 & \mathbf{A}^T + \mathbf{I} \frac{d}{dt} \\ 0 & \mathbf{R} & \mathbf{B}^T \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ \mathbf{U} \\ \mathbf{\Gamma} \end{bmatrix} = 0,$$

which contains  $2n$  simultaneous differential equations and  $m$  algebraic equations. Eliminating  $\mathbf{U}$  gives

$$(13.20) \quad \begin{bmatrix} \mathbf{A} - \mathbf{I} \frac{d}{dt} & -\mathbf{B} \mathbf{R}^{-1} \mathbf{B}^T \\ \mathbf{Q} & \mathbf{A}^T + \mathbf{I} \frac{d}{dt} \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ \mathbf{\Gamma} \end{bmatrix} = 0.$$

This is called the Hamiltonian differential equation of the problem, and consists of  $2n$  differential equations, which must be solved subject to the  $n$  initial conditions given by  $\mathbf{X}(t_0)$  and the  $n$  final conditions given by (13.15).

The above development solves the simplest multi-input optimal control problem. The optimal input function  $\mathbf{U}(t)$  is obtained by solving (13.20) over the required interval, subject to the given boundary conditions, and then substituting in (13.18). This, however, is open-loop control. A more desirable closed-loop solution can be obtained for linear systems.

A generalized formulation that allows for the many variations in cost function is given in Sections 2.1 and 2.2, which follow.

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**Example 4**  
Minimizing  
speed

A spacecraft is to be slowed as much as possible in  $t_f$  seconds by a jet of plasma. The released mass of the plasma is negligible compared to the mass of the spacecraft. The impulse (the integral of the force over time) of the jet is to be minimized, and the optimal force is to be found as a function of time.

Let  $x$  be the momentum of the craft, which is proportional to velocity since the change in mass is negligible. The rate of change of momentum is  $\dot{x} = u$ , where  $u(t)$  is the force. The function to be optimized is

$$\phi = sx^2(t_f) + \int_0^{t_f} u^2(t) dt,$$

where the design constant  $s$  weights the momentum at  $t_f$ . Thus the problem matrices are  $\mathbf{A} = 0$ ,  $\mathbf{B} = 1$ ,  $\mathbf{Q} = 0$ ,  $\mathbf{R} = 1$ , and  $\mathbf{S} = s$ . Equation (13.20) is then

$$\begin{bmatrix} \dot{x} \\ \dot{\gamma} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \gamma \end{bmatrix}$$

with  $x(0)$  fixed and  $\gamma(t_f) = sx(t_f)$ . Denoting the matrix on the right of the above equation as  $\mathbf{H}$ , write the solution as

$$\begin{bmatrix} x(t) \\ \gamma(t) \end{bmatrix} = e^{t\mathbf{H}} \begin{bmatrix} x(0) \\ \gamma(0) \end{bmatrix} = \begin{bmatrix} 1 & -t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x(0) \\ \gamma(0) \end{bmatrix},$$

where  $\gamma(0)$  is unknown but where  $\gamma(t_f) = sx(t_f)$ . The value of  $\gamma(t)$ , a constant in this example, is obtained from these equations as

$$\gamma(t) = \frac{sx(0)}{1 + st_f}.$$

Substituting this into Equation (13.18) gives the optimum input function  $u(t)$  as

$$u(t) = -\frac{sx(0)}{1 + st_f}.$$

### 2.1 A generalized formulation

In order to avoid treating the many cases individually, temporarily define the composite matrices

$$(13.21) \quad \mathbf{M} = \begin{bmatrix} \mathbf{A} & 0 & \mathbf{B} \\ \mathbf{C} & -\mathbf{I} & \mathbf{D} \end{bmatrix}, \quad \mathbf{N} = \begin{bmatrix} \mathbf{I} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \tilde{\mathbf{X}} = \begin{bmatrix} \mathbf{X} \\ \mathbf{Y} \\ \mathbf{U} \end{bmatrix}$$

and consider the simplified and generalized problem of minimizing

$$(13.22) \quad \phi = \frac{1}{2} \tilde{\mathbf{X}}(t)^T \tilde{\mathbf{S}}(t) \tilde{\mathbf{X}}(t) \Big|_{t_0}^{t_f} + \frac{1}{2} \int_{t_0}^{t_f} \tilde{\mathbf{X}}(t)^T \tilde{\mathbf{Q}} \tilde{\mathbf{X}}(t) dt$$

subject to constraint equation

$$(13.23) \quad \mathbf{N} \frac{d}{dt} \tilde{\mathbf{X}} = \mathbf{M} \tilde{\mathbf{X}},$$

where  $\tilde{\mathbf{Q}}$  and  $\tilde{\mathbf{S}}$  are square real matrices and  $\tilde{\mathbf{S}}(t_f)$  and  $-\tilde{\mathbf{S}}(t_0)$  are symmetric and positive semidefinite.

The solution can be developed similarly to the previous section. Let  $\tilde{\Gamma}(t) \in \mathbb{R}^q$  be an arbitrary vector. Include constraint (13.23) pre-multiplied by  $\tilde{\Gamma}^T$  in cost function (13.22):

$$(13.24) \quad \phi = \frac{1}{2} \tilde{\mathbf{X}}^T \tilde{\mathbf{S}} \tilde{\mathbf{X}} \Big|_{t_0}^{t_f} + \int_{t_0}^{t_f} \left( \frac{1}{2} \tilde{\mathbf{X}}^T \tilde{\mathbf{Q}} \tilde{\mathbf{X}} + \tilde{\Gamma}^T (\mathbf{M} \tilde{\mathbf{X}} - \mathbf{N} \dot{\tilde{\mathbf{X}}}) \right) dt.$$

By the calculus of variations, at a minimum the change in  $\phi$  with respect to arbitrary differential  $\delta \tilde{\Gamma}(\cdot)$  in  $\tilde{\Gamma}(\cdot)$  must be zero:

$$(13.25) \quad 0 = \int_{t_0}^{t_f} \delta \tilde{\Gamma}^T (\mathbf{M} \tilde{\mathbf{X}} - \mathbf{N} \dot{\tilde{\mathbf{X}}}) dt,$$

and this yields the original constraint Equation (13.23).

To take the differential with respect to  $\tilde{\mathbf{X}}$ , integrate the term  $-\tilde{\Gamma}^T \mathbf{N} \dot{\tilde{\mathbf{X}}}$  by parts to get

$$(13.26) \quad \phi = \left( \frac{1}{2} \tilde{\mathbf{X}}^T \tilde{\mathbf{S}} \tilde{\mathbf{X}} - \tilde{\Gamma}^T \mathbf{N} \tilde{\mathbf{X}} \right) \Big|_{t_0}^{t_f} + \int_{t_0}^{t_f} \left( \frac{1}{2} \tilde{\mathbf{X}}^T \tilde{\mathbf{Q}} \tilde{\mathbf{X}} + \tilde{\Gamma}^T \mathbf{M} \tilde{\mathbf{X}} + \dot{\tilde{\Gamma}}^T \mathbf{N} \tilde{\mathbf{X}} \right) dt.$$

Now equate the first variation of  $\phi$  with respect to variation  $\delta \tilde{\mathbf{X}}(\cdot)$  in  $\tilde{\mathbf{X}}(\cdot)$  to 0:

$$(13.27) \quad 0 = \delta \tilde{\mathbf{X}}^T (\tilde{\mathbf{S}} \tilde{\mathbf{X}} - \mathbf{N}^T \tilde{\Gamma}) \Big|_{t_0}^{t_f} + \int_{t_0}^{t_f} \delta \tilde{\mathbf{X}}^T (\tilde{\mathbf{Q}} \tilde{\mathbf{X}} + \mathbf{M}^T \tilde{\Gamma} + \mathbf{N}^T \dot{\tilde{\Gamma}}) dt.$$

Since  $\delta \tilde{\mathbf{X}}$  is arbitrary in  $(t_0, t_f)$ , the equation

$$(13.28) \quad \tilde{\mathbf{Q}} \tilde{\mathbf{X}} + \mathbf{M}^T \tilde{\Gamma} + \mathbf{N}^T \dot{\tilde{\Gamma}} = 0$$

must hold. Together, Equations (13.28) and (13.23) give

$$(13.29) \quad \begin{bmatrix} \mathbf{M} - \mathbf{N} \frac{d}{dt} & 0 \\ \tilde{\mathbf{Q}} & \mathbf{M}^T + \mathbf{N}^T \frac{d}{dt} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{X}} \\ \tilde{\Gamma} \end{bmatrix} = 0,$$



which is a set of differential and algebraic equations, in general. From the first term of (13.27), these equations are subject to transversality (boundary) conditions defined at  $t = 0$  and  $t = t_f$  by

$$(13.30) \quad \delta \tilde{\mathbf{X}}(t)^T (\tilde{\mathbf{S}} \tilde{\mathbf{X}}(t) - \mathbf{N}^T \tilde{\mathbf{\Gamma}}(t)) = 0.$$

With  $\mathbf{N}$  defined in (13.21), normally the term  $\tilde{\mathbf{X}}^T \tilde{\mathbf{S}} \tilde{\mathbf{X}}$  is chosen to be  $\mathbf{X}^T \mathbf{S}_{xx} \mathbf{X}$  or, to include  $\mathbf{Y}$  at  $t_f$  in the cost when  $\mathbf{D} = 0$ , to be  $\mathbf{Y}^T \mathbf{S}_{yy} \mathbf{Y} = \mathbf{X}^T \mathbf{C}^T \mathbf{S}_{yy} \mathbf{C} \mathbf{X}$  with  $\mathbf{S}_{xx}$  and  $\mathbf{S}_{yy}$  positive-semidefinite. Then for example, if  $t_0, t_f$  are finite and  $\mathbf{X}(t_0) = \mathbf{X}_0$  is a fixed initial vector and  $\mathbf{X}(t_f)$  is free, then  $\delta \mathbf{X}(t_0) = 0$  and the boundary conditions are

$$(13.31a) \quad \mathbf{X}(t_0) = \mathbf{X}_0$$

$$(13.31b) \quad \mathbf{S} \mathbf{X}(t_f) = \mathbf{\Gamma}(t_f),$$

where  $\mathbf{S}$  is either  $\mathbf{S}_{xx}$  or  $\mathbf{C}^T \mathbf{S}_{yy} \mathbf{C}$  as appropriate.

### Example 5 Generalized systems

Not only does Equation (13.23) represent the general class of state-space systems, it can also model systems with improper transfer matrices, which therefore do not have state-space models. Consider the differentiator

$$y(t) = \frac{d}{dt} u(t),$$

for which Equation (13.23) is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} x \\ y \\ u \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ u \end{bmatrix}.$$

The above development implies that an optimal design can be obtained by solving (13.29) subject to suitable boundary conditions, and that this method is analogous to the classical optimal control design method for state-space systems.

## 2.2 Particular cases

When Equations 13.23 represent a state-space model as in Equation (13.21) and only the initial and final states appear in the first term of the cost function, then the following substitutions are required:

$$(13.32) \quad \begin{bmatrix} \mathbf{Q}_{xx} & \mathbf{Q}_{xy} & \mathbf{Q}_{xu} \\ \mathbf{Q}_{xy}^T & \mathbf{Q}_{yy} & \mathbf{Q}_{yu} \\ \mathbf{Q}_{xu}^T & \mathbf{Q}_{yu}^T & \mathbf{Q}_{uu} \end{bmatrix} \leftarrow \tilde{\mathbf{Q}}, \quad \begin{bmatrix} \mathbf{S} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \leftarrow \tilde{\mathbf{S}}, \quad \begin{bmatrix} \mathbf{\Gamma} \\ \mathbf{V} \end{bmatrix} \leftarrow \tilde{\mathbf{\Gamma}}.$$

These substitutions correspond to minimization of the cost function

$$(13.33) \quad \phi = \frac{1}{2} \mathbf{X}^T \mathbf{S} \mathbf{X} \Big|_{t_0}^{t_f} + \frac{1}{2} \int_{t_0}^{t_f} [\mathbf{X}^T, \mathbf{Y}^T, \mathbf{U}^T] \tilde{\mathbf{Q}} \begin{bmatrix} \mathbf{X} \\ \mathbf{Y} \\ \mathbf{U} \end{bmatrix} dt$$

subject to (13.1) and (13.6). Special cases of the the above general matrix  $\tilde{\mathbf{Q}}$  are used in practice. With the above substitutions, (13.29) becomes

$$(13.34) \quad \begin{bmatrix} \mathbf{A} - \mathbf{I} \frac{d}{dt} & 0 & \mathbf{B} & 0 & 0 \\ \mathbf{C} & -\mathbf{I} & \mathbf{D} & 0 & 0 \\ \mathbf{Q}_{xx} & \mathbf{Q}_{xy} & \mathbf{Q}_{xu} & \mathbf{A}^T + \mathbf{I} \frac{d}{dt} & \mathbf{C}^T \\ \mathbf{Q}_{xy}^T & \mathbf{Q}_{yy} & \mathbf{Q}_{yu} & 0 & -\mathbf{I} \\ \mathbf{Q}_{xu}^T & \mathbf{Q}_{yu}^T & \mathbf{Q}_{uu} & \mathbf{B}^T & \mathbf{D}^T \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ \mathbf{Y} \\ \mathbf{U} \\ \mathbf{\Gamma} \\ \mathbf{V} \end{bmatrix} = 0$$

which is a set of simultaneous algebraic and differential equations, as is (13.29) in general. Adding multiples of the second and fifth rows of (13.34) to other rows in order to eliminate  $\mathbf{Y}$  and  $\mathbf{V}$  produces

$$(13.35) \quad \begin{bmatrix} \mathbf{A} - \mathbf{I} \frac{d}{dt} & 0 & \mathbf{B} & 0 & 0 \\ \mathbf{C} & -\mathbf{I} & \mathbf{D} & 0 & 0 \\ \mathbf{Q}'_{xx} & 0 & \mathbf{Q}'_{xu} & \mathbf{A}^T + \mathbf{I} \frac{d}{dt} & 0 \\ \mathbf{Q}'_{xy} & \mathbf{Q}_{yy} & \mathbf{Q}_{yu} & 0 & -\mathbf{I} \\ \mathbf{Q}'_{xu} & 0 & \hat{\mathbf{R}} & \mathbf{B}^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ \mathbf{Y} \\ \mathbf{U} \\ \mathbf{\Gamma} \\ \mathbf{V} \end{bmatrix} = 0$$

where

$$(13.36a) \quad \hat{\mathbf{R}} = \mathbf{Q}_{uu} + \mathbf{D}^T \mathbf{Q}_{yu} + \mathbf{Q}_{yu}^T \mathbf{D} + \mathbf{D}^T \mathbf{Q}_{yy} \mathbf{D}$$

$$(13.36b) \quad \mathbf{Q}'_{xx} = \mathbf{Q}_{xx} + \mathbf{C}^T \mathbf{Q}_{xy}^T + \mathbf{Q}_{xy} \mathbf{C} + \mathbf{C}^T \mathbf{Q}_{yy} \mathbf{C}$$

$$(13.36c) \quad \mathbf{Q}'_{xu} = \mathbf{Q}_{xu} + \mathbf{C}^T \mathbf{Q}_{yu} + \mathbf{Q}_{xy} \mathbf{D} + \mathbf{C}^T \mathbf{Q}_{yy} \mathbf{D}.$$

Deleting  $\mathbf{Y}$  and  $\mathbf{V}$  from these equations leaves the following:

$$(13.37) \quad \begin{bmatrix} \mathbf{A} - \mathbf{I} \frac{d}{dt} & \mathbf{B} & 0 \\ \mathbf{Q}'_{xx} & \mathbf{Q}'_{xu} & \mathbf{A}^T + \mathbf{I} \frac{d}{dt} \\ \mathbf{Q}'_{xu} & \hat{\mathbf{R}} & \mathbf{B}^T \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ \mathbf{U} \\ \mathbf{\Gamma} \end{bmatrix} = 0.$$

Provided that  $\tilde{\mathbf{Q}}$  in (13.33) is chosen such that  $\hat{\mathbf{R}}$  in (13.36a) is nonsingular, then  $\mathbf{U}$  can be solved as

$$(13.38) \quad \mathbf{U} = -\hat{\mathbf{R}}^{-1} (\mathbf{Q}'_{xu} \mathbf{X} + \mathbf{B}^T \mathbf{\Gamma}),$$

and substituted into (13.37) to leave the simplified equations

$$(13.39) \quad \begin{bmatrix} \hat{\mathbf{A}} - \mathbf{I} \frac{d}{dt} & -\mathbf{B} \hat{\mathbf{R}}^{-1} \mathbf{B}^T \\ \hat{\mathbf{Q}} & \hat{\mathbf{A}}^T + \mathbf{I} \frac{d}{dt} \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ \mathbf{\Gamma} \end{bmatrix} = 0$$

where

$$(13.40a) \quad \hat{\mathbf{A}} = \mathbf{A} - \mathbf{B} \hat{\mathbf{R}}^{-1} \mathbf{Q}'_{xu}$$

$$(13.40b) \quad \hat{\mathbf{Q}} = \mathbf{Q}'_{xx} - \mathbf{Q}'_{xu} \hat{\mathbf{R}}^{-1} \mathbf{Q}'_{xu}.$$

Equation (13.39) is identical in form to Equation (13.20) of the basic formulation.

With the above general development in hand, the solution of (13.29) subject to (13.30) is most simply approached in its particular cases by simplification, as follows.

**Basic case** Setting all off-diagonal submatrices of  $\mathbf{Q}$  in (13.32) to zero, with  $\mathbf{Q}_{yy} = 0$ ,  $\mathbf{Q}_{uu} = \mathbf{R}$ , and  $\mathbf{Q}_{xx}$  written simply as  $\mathbf{Q}$  corresponds to minimizing (13.2). Then the expression (13.38) for the control vector becomes, as before,

$$(13.41) \quad \mathbf{U} = -\mathbf{R}^{-1}\mathbf{B}^T\mathbf{T}$$

and the submatrices in (13.39) are

$$(13.42a) \quad \hat{\mathbf{R}} = \mathbf{R},$$

$$(13.42b) \quad \hat{\mathbf{A}} = \mathbf{A},$$

$$(13.42c) \quad \hat{\mathbf{Q}} = \mathbf{Q}.$$

**Y in the cost** Minimizing (13.2) with  $\mathbf{D} = 0$ , the term  $\mathbf{X}^T\mathbf{Q}\mathbf{X}$  replaced by  $\mathbf{Y}^T\mathbf{Q}\mathbf{Y}$ , and  $\mathbf{Y} = \mathbf{C}\mathbf{X}$  corresponds to setting the off-diagonal submatrices of  $\hat{\mathbf{Q}}$  equal to zero as before and writing  $\mathbf{Q}_{uu} = \mathbf{R}$ ,  $\mathbf{Q}_{xx} = 0$ , with  $\mathbf{Q}_{yy}$  written simply as  $\mathbf{Q}$ . Then  $\hat{\mathbf{R}}$ ,  $\mathbf{U}$ , and  $\hat{\mathbf{A}}$  are the same as for the basic case, but

$$(13.43) \quad \hat{\mathbf{Q}} = \mathbf{C}^T\mathbf{Q}\mathbf{C}.$$

**Nonzero D** If the above conditions hold but  $\mathbf{D} \neq 0$  then the control vector is

$$(13.44) \quad \mathbf{U} = -\hat{\mathbf{R}}^{-1}(\mathbf{D}^T\mathbf{C}\mathbf{X} + \mathbf{B}^T\mathbf{T})$$

and equations (13.42) become

$$(13.45a) \quad \hat{\mathbf{R}} = \mathbf{R} + \mathbf{D}^T\mathbf{D},$$

$$(13.45b) \quad \hat{\mathbf{A}} = \mathbf{A} - \mathbf{B}\hat{\mathbf{R}}^{-1}\mathbf{D}^T\mathbf{C},$$

$$(13.45c) \quad \hat{\mathbf{Q}} = \mathbf{C}^T(\mathbf{C} - \mathbf{D}\hat{\mathbf{R}}^{-1}\mathbf{D}^T)\mathbf{C}.$$

Other cases can be simplified similarly. In addition to the above examples, in practice  $\mathbf{R}$ ,  $\mathbf{Q}_{yy}$  and  $\mathbf{Q}_{xx}$  are often chosen to be diagonal when the cost associated with the entries of  $\mathbf{U}$  and of  $\mathbf{X}$  or  $\mathbf{Y}$  can be considered independently.

**Example 6**  
Equivalent  
optimization

We wish to find the matrices in (13.39) given the system matrices  $\mathbf{A} = 0$ ,  $\mathbf{B} = 1$ ,  $\mathbf{C} = 1$ ,  $\mathbf{D} = 1$ , and cost-function matrices

$$\tilde{\mathbf{S}} = \begin{bmatrix} s & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \tilde{\mathbf{Q}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & r \end{bmatrix}.$$

Substitutions (13.36) give

$$\hat{\mathbf{R}} = r + 0 + 0 + q = r + q, \quad \mathbf{Q}'_{xx} = 0 + 0 + 0 + q = q, \quad \mathbf{Q}'_{xu} = 0 + 0 + 0 + q = q.$$

From (13.38) the optimal input is

$$u(t) = -(qx(t) + \gamma(t))/(q + r),$$

and (13.39) must be solved for matrices given by (13.40b) as

$$\hat{\mathbf{A}} = 0 - q/(q + r) = -q/(q + r), \quad \hat{\mathbf{Q}} = q - q^2/(q + r).$$

### 2.3 Constructing a state feedback solution

The optimal input function  $\mathbf{U}$  in (13.38) is a linear combination of state vector  $\mathbf{X}$  and *co-state* vector  $\mathbf{\Gamma}$ . To obtain these vectors, (13.39) must be solved subject to constraints derived from (13.30), according to the problem at hand. Not only can such solutions normally be obtained, but it turns out that a linear expression for  $\mathbf{\Gamma}$  as a function of  $\mathbf{X}$  is obtainable for all  $\mathbf{X}_0$ , so that  $\mathbf{U}(t)$  is a linear function of  $\mathbf{X}(t)$  at any time  $t$ , giving a closed-loop state-feedback control configuration. Equation (13.39) is of identical form to Equation (13.20); therefore in the following, the caret on the matrices  $\hat{\mathbf{A}}$ ,  $\hat{\mathbf{Q}}$ , and  $\hat{\mathbf{R}}$  will be dropped for simplicity of notation.

Suppose that  $\mathbf{\Gamma}$  is a linear function of  $\mathbf{X}$ , as follows:

$$(13.46) \quad \mathbf{\Gamma}(t) = \mathbf{P}(t)\mathbf{X}(t).$$

Then from (13.38), a feedback solution results, with  $\mathbf{U}$  proportional to  $\mathbf{X}$  for all time.

The matrix  $\mathbf{P}(t)$  can be found as follows. Substitute formula (13.46) into the bottom block equation of (13.20) to get

$$(13.47) \quad \mathbf{Q}\mathbf{X} + \mathbf{A}^T\mathbf{P}\mathbf{X} + \dot{\mathbf{P}}\mathbf{X} + \mathbf{P}\dot{\mathbf{X}} = 0$$

and substitute for  $\dot{\mathbf{X}}$  from the top block equation to obtain

$$(13.48) \quad (\dot{\mathbf{P}} + \mathbf{P}\mathbf{A} + \mathbf{A}^T\mathbf{P} - \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P} + \mathbf{Q})\mathbf{X} = 0.$$

This equation is to be true for all  $\mathbf{X}(t)$ , which can take on any value since  $\mathbf{X}(t_0)$  is arbitrary. Therefore, the parenthesized factor in this equation is zero and the matrix  $\mathbf{P}(t)$  relating  $\mathbf{\Gamma}(t)$  to  $\mathbf{X}(t)$  satisfies

$$(13.49) \quad -\dot{\mathbf{P}} = \mathbf{P}\mathbf{A} + \mathbf{A}^T\mathbf{P} - \mathbf{P}(\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T)\mathbf{P} + \mathbf{Q},$$

subject to boundary conditions derived from (13.30). For example, (13.46) and (13.31) imply that

$$(13.50) \quad \mathbf{P}(t_f) = \mathbf{S}(t_f),$$

which is the boundary condition that allows (13.49) to be solved backwards in time over the interval  $[t_0, t_f]$ .

#### Riccati differential equation

Equation (13.49) is of a form known as a *Riccati differential equation*. The computational requirement of the optimization has been reduced to the backward solution of this nonlinear differential equation subject to a final condition, instead of solving the  $2n$  linear differential equations given by (13.20) subject to boundary conditions.

The right-hand side of (13.49) is locally Lipschitz (see Section 2.1 of Chapter 2), its solutions are therefore unique for given boundary conditions, and from the symmetry of (13.49) and (13.50),  $\mathbf{P}$  is symmetric. Thus a symmetric matrix  $\mathbf{S}(t_f) \geq 0$  has to be specified as part of the problem specification.

It can also be shown that under reasonable assumptions, namely the stabilizability of (13.1), positive-definiteness of  $\mathbf{R}$ , and observability of (13.1) by output vector  $\mathbf{Q}^{1/2}\mathbf{X}$  where  $\mathbf{Q} = (\mathbf{Q}^{1/2})^2$ , that the solution  $\mathbf{P}$  to (13.49) exists and is bounded and positive-definite for all  $t$  in  $(t_0, t_f)$  [57, 18]. These conditions can be relaxed somewhat; if

(13.1) is merely stabilizable by inputs  $\mathbf{U}$  and detectable by output vector  $\mathbf{Q}^{1/2}\mathbf{X}$ , then  $\mathbf{P}(t)$  is positive semi-definite and the uncontrollable modes of the system, which are asymptotically stable, are unchanged by the addition of feedback.

**Feedback matrix** The optimal control vector is obtained by substituting for  $\mathbf{\Gamma}(t)$  in (13.38) to obtain a linear function of  $\mathbf{X}(t)$  as follows:

$$(13.51) \quad \mathbf{U}(t) = -\hat{\mathbf{R}}^{-1}(\mathbf{Q}'_{xu} + \mathbf{B}^T \mathbf{P}(t)) \mathbf{X}(t) = -\mathbf{K}(t) \mathbf{X}(t).$$

This is the required state feedback, with time-varying feedback matrix

$$(13.52) \quad \mathbf{K}(t) = \hat{\mathbf{R}}^{-1}(\mathbf{Q}'_{xu} + \mathbf{B}^T \mathbf{P}(t)).$$

The term  $\mathbf{Q}'_{xu}$  in this formula is often zero, as can be seen by examining (13.36c), which shows that  $\mathbf{Q}'_{xu} \neq 0$  only if non-zero diagonal blocks have been specified in  $\tilde{\mathbf{Q}}$  or if  $\mathbf{C}^T \mathbf{Q}_{yy} \mathbf{D} \neq 0$ . Thus, in the basic case of optimizing the cost in (13.2) subject to constraint (13.1), the matrix  $\mathbf{Q}'_{xu}$  is zero.

**Summary** To optimize (13.24) over  $[t_0, t_f]$  using state feedback, specify  $\tilde{\mathbf{Q}}$  and  $\tilde{\mathbf{S}}$  as in (13.32) with matrix  $\mathbf{S} = \mathbf{S}_{xx}$  to weight  $\mathbf{X}(t_f)$  or  $\mathbf{S} = \mathbf{C}^T \mathbf{S}_{yy} \mathbf{C}$  to weight  $\mathbf{Y}(t_f)$  (assuming  $\mathbf{D} = 0$ ). Calculate the substitutions (13.36) and (13.40) to obtain a system of the form of (13.20). Solve (13.49) backwards in time over  $[t_0, t_f]$  with starting condition (13.50). Then the optimal state feedback matrix is given by (13.52).

For the basic case (13.2), simply specify  $\mathbf{S}(t_f)$ ,  $\mathbf{Q}$  and  $\mathbf{R}$ , integrate (13.20) backwards in time subject to (13.50) and then, from (13.52), the optimal state feedback matrix is given by

$$(13.53) \quad \mathbf{K}(t) = \mathbf{R}^{-1} \mathbf{B}^T \mathbf{P}(t).$$

### Example 7

Backward solutions

$$(13.54) \quad \frac{d}{dt} \mathbf{X}(t) = \frac{d}{d(t_f - s)} \mathbf{X}(t_f - s) = -\frac{d}{ds} \tilde{\mathbf{X}}(s)$$

$$= \mathbf{F}(\mathbf{X}(t), t) = \mathbf{F}(\mathbf{X}(t_f - s), t_f - s) = \mathbf{F}(\tilde{\mathbf{X}}(s), t_f - s).$$

Negating both sides of this equation gives

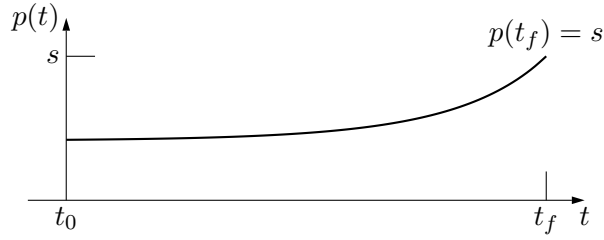
$$(13.55) \quad \frac{d}{ds} \tilde{\mathbf{X}}(s) = -\mathbf{F}(\tilde{\mathbf{X}}(s), t_f - s),$$

which is a differential equation that can be solved forward in the usual way from  $s = 0$  to  $s = t_f - t_0$  as an initial value problem, with initial condition  $\tilde{\mathbf{X}}(0) = \boldsymbol{\xi}$ . Given  $\tilde{\mathbf{X}}(s)$  over  $[0, t_f - t_0]$ , substituting  $\mathbf{X}(t) = \tilde{\mathbf{X}}(t_f - t)$  gives  $\mathbf{X}(t)$  over  $[t_0, t_f]$ .

### Example 8

Optimal state feedback for Example 6

The matrix  $\mathbf{P}(t)$  (written  $p(t)$  since it is a scalar) for the optimization defined in Example 6 is graphed in Figure 13.2 for  $t_0 = 0$ ,  $t_f = 3$ ,  $q = r = s$ . The solution exhibits an initial transient starting at  $t = t_f$  and approaches a steady-state value from right to left. From (13.52), the optimal state-feedback matrix is  $\mathbf{K}(t) = (1 + p(t))/2$ .



**Fig. 13.2** The solution of the Riccati differential equation calculated numerically from right to left for the system defined in Example 6.

## 2.4 Properties of the optimal feedback solution

The basic problem of minimizing (13.2) subject to (13.1) with  $\mathbf{S}(0) = 0$  and  $\mathbf{S}(t_f)$  written as  $\mathbf{S}$  for notational simplicity will be considered here. The consequences of using the Riccati differential equation (13.49) to obtain a feedback solution will be investigated. We shall show that  $\mathbf{P}(t_0)$  defines the optimal cost as a function of the initial state (see also Example 8 of Chapter 8), and that as  $t_f \rightarrow \infty$ ,  $\mathbf{P}(0)$  approaches a constant as observed in Example 8.

**Optimal cost** To calculate the optimal cost, let the system input be given by

$$(13.56) \quad \mathbf{U}(t) = -\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}(t)\mathbf{X} + \mathbf{V}(t)$$

with no loss of generality, where  $\mathbf{P}(t)$  satisfies Equation (13.49) and  $\mathbf{V}(t)$  is to be determined. Then the closed-loop equation for the state vector is

$$(13.57) \quad \dot{\mathbf{X}} = (\mathbf{A} - \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P})\mathbf{X} + \mathbf{B}\mathbf{V}.$$

Writing  $\mathbf{U}(t)$  in the form of (13.56) allows the integrand in the cost function (13.2) to be written as

$$(13.58) \quad \mathbf{X}^T\mathbf{Q}\mathbf{X} + \mathbf{U}^T\mathbf{R}\mathbf{U} = -\frac{d}{dt}(\mathbf{X}^T\mathbf{P}\mathbf{X}) + \mathbf{V}^T\mathbf{R}\mathbf{V},$$

which will be shown as follows. Consider the quadratic  $\mathbf{X}^T\mathbf{P}\mathbf{X}$  and its derivative:

$$(13.59) \quad \begin{aligned} \frac{d}{dt}(\mathbf{X}^T\mathbf{P}\mathbf{X}) &= \dot{\mathbf{X}}^T\mathbf{P}\mathbf{X} + \mathbf{X}^T\dot{\mathbf{P}}\mathbf{X} + \mathbf{X}^T\mathbf{P}\dot{\mathbf{X}} \\ &= \mathbf{X}^T(\mathbf{A} - \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P})^T\mathbf{P}\mathbf{X} + \mathbf{X}^T\dot{\mathbf{P}}\mathbf{X} + \mathbf{X}^T\mathbf{P}(\mathbf{A} - \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P})\mathbf{X} \\ &\quad + \mathbf{V}^T\mathbf{B}^T\mathbf{P}\mathbf{X} + \mathbf{X}^T\mathbf{P}\mathbf{B}\mathbf{V} \\ &= \mathbf{X}^T(\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} - 2\mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P} + \dot{\mathbf{P}})\mathbf{X} + 2\mathbf{V}^T\mathbf{B}^T\mathbf{P}\mathbf{X} \\ &= \mathbf{X}^T(-\mathbf{Q} - \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P})\mathbf{X} + 2\mathbf{V}^T\mathbf{B}^T\mathbf{P}\mathbf{X} \end{aligned}$$

where the last equality is obtained using (13.49). Solving for the quantity  $\mathbf{X}^T\mathbf{Q}\mathbf{X}$  from this expression gives

$$(13.60) \quad \mathbf{X}^T\mathbf{Q}\mathbf{X} = -\frac{d}{dt}(\mathbf{X}^T\mathbf{P}\mathbf{X}) - \mathbf{X}^T\mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}\mathbf{X} + 2\mathbf{V}^T\mathbf{B}^T\mathbf{P}\mathbf{X}.$$

Now apply the formula (13.56) for  $\mathbf{U}$  to the term  $\mathbf{U}^T\mathbf{R}\mathbf{U}$  in (13.58) to obtain

$$(13.61) \quad \begin{aligned} \mathbf{U}^T\mathbf{R}\mathbf{U} &= (\mathbf{V}^T - \mathbf{X}^T\mathbf{P}\mathbf{B}\mathbf{R}^{-1})\mathbf{R}(-\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}\mathbf{X} + \mathbf{V}) \\ &= \mathbf{X}^T\mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}\mathbf{X} - 2\mathbf{V}^T\mathbf{B}^T\mathbf{P}\mathbf{X} + \mathbf{V}^T\mathbf{R}\mathbf{V}. \end{aligned}$$

Then adding (13.60) and (13.61) produces (13.58) as required.

The consequence of (13.58) is that the integral in the formula (13.2) can be calculated, giving an expression for the closed-loop cost  $\phi_{\text{CL}}$  as shown:

$$(13.62) \quad \phi_{\text{CL}} = \frac{1}{2} \mathbf{X}(t_f) \mathbf{S} \mathbf{X}(t_f) + \frac{1}{2} \int_{t_0}^{t_f} \left( -\frac{d}{dt} (\mathbf{X}^T \mathbf{P} \mathbf{X}) + \mathbf{V}^T \mathbf{R} \mathbf{V} \right) dt$$

$$(13.63) \quad = \frac{1}{2} \left( \mathbf{X}(t_f) \mathbf{S} \mathbf{X}(t_f) - \mathbf{X}(t_f) \mathbf{P}(t_f) \mathbf{X}(t_f) + \mathbf{X}(t_0) \mathbf{P}(t_0) \mathbf{X}(t_0) + \int_{t_0}^{t_f} \mathbf{V}^T \mathbf{R} \mathbf{V} dt \right).$$

However,  $\mathbf{P}(t_f) = \mathbf{S}$ , so this reduces to

$$(13.64) \quad \phi_{\text{CL}} = \frac{1}{2} \left( \mathbf{X}(t_0) \mathbf{P}(t_0) \mathbf{X}(t_0) + \int_{t_0}^{t_f} \mathbf{V}^T \mathbf{R} \mathbf{V} dt \right),$$

where  $\mathbf{V}(t)$  is still to be determined. Since  $\mathbf{R}$  is a positive-definite matrix, the integral is non-negative, and  $\phi_{\text{CL}}$  will be minimized by setting  $\mathbf{V}(t)$  to zero.

The optimal cost  $\phi_{\text{min}}$  is therefore given by

$$(13.65) \quad \phi_{\text{min}} = \frac{1}{2} \mathbf{X}(t_0) \mathbf{P}(t_0) \mathbf{X}(t_0).$$

### Increasing $t_f$

The important case of infinite  $t_f$  will be investigated. Most of the development to this point applies to time-varying systems as well as to LTI systems. In the following, assume that  $\mathbf{A}$ ,  $\mathbf{B}$ ,  $\mathbf{Q}$ ,  $\mathbf{R}$ ,  $\mathbf{S}$ , and  $t_0$  are constant. Then the Riccati differential equation (13.49) is time-invariant (see Section 3 of Chapter 1) and its solution  $\mathbf{P}(t)$  depends on  $t_f - t$  but not directly on the absolute values of  $t_f$  or  $t$ .

We shall show that the optimal cost  $\phi$  in (13.2) is bounded and non-decreasing with increasing  $t_f$ , and must therefore have a limit as  $t_f \rightarrow \infty$ . First, (13.49) is time-invariant and the integrand in (13.2) is nonnegative; therefore  $\phi$  is non-decreasing with increasing  $t_f$ . Now since the plant has been assumed to be stabilizable, a constant state feedback  $\mathbf{K}$  matrix can be found such that the closed-loop system eigenvalues  $\lambda_i$  are in the left half-plane, in which case the entries of  $\mathbf{X}$  and  $\mathbf{U}$  are linear combinations of terms of the form  $t^\alpha e^{\lambda_i t}$  (see Example 17 of Chapter 7), where  $\alpha$  is a nonnegative integer. The integral of such terms over  $[t_0, \infty)$  is finite. Therefore, the optimal cost is bounded.

The most important engineering case, for which  $t_f$  is infinity, will be investigated next.

## 2.5

### Infinite time interval and the Hamiltonian matrix

#### Algebraic Riccati equation (ARE)

Since the minimum cost given by (13.65) has a limit as  $t_f \rightarrow \infty$ , solutions  $\mathbf{P}(t)$  of (13.49) become asymptotically constant for  $t$  far to the left of  $t_f$ . Hence for  $t_f$  arbitrarily large,  $\mathbf{P}$  can be assumed a constant matrix that satisfies the *algebraic Riccati equation* (abbreviated ARE) obtained from (13.49) by setting  $\dot{\mathbf{P}} = 0$ :

$$(13.66) \quad 0 = \mathbf{P} \mathbf{A} + \mathbf{A}^T \mathbf{P} - \mathbf{P} \mathbf{B} \mathbf{R}^{-1} \mathbf{B}^T \mathbf{P} + \mathbf{Q}.$$

Solution of the optimal feedback problem for constant matrices and  $t_f = \infty$  reduces to finding a positive-definite symmetric solution of this nonlinear algebraic equation rather than the solution of the Riccati differential equation (13.49).

Since  $\mathbf{P}$  is constant, Equation (13.53) gives a state feedback minimizing (13.2) subject to (13.1) as

$$(13.67) \quad \mathbf{U}(t) = -\mathbf{K}\mathbf{X}(t) = -\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}\mathbf{X}(t)$$

with constant state feedback matrix  $\mathbf{K}$ . Furthermore, the resulting closed-loop system

$$(13.68) \quad \dot{\mathbf{X}} = (\mathbf{A} - \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P})\mathbf{X}$$

is asymptotically stable for constant  $\mathbf{P}$ , for if it were not, a stabilizing state feedback  $\mathbf{U} = \mathbf{K}\mathbf{X}$ , with  $\mathbf{K}$  constant, could be found to give a finite, hence lower, cost  $\phi$ .

A constant  $\mathbf{P}$  allows some properties of the optimal feedback to be easily found. In particular, equation (13.20) becomes

$$(13.69) \quad \left( \begin{bmatrix} \mathbf{A} & -\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T \\ -\mathbf{Q} & -\mathbf{A}^T \end{bmatrix} - \mathbf{I}\frac{d}{dt} \right) \begin{bmatrix} \mathbf{I} \\ \mathbf{P} \end{bmatrix} \mathbf{X} = 0$$

where the matrix

$$(13.70) \quad \mathbf{H} = \begin{bmatrix} \mathbf{A} & -\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T \\ -\mathbf{Q} & -\mathbf{A}^T \end{bmatrix}$$

is called the *Hamiltonian matrix*. Two simple but very important properties of  $\mathbf{H}$  will be demonstrated. First, apply the nonsingular matrix

$$(13.71) \quad \mathbf{Z} = \begin{bmatrix} 0 & \mathbf{I} \\ -\mathbf{I} & 0 \end{bmatrix}$$

as a similarity transformation to  $\mathbf{H}^T$  to obtain

$$(13.72) \quad \mathbf{Z}^{-1}\mathbf{H}^T\mathbf{Z} = -\mathbf{H}.$$

The eigenvalues of a matrix are invariant under transposition and similarity transformations, so taking the eigenvalues of the matrices on both sides of this equation leads to the conclusion that the eigenvalues of  $\mathbf{H}$  are eigenvalues of  $-\mathbf{H}$ . However, negating a matrix negates its eigenvalues; that is,  $\lambda_i(-\mathbf{H}) = -\lambda_i(\mathbf{H})$ , so one must conclude that the eigenvalues of  $\mathbf{H}$  are distributed symmetrically across the imaginary axis of the complex plane.

To show the second property of  $\mathbf{H}$ , apply

$$(13.73) \quad \hat{\mathbf{Z}} = \begin{bmatrix} \mathbf{I} & 0 \\ -\mathbf{P} & \mathbf{I} \end{bmatrix}$$

as a similarity transformation to obtain

$$(13.74) \quad \hat{\mathbf{Z}}\mathbf{H}\hat{\mathbf{Z}}^{-1} = \begin{bmatrix} \mathbf{A} - \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P} & -\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T \\ -\mathbf{Q} - \mathbf{A}^T\mathbf{P} - \mathbf{P}\mathbf{A} + \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P} & -\mathbf{A}^T + \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T \end{bmatrix}.$$

Provided  $\mathbf{P}$  in  $\hat{\mathbf{Z}}$  satisfies (13.66), the lower-left block of this matrix is zero. Therefore, the eigenvalues of this matrix, and hence of  $\mathbf{H}$ , are those of the upper-left block together with those of the lower-right block. The upper-left block is the closed-loop



matrix  $\mathbf{A}_{CL} = \mathbf{A} - \mathbf{BK}$  defined previously. The lower-right block is the negative transpose of  $\mathbf{A}_{CL}$  and hence its eigenvalues are the negative of the eigenvalues of  $\mathbf{A}_{CL}$ . Therefore, the eigenvalues of  $\mathbf{H}$  are the stable closed-loop optimal system eigenvalues and their right half-plane mirror images.

### Closed-loop eigenvalues

An important design simplification is obtained from the above properties of  $\mathbf{H}$ . The closed-loop system poles are readily obtained by constructing  $\mathbf{H}$  and finding its left half-plane eigenvalues. Solving for  $\mathbf{P}$  or  $\mathbf{K}$  is not required.

### Summary

To optimize (13.2) over  $[t_0, \infty)$ , specify the matrices  $\mathbf{R}$ , and  $\mathbf{Q}$ . Then find for the positive-semidefinite, symmetric matrix  $\mathbf{P}$  that solves (13.66). Then the optimal constant-gain feedback control vector is given by  $\mathbf{U}(t) = -\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}\mathbf{X}(t) = -\mathbf{K}\mathbf{X}(t)$ .

To optimize (13.33) over  $[t_0, \infty)$ , compute  $\hat{\mathbf{R}}$  from (13.36a),  $\hat{\mathbf{A}}$  from (13.40a),  $\hat{\mathbf{Q}}$  from (13.40b), and substitute these values for  $\mathbf{R}$ ,  $\mathbf{A}$ , and  $\mathbf{Q}$  in (13.66). Then  $\mathbf{U}(t)$  is given by (13.38).

### Example 9

Hamiltonian matrix for Example 1

For the system described in Example 1, the Hamiltonian matrix is given by

$$\mathbf{H} = \begin{bmatrix} 1 & -1 \\ -q & -1 \end{bmatrix},$$

where  $\mathbf{Q} = q$  is the weight assigned to the state in the cost function. The eigenvalues of this matrix are given by  $\lambda = \pm\sqrt{1+q}$  and are placed symmetrically with respect to the imaginary axis. Setting  $q = 0$ , for example, stabilizes the original system using a minimal control signal and gives a closed-loop system with pole at  $-1$ , as in Example 1.

### Example 10

Stabilizing control

Sometimes the performance of a closed-loop system is not particularly important but the system is required to be asymptotically stable. Then, provided the system is observable, setting  $\mathbf{Q} = 0$  in the basic control configuration results in a Hamiltonian matrix (13.70) that has the eigenvalues of the open-loop system and their imaginary-axis mirror images. The mirror images of the unstable open-loop eigenvalues are moved to the left-half plane and the others are unchanged in the closed-loop result. Thus, for example, for the matrices

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 0 & -2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{R} = 1, \quad \mathbf{Q} = 0,$$

the Hamiltonian matrix is

$$\mathbf{H} = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & -2 & 0 & -1 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & -1 & 2 \end{bmatrix},$$

from which, by inspection, the closed-loop system will have two eigenvalues at  $-2$ .

**Example 11**  
Inverse  
pendulum

For a linearization of the system described in Example 19 of Chapter 1, a constant state-feedback matrix was obtained in Example 4 of Chapter 12 to assign the closed-loop eigenvalues to specified locations. Since optimal control with infinite time interval results in a constant state-feedback matrix, the two methods can produce identical results (see Problem 16). However, optimal control is most often applied numerically and iteratively to obtain the desired responses. Suppose that the example parameters are  $M = 1$  kg,  $m = 0.2$  kg,  $\ell = 1$  m. Then the linearized model is given below, written using the physical variables in the state vector:

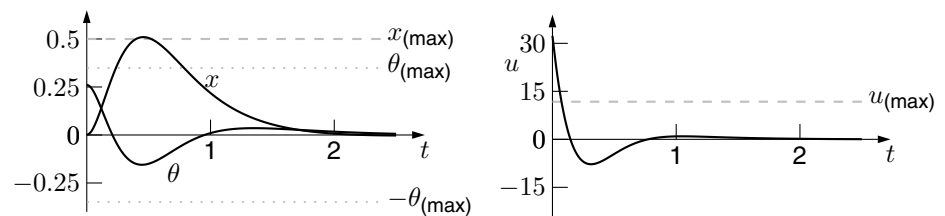
$$(13.75) \quad \frac{d}{dt} \begin{bmatrix} x \\ v \\ \theta \\ \omega \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1.96 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 11.76 & 0 \end{bmatrix} \begin{bmatrix} x \\ v \\ \theta \\ \omega \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} u.$$

We wish to obtain a state feedback such that, starting at  $x = 0$ ,  $v = dx/dt = 0$ ,  $\theta = 15^\circ = 0.26$  rad,  $\omega = d\theta/dt = 0$ , the cart displacement should be limited to  $|x(t)| < 0.5$  m and the angle to  $|\theta(t)| < 20^\circ = 0.35$  rad. The maximum force is not specified but an initial estimate might be the weight of the cart and pendulum; that is,  $g(m + M) = 11.76$  N. Then, by application of the rule of thumb given by Equation (13.7), the nonzero entries in the cost matrices are obtained as  $q_{11} = 4$  m<sup>-2</sup>,  $q_{33} = 8.2$  rad<sup>-2</sup>, and  $\mathbf{R} = 7.2 \times 10^{-3}$  N<sup>-2</sup>. The state-feedback matrix that results from these values is

$$(13.76) \quad \mathbf{K} = [-13.52, -23.16, -123.5, -37.30]$$

and the eigenvalues of the closed-loop matrix  $\mathbf{A} - \mathbf{BK}$  are at  $-4.99 \pm j4.02$  and  $-2.08 \pm j1.13$ .

The responses are shown in Figure 13.3; the figure shows that the bounds on displacement and angle are met but that the initial force is considerably higher than estimated. If such a force is not obtainable, then the value of  $\mathbf{R}$  must be increased, reducing the force but resulting in larger transients. In fact, there is no guarantee that a set of desired bounds can be met without changing the masses  $m$  and  $M$  or pendulum length  $\ell$ .



**Fig. 13.3** Closed-loop responses of cart position  $x$ , angle  $\theta$  of the inverse pendulum, and input force  $u$ . The desired bounds  $x_{\max}$ ,  $\theta_{\max}$ , and  $u_{\max}$  are shown.

**2.6****Solving for constant P: the continuous-time case**

Equation (13.66) is nonlinear in the unknown matrix  $\mathbf{P}$ . From the form of the equation,  $\mathbf{P}$  can be taken to be symmetric and must be positive-definite or, for the relaxed conditions discussed in Section 2.3, positive-semidefinite. There is more than one solution

of (13.66) in general but only one will satisfy this condition. The required solution of (13.66) for  $\mathbf{P}$  can be obtained in several ways.

First, the solution of nonlinear algebraic equations requires, in general, the use of potentially infinite processes that converge to an exact solution. It will be shown that solving for  $\mathbf{P}$  is closely connected with the problem of finding eigenvectors, and hence eigenvalues, of the Hamiltonian matrix. Finite methods do not exist for finding the roots of algebraic equations of arbitrary order, and such roots must be found by successive approximation, a process requiring convergence of the solution estimate.

Second and consequential to the above, no single method of solution can be expected to be best in all circumstances. However, reliable methods are available for situations in which the required matrices can be assumed to be exactly known and no larger than several tens of rows or columns.

The following examples are either of expository interest or illustrate the principles used by reliable library algorithms. Methods based on direct solution of the Riccati differential equation (13.49), Newton methods, and methods based on eigenspaces of the Hamiltonian matrix are mentioned.

---

**Example 12**  
The ARE for  
Example 1

The ARE can be solved directly for very small problems, since this equation is quadratic in the unknown matrix  $\mathbf{P}$ . The system of Example 1 with weight  $\mathbf{Q} = q$  and unknown matrix  $\mathbf{P} = p$  has algebraic Riccati equation

$$\mathbf{P}\mathbf{A} + \mathbf{A}^T\mathbf{P} - \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P} + \mathbf{Q} = p \cdot 1 + 1 \cdot p - p \cdot 1 \cdot p + q = 0,$$

the solutions of which are  $p = 1 \pm \sqrt{1+q}$ . The required positive semidefinite solution is  $p = 1 + \sqrt{1+q}$ , and results in the constant state feedback matrix  $\mathbf{K} = \mathbf{R}^{-1}\mathbf{B}^T\mathbf{P} = 1 + \sqrt{1+q}$  containing parameter  $q$ . This result shows that increasing the weight  $\mathbf{Q}$  on the state vector in the cost increases the state feedback gain given by  $\mathbf{K}$ .

---

**Example 13**  
Solving (13.49)

Library integration routines may be employed to solve (13.49) backward in time (see Example 7) from a final condition  $\mathbf{P}(t_f) = \mathbf{S}$  obtained from (13.50). Because  $\mathbf{P}$  is symmetric, only  $n(n+1)/2$  simultaneous differential equations need be solved, but difficulties of choosing  $\mathbf{S}$  and the extreme length of integration interval sometimes required for adequate convergence of  $\mathbf{P}$  make this method generally unsuitable.

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**Example 14**  
Solving (13.69)

Equation (13.69) may be solved backward in time. Thus for

$$(13.77) \quad \frac{d}{dt} \begin{bmatrix} \mathbf{X} \\ \mathbf{\Gamma} \end{bmatrix} = \mathbf{H} \begin{bmatrix} \mathbf{X} \\ \mathbf{\Gamma} \end{bmatrix}; \quad \mathbf{\Gamma}(t_f) = \mathbf{S}\mathbf{X}(t_f)$$

where  $\mathbf{H}$  is the Hamiltonian matrix defined in (13.70), the solution is

$$(13.78) \quad \begin{bmatrix} \mathbf{X}(t) \\ \mathbf{\Gamma}(t) \end{bmatrix} = \Phi(t-t_f) \begin{bmatrix} \mathbf{X}(t_f) \\ \mathbf{\Gamma}(t_f) \end{bmatrix}$$

where

$$(13.79) \quad \Phi(t) = e^{t\mathbf{H}}$$

for which the inverse  $\Phi(t)^{-1} = \Phi(-t)$  always exists (see Example 16 of Chapter 2). The solution at time  $t = t_f - \Delta$  is

$$(13.80) \quad \begin{bmatrix} \mathbf{X}(t_f - \Delta) \\ \mathbf{\Gamma}(t_f - \Delta) \end{bmatrix} = \Phi(-\Delta) \begin{bmatrix} \mathbf{X}(t_f) \\ \mathbf{SX}(t_f) \end{bmatrix} = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \begin{bmatrix} \mathbf{X}(t_f) \\ \mathbf{SX}(t_f) \end{bmatrix}$$

where  $\Phi(-\Delta)$  has been partitioned into  $n \times n$  blocks as shown, so that

$$(13.81) \quad \mathbf{\Gamma}(t_f - \Delta) = (\Phi_{21} + \Phi_{22}\mathbf{S})(\Phi_{11} + \Phi_{12}\mathbf{S})^{-1}\mathbf{X}(t_f - \Delta).$$

However, by definition,

$$(13.82) \quad \mathbf{\Gamma}(t_f - \Delta) = \mathbf{P}(t_f - \Delta)\mathbf{X}(t_f - \Delta),$$

so that

$$(13.83) \quad \mathbf{P}(t_f - \Delta) = (\Phi_{21} + \Phi_{22}\mathbf{S})(\Phi_{11} + \Phi_{12}\mathbf{S})^{-1}.$$

Repeated use of the above steps gives a sequence  $\mathbf{P}(t_f) = \mathbf{S}, \mathbf{P}(t_f - \Delta), \mathbf{P}(t_f - 2\Delta) \dots$ , starting at an assumed value  $\mathbf{S}$ . The essential requirement for this method is accurate computation of  $e^{(-\Delta\mathbf{H})}$ , but even with an exact result,  $\mathbf{P}$  may converge extremely slowly. Consequently, this method, although once widely applied, is normally rejected for general use. Its main value is as a simple algorithm for approximate step-wise solution of time-varying problems, for which  $\Phi(-\Delta)$  typically must be re-calculated at each step as the entries of  $\mathbf{H}$  vary over time.

---

**Example 15**  
Newton methods

Another method relies on a Newton-like sequence of approximations to solve (13.66) directly. The object is to minimize

$$(13.84) \quad q = \|\mathbf{PA} + \mathbf{A}^T\mathbf{P} - \mathbf{PLP} + \mathbf{Q}\|$$

in a suitable norm, where for brevity  $\mathbf{L}$  replaces  $\mathbf{BR}^{-1}\mathbf{B}^T$ . For a symmetric perturbation  $\Delta\mathbf{P}$  of  $\mathbf{P}$ ,

$$(13.85) \quad q + \Delta q = \|\mathbf{PA} + \mathbf{A}^T\mathbf{P} - \mathbf{PLP} + \mathbf{Q} + \Delta\mathbf{P}(\mathbf{A} + \mathbf{LP}) + (\mathbf{A} + \mathbf{LP})^T\Delta\mathbf{P} + \Delta\mathbf{P}\mathbf{L}\Delta\mathbf{P}\|,$$

from which,  $q$  may be reduced for any  $\mathbf{P}_i$  not satisfying (13.66) by solving

$$(13.86) \quad \mathbf{X}_i(\mathbf{A} + \mathbf{LP}_i) + (\mathbf{A} + \mathbf{LP}_i)^T\mathbf{X}_i + (\mathbf{P}_i\mathbf{A} + \mathbf{A}^T\mathbf{P}_i - \mathbf{P}_i\mathbf{LP}_i + \mathbf{Q}) = 0$$

for  $\mathbf{X}_i$ , and letting  $\mathbf{P}_{i+1} = \mathbf{P}_i + \mu\mathbf{X}_i$  for sufficiently small  $\mu$ . This method and its variants suffer from the general defects of Newton algorithms, namely relatively slow convergence and the difficulties of choosing the initial guess  $\mathbf{P}_0$  sufficiently close to the solution to guarantee convergence. One use of the method is for iterative refinement of approximate solutions obtained by other methods. Each iteration requires the solution of the above continuous-time Lyapunov equation discussed in Section 3.2 and Section 3.3 of Chapter 8.

**Example 16**  
Eigenvector  
methods

Useful algorithms can be based on computing a set of basis vectors for the stable eigenspace of (13.69) or, in general, of (13.29). All such algorithms contain a potentially infinite iterative process for isolating the stable eigenvalues and corresponding eigenvectors, which constitute a basis for the stable eigenspace.

Construct the Hamiltonian matrix  $\mathbf{H}$  discussed in Section 2.5 and the nonsingular matrix  $\mathbf{V}$  partitioned using the notation  $\mathbf{V} = [\mathbf{V}_-, \mathbf{V}_+]$  such that

$$(13.87) \quad \mathbf{V}^{-1}\mathbf{H}\mathbf{V} = \begin{bmatrix} \Lambda_- & 0 \\ 0 & \Lambda_+ \end{bmatrix},$$

where the eigenvalues of  $\Lambda_-$  are in the left half-plane and the eigenvalues of  $\Lambda_+$  are their unstable mirror images. The matrices  $\Lambda_-$  and  $\Lambda_+$  need not be diagonal. By defining the transformation of variables

$$(13.88) \quad \begin{bmatrix} \mathbf{X}(t) \\ \mathbf{\Gamma}(t) \end{bmatrix} = [\mathbf{V}_-, \mathbf{V}_+] \begin{bmatrix} \mathbf{W}_1(t) \\ \mathbf{W}_2(t) \end{bmatrix},$$

and by setting  $t_0 = 0$  for convenience, we obtain the solution of (13.20) as

$$(13.89) \quad \begin{bmatrix} \mathbf{X}(t) \\ \mathbf{\Gamma}(t) \end{bmatrix} = e^{t\mathbf{H}} \begin{bmatrix} \mathbf{X}(0) \\ \mathbf{\Gamma}(0) \end{bmatrix} = \mathbf{V} e^{t\mathbf{V}^{-1}\mathbf{H}\mathbf{V}} \mathbf{V}^{-1} \begin{bmatrix} \mathbf{W}_1(0) \\ \mathbf{W}_2(0) \end{bmatrix} \\ = [\mathbf{V}_-, \mathbf{V}_+] \begin{bmatrix} e^{t\Lambda_-} & 0 \\ 0 & e^{t\Lambda_+} \end{bmatrix} \begin{bmatrix} \mathbf{W}_1(0) \\ \mathbf{W}_2(0) \end{bmatrix} = \mathbf{V}_- e^{t\Lambda_-} \mathbf{W}_1(0) + \mathbf{V}_+ e^{t\Lambda_+} \mathbf{W}_2(0).$$

The solution  $\mathbf{X}(t)$  must be asymptotically stable, otherwise (13.2) is not a minimum. Therefore,  $\mathbf{\Gamma} = \mathbf{P}\mathbf{X}$  must also be asymptotically stable. Since  $e^{t\Lambda_+}$  is nonsingular and  $\mathbf{V}_-$  has full column rank,  $\mathbf{W}_2(0)$  must be 0 and the solution reduces to

$$(13.90) \quad \begin{bmatrix} \mathbf{X}(t) \\ \mathbf{\Gamma}(t) \end{bmatrix} = \mathbf{V}_- (e^{t\Lambda_-} \mathbf{W}_1(0)) = \begin{bmatrix} \mathbf{I} \\ \mathbf{P} \end{bmatrix} \mathbf{X}(t),$$

where the rightmost term uses the substitution of (13.69). This equation shows that the columns of  $[\mathbf{I}, \mathbf{P}]^T$  are linear combinations of the columns of  $\mathbf{V}_-$ , in which case there is a nonsingular matrix  $\hat{\mathbf{M}}$  that satisfies

$$(13.91) \quad \begin{bmatrix} \mathbf{I} \\ \mathbf{P} \end{bmatrix} = \mathbf{V}_- \hat{\mathbf{M}} = \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} \hat{\mathbf{M}},$$

where  $\mathbf{V}_-$  has been partitioned into  $\mathbf{V}_1$  and  $\mathbf{V}_2$  as shown. Solving for  $\hat{\mathbf{M}} = \mathbf{V}_1^{-1}$  in this equation leads to the equation

$$(13.92) \quad \mathbf{V}_2 = \mathbf{P}\mathbf{V}_1$$

to be solved for the required  $\mathbf{P}$ .

The above argument depends on classifying the eigenvalues of  $\mathbf{H}$  into sets of values in the negative and positive half-planes; if there are eigenvalues on the imaginary axis, the optimization problem is singular. In the simplest case,  $\mathbf{V}_-$  can contain the eigenvectors of  $\mathbf{H}$  corresponding to stable eigenvalues. More generally, the columns of  $\mathbf{V}_-$  need only span the range of these eigenvectors.

As a concrete example, consider the matrix  $\mathbf{H}$  containing parameter  $q$  given in Example 9. An eigenvector corresponding to stable eigenvalue  $\sqrt{1+q}$  is given by

$$\mathbf{V}_- = \begin{bmatrix} 1 \\ 1 + \sqrt{1+q} \end{bmatrix} = \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix},$$

to which (13.92) can be applied to get  $p = 1 + \sqrt{1+q}$  as in Example 12.

---

**Example 17**  
The Schur matrix

The eigenvector method described in Example 16, although conceptually simple, can be improved at very little cost to avoid the computation of actual eigenvectors, resulting in one of the premier methods available for solving the matrix Riccati equation for systems up to medium size with no special structure. The essential requirement is the computation of  $\mathbf{V}_-$ , a basis for the stable eigen-space of  $\mathbf{H}$ . By a minor modification of the almost universally used QR-algorithm for computing matrix eigenvalues, an orthogonal matrix  $\mathbf{V}$  can be constructed such that

$$(13.93) \quad \mathbf{V}^T \mathbf{H} \mathbf{V} = \begin{bmatrix} \mathbf{L}_1 & \cdots & \\ & \ddots & \vdots \\ & & \mathbf{L}_t \end{bmatrix},$$

is in *real Schur form*, which is upper block-triangular, with diagonal blocks  $\mathbf{L}_i$  that are either scalar, equal to real eigenvalues of  $\mathbf{H}$ , or of size  $2 \times 2$ , with eigenvalues equal to complex-conjugate eigenvalue pairs of  $\mathbf{H}$  (see Problem 7 of Chapter 7). Furthermore, the diagonal blocks of the resulting  $\mathbf{V}^T \mathbf{H} \mathbf{V}$  can be ordered such that the eigenvalues of the upper-left  $n \times n$  submatrix are the left half-plane eigenvalues of  $\mathbf{H}$ . Then a suitable matrix  $\mathbf{V}_-$  is given by the leftmost  $n$  columns of  $\mathbf{V}$  as in Example 16.

The matrix  $\mathbf{H}$  of Example 9 is put into real Schur form by the orthogonal matrix

$$\mathbf{V} = \frac{1}{(1 + (1 + \sqrt{1+q})^2)^{1/2}} \begin{bmatrix} 1 & -(1 + \sqrt{1+q}) \\ 1 + \sqrt{1+q} & 1 \end{bmatrix},$$

for example, to get

$$\mathbf{V}^T \mathbf{H} \mathbf{V} = \begin{bmatrix} -\sqrt{1+q} & q-1 \\ 0 & \sqrt{1+q} \end{bmatrix}.$$

Since  $\mathbf{V}_-$  is the first column of  $\mathbf{V}$  in this example, (13.92) can be applied to obtain  $\mathbf{P}$  as previously.

---

**Example 18**  
The matrix sign function

Computing the sign function of the Hamiltonian matrix is the core of a method of considerable simplicity and speed for solving the ARE. This function is defined consistently with the sign function of complex scalars  $z$ , which is

$$(13.94) \quad \text{sign}(z) = \begin{cases} 1, & \text{Re}(z) > 0 \\ \text{undefined}, & \text{Re}(z) = 0 \\ -1, & \text{Re}(z) < 0 \end{cases}.$$

For a Jordan block  $\mathbf{J}_{\delta_i}$  of size  $\delta_i$  and eigenvalue  $\lambda_i$  as described in Section 3 of Chapter 7, define the sign function to be the diagonal matrix

$$(13.95) \quad \text{sign}(\mathbf{J}_{\delta_i}) = \text{diag}[\text{sign}(\lambda_i)]$$

of size  $\delta_i$ . For any real matrix  $\mathbf{A}$ , let  $\mathbf{W}^{-1}\mathbf{A}\mathbf{W} = \mathbf{J} = \text{diag}[\mathbf{J}_{\delta_i}]$  be its Jordan form and, as in Equation (7.33), define

$$(13.96) \quad \text{sign}(\mathbf{A}) = \mathbf{W} \text{sign}(\mathbf{J}) \mathbf{W}^{-1}.$$

The solution of the ARE can be found by calculating  $\text{sign}(\mathbf{H})$ , as follows. Let  $\mathbf{W}$  be the identically non-singular matrix

$$(13.97) \quad \mathbf{W} = \begin{bmatrix} 0 & \mathbf{I} \\ -\mathbf{I} & \mathbf{P} \end{bmatrix} \begin{bmatrix} \mathbf{I} & 0 \\ -\mathbf{V} & \mathbf{I} \end{bmatrix} = \begin{bmatrix} \mathbf{V} & \mathbf{I} \\ -\mathbf{I} + \mathbf{P}\mathbf{V} & \mathbf{P} \end{bmatrix}$$

where  $\mathbf{P}$  satisfies the algebraic Riccati equation (13.66) and where  $\mathbf{V}$  satisfies the Lyapunov equation

$$(13.98) \quad \mathbf{A}_{\text{CL}}\mathbf{V} + \mathbf{V}\mathbf{A}_{\text{CL}}^T + \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T = 0$$

with  $\mathbf{A}_{\text{CL}} = \mathbf{A} - \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}$ . Then matrix multiplication and simplification results in

$$(13.99) \quad \mathbf{W}^{-1}\mathbf{H}\mathbf{W} = \begin{bmatrix} -\mathbf{A}_{\text{CL}}^T & 0 \\ 0 & \mathbf{A}_{\text{CL}} \end{bmatrix}.$$

If all eigenvalues of  $\mathbf{A}_{\text{CL}}$  are in the open left half-plane then, from the form of the matrix above, the rightmost  $n$  vectors  $\begin{bmatrix} \mathbf{I} \\ \mathbf{P} \end{bmatrix}$  of  $\mathbf{W}$  span the image of the eigenvectors of  $\mathbf{H}$  corresponding to left half-plane eigenvalues. Furthermore,  $\text{sign}(\mathbf{A}_{\text{CL}}) = -\mathbf{I}$  and

$$(13.100) \quad \text{sign}(\mathbf{H}) = \mathbf{W} \text{sign}(\mathbf{W}^{-1}\mathbf{H}\mathbf{W}) \mathbf{W}^{-1} = \mathbf{W} \begin{bmatrix} \mathbf{I} & 0 \\ 0 & -\mathbf{I} \end{bmatrix} \mathbf{W}^{-1}.$$

Now define matrix  $\mathbf{F}$  to be

$$(13.101) \quad \begin{aligned} \mathbf{F} &= \mathbf{I} + \text{sign}(\mathbf{H}) = \mathbf{W} \left( \begin{bmatrix} \mathbf{I} & 0 \\ 0 & \mathbf{I} \end{bmatrix} + \begin{bmatrix} \mathbf{I} & 0 \\ 0 & -\mathbf{I} \end{bmatrix} \right) \mathbf{W}^{-1} \\ &= \begin{bmatrix} \mathbf{V} & \mathbf{I} \\ -\mathbf{I} + \mathbf{P}\mathbf{V} & \mathbf{P} \end{bmatrix} \begin{bmatrix} 2\mathbf{I} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{P} & -\mathbf{I} \\ \mathbf{I} - \mathbf{V}\mathbf{P} & \mathbf{V} \end{bmatrix} \\ &= 2 \begin{bmatrix} \mathbf{V} \\ -\mathbf{I} + \mathbf{P}\mathbf{V} \end{bmatrix} [\mathbf{P}, -\mathbf{I}] = [\mathbf{F}_1, \mathbf{F}_2] \end{aligned}$$

which has been partitioned as  $[\mathbf{F}_1, \mathbf{F}_2]$  as shown, where  $\mathbf{F}_1$  and  $\mathbf{F}_2$  are  $2n \times n$ . If  $\mathbf{F}$  is known, then  $\mathbf{P}$  can be found by solving

$$(13.102) \quad \mathbf{F}_2\mathbf{P} = -\mathbf{F}_1.$$

This equation is overdetermined but has a solution since  $\mathbf{F}_1$  is in the column range of  $\mathbf{F}_2$ . Furthermore,  $\mathbf{F}_2$  is the first block column of the identically nonsingular matrix  $\mathbf{W}$  and is therefore of full column rank, so the solution is unique. The computation  $\mathbf{P}$  has now been reduced to one of solving this set of linear equations, together with the computation of  $\text{sign}(\mathbf{H})$ , which will be investigated next.

Let  $f(z) = z^2 - 1$  so that, provided  $\operatorname{Re}(z_0) \neq 0$ , the Newton sequence

$$(13.103) \quad z_{k+1} = z_k - \frac{f(z_k)}{f'(z_k)} = \frac{1}{2} \left( z_k + \frac{1}{z_k} \right)$$

converges rapidly to  $\operatorname{sign}(z_0)$ . The matrix algorithm is directly analogous to the scalar algorithm:

$$(13.104) \quad \mathbf{H}_0 = \mathbf{H}, \quad \mathbf{H}_{k+1} = \frac{1}{2}(\mathbf{H}_k + \mathbf{H}_k^{-1}).$$

This sequence of matrices  $\mathbf{H}_k$  converges rapidly to  $\operatorname{sign}(\mathbf{H})$  and does not require the direct calculation of eigenvectors or their images. Library algorithms accelerate convergence by scaling the result of each step.

**Example 19**  
Sign function for  
Example 9

To apply the matrix-sign method to the parameterized  $\mathbf{H}$  given in Example 9, a matrix containing eigenvectors, such as

$$\mathbf{W} = \begin{bmatrix} 1 & 1 \\ 1 - \sqrt{1+q} & 1 + \sqrt{1+q} \end{bmatrix}$$

can be employed to transform  $\mathbf{H}$  to Jordan form  $\mathbf{J} = \mathbf{W}^{-1}\mathbf{H}\mathbf{W} = \begin{bmatrix} \sqrt{1+q} & 0 \\ 0 & -\sqrt{1+q} \end{bmatrix}$

from which, by inspection,  $\operatorname{sign}(\mathbf{J}) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$  and

$$\mathbf{F} = \mathbf{I} + \mathbf{W} \operatorname{sign}(\mathbf{J}) \mathbf{W}^{-1} = \frac{1}{\sqrt{1+q}} \begin{bmatrix} 1 + \sqrt{1+q} & -1 \\ -q & \sqrt{1+q} - 1 \end{bmatrix} = [\mathbf{F}_1, \mathbf{F}_2].$$

Solving (13.102) gives  $p = 1 + \sqrt{1+q}$  as before.

**3**

**Discrete-time optimization**

Much of the previous discussion of continuous-time optimization is applicable to the discrete-time optimization problem, but the form of the equations is different. The basic optimization problem is given by Equations (13.4) and (13.5). A general formulation will be considered and then simplified as necessary to handle the many variations of cost function, as in Section 2.1. Analogous to (13.22) and (13.23), the cost function

$$(13.105) \quad \phi = \frac{1}{2} \sum_{t=t_0}^{t_f} \tilde{\mathbf{X}}(t)^T \tilde{\mathbf{Q}} \tilde{\mathbf{X}}(t)$$

is to be minimized, subject to

$$(13.106) \quad \mathbf{N}\tilde{\mathbf{X}}(t+1) = \mathbf{M}\tilde{\mathbf{X}}(t), \quad t = t_0, \dots, t_f-1.$$

This constrained algebraic optimization problem can be solved by the method of Lagrange multipliers without invoking the calculus of variations. Time-varying matrices



can be allowed with minor limitations, but the development will assume the matrices to be constant.

Introducing a time-indexed sequence of Lagrange multiplier vectors  $\tilde{\Gamma}(t+1)$ ,  $t = t_0, \dots, t_f-1$  and adding the products of these multiplied by the constraint equations to (13.105) gives

$$(13.107) \quad \phi = \sum_{t=t_0}^{t_f-1} \frac{1}{2} \tilde{\mathbf{X}}(t)^T \tilde{\mathbf{Q}} \tilde{\mathbf{X}}(t) + \tilde{\Gamma}(t+1)^T (\mathbf{M} \tilde{\mathbf{X}}(t) - \mathbf{N} \tilde{\mathbf{X}}(t+1)) + \frac{1}{2} \tilde{\mathbf{X}}(t_f)^T \tilde{\mathbf{Q}} \tilde{\mathbf{X}}(t_f).$$

At a minimum, the derivative of this function with respect to all variables is zero. Differentiating gives the following:

$$(13.108a) \quad \frac{\partial \phi}{\partial \tilde{\mathbf{X}}(t)} = \tilde{\mathbf{Q}} \tilde{\mathbf{X}}(t) + \mathbf{M}^T \tilde{\Gamma}(t+1) - \mathbf{N}^T \tilde{\Gamma}(t), \quad t = t_0, \dots, t_f-1$$

$$(13.108b) \quad \frac{\partial \phi}{\partial \tilde{\mathbf{X}}(t_f)} = \tilde{\mathbf{Q}} \tilde{\mathbf{X}}(t_f) - \mathbf{N}^T \tilde{\Gamma}(t_f),$$

$$(13.108c) \quad \frac{\partial \phi}{\partial \tilde{\Gamma}(t+1)} = \mathbf{M} \tilde{\mathbf{X}}(t) - \mathbf{N} \tilde{\mathbf{X}}(t+1), \quad t = t_0, \dots, t_f-1.$$

Equating (13.108a) and (13.108c) to zero gives

$$(13.109) \quad \begin{bmatrix} \mathbf{N} & 0 \\ 0 & \mathbf{M}^T \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{X}}(t+1) \\ \tilde{\Gamma}(t+1) \end{bmatrix} = \begin{bmatrix} \mathbf{M} & 0 \\ -\tilde{\mathbf{Q}} & \mathbf{N}^T \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{X}}(t) \\ \tilde{\Gamma}(t) \end{bmatrix}, \quad t = t_0, \dots, t_f-1.$$

These equations are analogous to (13.29). The initial condition  $\tilde{\mathbf{X}}(t_0)$  is typically fixed but arbitrary, and the final condition is obtained by equating (13.108b) to zero.

The state-space substitutions given by (13.21) and (13.32) will be applied to Equation (13.109), producing

$$(13.110) \quad \begin{bmatrix} \mathbf{I} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{A}^T & \mathbf{C}^T \\ 0 & 0 & 0 & 0 & -\mathbf{I} \\ 0 & 0 & 0 & \mathbf{B}^T & \mathbf{D}^T \end{bmatrix} \begin{bmatrix} \mathbf{X}(t+1) \\ \mathbf{Y}(t+1) \\ \mathbf{U}(t+1) \\ \Gamma(t+1) \\ \mathbf{V}(t+1) \end{bmatrix} = \begin{bmatrix} \mathbf{A} & 0 & \mathbf{B} & 0 & 0 \\ \mathbf{C} & -\mathbf{I} & \mathbf{D} & 0 & 0 \\ -\mathbf{Q}_{xx} & -\mathbf{Q}_{xy} & -\mathbf{Q}_{xu} & \mathbf{I} & 0 \\ -\mathbf{Q}_{xy}^T & -\mathbf{Q}_{yy} & -\mathbf{Q}_{yu} & 0 & 0 \\ -\mathbf{Q}_{xy}^T & -\mathbf{Q}_{yu}^T & -\mathbf{Q}_{uu} & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{X}(t) \\ \mathbf{Y}(t) \\ \mathbf{U}(t) \\ \Gamma(t) \\ \mathbf{V}(t) \end{bmatrix}.$$

Eliminating  $\mathbf{Y}(t)$  and  $\mathbf{V}(t)$  leaves the equation

$$(13.111) \quad \begin{bmatrix} \mathbf{I} & 0 & 0 \\ 0 & 0 & \mathbf{A}^T \\ 0 & 0 & \mathbf{B}^T \end{bmatrix} \begin{bmatrix} \mathbf{X}(t+1) \\ \mathbf{U}(t+1) \\ \Gamma(t+1) \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} & 0 \\ -\mathbf{Q}'_{xx} & -\mathbf{Q}'_{xu} & \mathbf{I} \\ -\mathbf{Q}'_{xu} & -\hat{\mathbf{R}} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{X}(t) \\ \mathbf{U}(t) \\ \Gamma(t) \end{bmatrix},$$

where the formulas for  $\hat{\mathbf{R}}$ ,  $\mathbf{Q}'_{xx}$ , and  $\mathbf{Q}'_{xu}$  are identical to the continuous-time case in Equations (13.36). If the optimization problem is nonsingular then  $\hat{\mathbf{R}}$  is invertible and  $\mathbf{U}(t)$  is obtained from

$$(13.112) \quad \hat{\mathbf{R}} \mathbf{U}(t) = -\mathbf{Q}'_{xu} \mathbf{X}(t) - \mathbf{B}^T \Gamma(t+1).$$

Finally, eliminating  $\mathbf{U}(t)$  from (13.111) gives the following, analogous to time-continuous equations (13.39):

$$(13.113) \quad \begin{bmatrix} \mathbf{I} & \hat{\mathbf{S}} \\ 0 & \hat{\mathbf{A}}^T \end{bmatrix} \begin{bmatrix} \mathbf{X}(t+1) \\ \Gamma(t+1) \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{A}} & 0 \\ -\hat{\mathbf{Q}} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{X}(t) \\ \Gamma(t) \end{bmatrix},$$

where  $\hat{\mathbf{A}}$  and  $\hat{\mathbf{Q}}$  are given as for the time-continuous case in (13.40a) and (13.40b) respectively, and where, for future simplicity,  $\hat{\mathbf{S}}$  is defined as

$$(13.114) \quad \hat{\mathbf{S}} = \mathbf{B}\hat{\mathbf{R}}^{-1}\mathbf{B}^T.$$

Equations (13.113) constitute a two-point boundary-value problem with initial constraint given by fixed  $\mathbf{X}(t_0)$  and final constraint obtained by setting (13.108b) to zero for the cost function at hand.

The formulas corresponding to the general cost function are rather cumbersome, but simplify drastically in the most common cases, as follows.

**Basic case** Minimizing (13.4) subject to (13.5) implies that  $\mathbf{Q}_{xy}$ ,  $\mathbf{Q}_{xu}$ ,  $\mathbf{Q}_{yy}$ , and  $\mathbf{Q}_{yu}$  are zero,  $\mathbf{Q}_{xx}$  and  $\mathbf{Q}_{uu}$  can be written simply as  $\mathbf{Q}$  and  $\mathbf{R}$  respectively, and the submatrices in (13.113) are given by formulas (13.42), with  $\mathbf{U}(t)$  given by

$$(13.115) \quad \mathbf{U}(t) = -\mathbf{R}^{-1}\mathbf{B}^T\mathbf{\Gamma}(t+1).$$

**Y in the cost** Substituting  $\mathbf{Q}$  in the basic case with  $\mathbf{C}^T\mathbf{Q}\mathbf{C}$  accounts for the replacement of the term  $\mathbf{X}(t)^T\mathbf{Q}\mathbf{X}(t)$  in the cost function by  $\mathbf{Y}(t)^T\mathbf{Q}\mathbf{Y}(t)$ , provided that  $\mathbf{D}$  is zero.

**Nonzero D** If  $\mathbf{Y}$  is included in the cost as above and  $\mathbf{D} \neq 0$ , then

$$(13.116) \quad \mathbf{U}(t) = -\hat{\mathbf{R}}^{-1}(\mathbf{D}^T\mathbf{Q}\mathbf{C}\mathbf{X}(t) + \mathbf{B}^T\mathbf{\Gamma}(t+1))$$

and the submatrices in (13.113) are given by formulas (13.45).

**Example 20**  
Scalar  
discrete-time  
system

For the basic system described by matrices  $\mathbf{A} = e$ ,  $\mathbf{B} = e - 1$ ,  $\mathbf{C} = 1$ ,  $\mathbf{D} = 0$  with fixed  $x(0)$ , let the optimization interval be  $t \in \{0, 1\}$ , and let the cost be

$$\phi = \frac{1}{2}(x^2(0) + x^2(1) + u^2(0)).$$

The optimal input  $u(0)$  is to be found by the method of Lagrange multipliers.

There is one constraint equation:

$$x(1) = \mathbf{A}x(0) + \mathbf{B}u(0)$$

and therefore one Lagrange multiplier  $\mathbf{\Gamma}(1) = \gamma(1)$  is required. The modified cost becomes

$$\phi = \frac{1}{2}(x^2(0) + x^2(1) + u^2(0)) + \gamma(1)(ex(0) + (e-1)u(0) - x(1)),$$

so that the cost is optimized by solving the following equations.

$$\frac{\partial\phi}{\partial x(1)} = x(1) - \gamma(1) = 0$$

$$\frac{\partial\phi}{\partial u(0)} = u(0) + \gamma(1)(e-1) = 0$$

$$\frac{\partial\phi}{\partial\gamma(1)} = ex(0) + (e-1)u(0) - x(1) = 0.$$

Solving for  $u(0)$  gives

$$u(0) = -\frac{(e-1)e}{1+(e-1)^2}x(0).$$

### 3.1 Feedback solutions

To obtain a feedback solution for  $\mathbf{U}(t)$ , let  $\mathbf{\Gamma}(t) = \mathbf{P}(t)\mathbf{X}(t)$  in (13.112), so that

$$(13.117) \quad \hat{\mathbf{R}}\mathbf{U}(t) = -\mathbf{B}^T\mathbf{P}(t+1)(\hat{\mathbf{A}}\mathbf{X}(t) + \mathbf{B}\mathbf{U}(t)) - \mathbf{Q}'_{xu}{}^T\mathbf{X}(t)$$

and solve for  $\mathbf{U}(t)$  to obtain

$$(13.118) \quad \mathbf{U}(t) = -(\hat{\mathbf{R}} + \mathbf{B}^T\mathbf{P}(t+1)\mathbf{B})^{-1}(\mathbf{B}^T\mathbf{P}(t+1)\hat{\mathbf{A}} + \mathbf{Q}'_{xu}{}^T)\mathbf{X}(t),$$

which has the form of a time-varying state-feedback matrix multiplying  $\mathbf{X}(t)$ .

To obtain the sequence of values of  $\mathbf{P}(t)$  required in (13.118), substitute  $\mathbf{\Gamma}(t) = \mathbf{P}(t)\mathbf{X}(t)$  into the top equation of (13.113) to get

$$(13.119) \quad (\mathbf{I} + \hat{\mathbf{S}}\mathbf{P}(t+1))\mathbf{X}(t+1) = \hat{\mathbf{A}}\mathbf{X}(t),$$

multiply by  $(\mathbf{I} + \hat{\mathbf{S}}\mathbf{P}(t+1))^{-1}$ , and substitute the resulting formula for  $\mathbf{X}(t+1)$  into the bottom equation of (13.113) to obtain

$$(13.120) \quad \hat{\mathbf{A}}^T\mathbf{P}(t+1)(\mathbf{I} + \hat{\mathbf{S}}\mathbf{P}(t+1))^{-1}\hat{\mathbf{A}}\mathbf{X}(t) = -\hat{\mathbf{Q}}\mathbf{X}(t) + \mathbf{P}(t)\mathbf{X}(t).$$

This equation is true by definition for all  $\mathbf{X}(t)$ ; therefore the following discrete-time equivalent of the Riccati differential equation (13.49) can be solved backward in time to obtain the required sequence of matrices  $\mathbf{P}(t)$ :

$$(13.121) \quad \mathbf{P}(t) = \hat{\mathbf{Q}} + \hat{\mathbf{A}}^T\mathbf{P}(t+1)(\mathbf{I} + \hat{\mathbf{S}}\mathbf{P}(t+1))^{-1}\hat{\mathbf{A}}.$$

Typically, the term  $\tilde{\mathbf{X}}(t_f)^T\tilde{\mathbf{Q}}\tilde{\mathbf{X}}(t_f)$  in (13.107) is simply  $\mathbf{X}(t_f)^T\mathbf{S}\mathbf{X}(t_f)$ , and the starting condition of this equation then is

$$(13.122) \quad \mathbf{P}(t_f) = \mathbf{S}.$$

#### Example 21

Feedback  
solution for  
Example 20

For the system of Example 20, the cost-function matrices are

$$\hat{\mathbf{R}} = 1, \quad \hat{\mathbf{Q}} = 1, \quad \hat{\mathbf{A}} = e, \quad \mathbf{S} = 1, \quad \hat{\mathbf{S}} = (e-1)^2.$$

The solution of (13.121) is required only at its final point; that is,  $\mathbf{P}(1) = \mathbf{S} = 1$ , so that the optimal input  $u(0)$  is given by (13.118) as

$$u(0) = -(1 + (e-1)^2)^{-1}(e-1)ex(0),$$

which is the solution previously found for this example.

#### Constant P

A reasoning similar to that for the time-continuous case can be given to show that  $\mathbf{P}$  can be assumed constant if the optimization interval is infinite. The required matrix  $\mathbf{P}$  is then the positive-definite, symmetric solution of the discrete algebraic Riccati equation (DARE) obtained by setting  $\mathbf{P}(t)$  constant in (13.121):

$$(13.123) \quad \hat{\mathbf{A}}^T\mathbf{P}(\mathbf{I} + \hat{\mathbf{S}}\mathbf{P})^{-1}\hat{\mathbf{A}} - \mathbf{P} + \hat{\mathbf{Q}} = 0.$$

Once  $\mathbf{P}$  is known, from (13.118) the constant state-feedback matrix  $\mathbf{K}$  is given by

$$(13.124) \quad \mathbf{K} = (\hat{\mathbf{R}} + \mathbf{B}^T\mathbf{P}\mathbf{B})^{-1}(\mathbf{B}^T\mathbf{P}\hat{\mathbf{A}} + \mathbf{Q}'_{xu}{}^T)$$

so that the closed-loop system matrix  $\mathbf{A}_{CL} = \mathbf{A} - \mathbf{B}\mathbf{K}$  is

$$(13.125) \quad \mathbf{A}_{CL} = \mathbf{A} - \mathbf{B}(\hat{\mathbf{R}} + \mathbf{B}^T\mathbf{P}\mathbf{B})^{-1}(\mathbf{B}^T\mathbf{P}\hat{\mathbf{A}} + \mathbf{Q}'_{xu}{}^T).$$

For the basic case of the optimization problem,  $\mathbf{Q}'_{xu}{}^T$  is zero in this formula.

**Example 22**  
Infinite  $t_f$  for  
Example 20

Extending the optimization interval to  $[0, \infty)$  in Example 20 requires constant  $\mathbf{P}$  to be found from Equation (13.123), which becomes, in this example,

$$ep(1 + (e-1)^2)^{-1}e - p + 1 = 0,$$

the positive solution of which is

$$p = \frac{-e + \sqrt{e^2 + 1}}{e - 1}.$$

Substituting this into (13.124) gives the constant state-feedback matrix  $\mathbf{K}$  as

$$\mathbf{K} = \frac{-e + e\sqrt{e^2 + 1}}{1 + (e-1)^2}.$$

**Example 23**  
Parameterized  
example

As in Example 1, let  $\mathbf{A} = 1$ ,  $\mathbf{B} = 1$ ,  $\mathbf{C} = 1$ ,  $\mathbf{D} = 0$ ,  $\mathbf{Q} = q$ , and  $\mathbf{R} = 1$ , with the optimization to be over the infinite interval  $[t_0, \infty)$ , but suppose that the system is discrete time. Then the discrete algebraic Riccati equation (13.123) is

$$1 \cdot (1 + 1 \cdot p)^{-1} \cdot 1 - p + q = 0,$$

and the positive solution of this is  $p = (q/2)(1 + \sqrt{1 + 4/q})$ . The constant state-feedback gain is given by (13.124) to be  $\mathbf{K} = p/(1 + p)$ . From (13.125), the closed-loop system matrix is

$$\mathbf{A}_{\text{CL}} = 1 - p/(1 + p) = 1 - \frac{q(1 + \sqrt{1 + 4/q})/2}{1 + q(1 + \sqrt{1 + 4/q})/2},$$

which has magnitude less than 1 and approaches zero as  $q \rightarrow \infty$ , implying faster transient decay of closed-loop solutions to the origin as  $q$  increases.

**3.2 Computation of P: the discrete-time case****Pencil of matrices**

Methods of solving (13.123) for constant matrix  $\mathbf{P}$  will be investigated briefly. Let  $\Gamma(t) = \mathbf{P}\mathbf{X}(t)$  and take the  $\mathcal{Z}$  transform of (13.113) to get

$$(13.126) \quad \left( \begin{bmatrix} \mathbf{I} & \hat{\mathbf{S}} \\ 0 & \hat{\mathbf{A}}^T \end{bmatrix} z - \begin{bmatrix} \hat{\mathbf{A}} & 0 \\ -\hat{\mathbf{Q}} & \mathbf{I} \end{bmatrix} \right) \begin{bmatrix} \mathbf{I} \\ \mathbf{P} \end{bmatrix} \mathbf{X}(z) = (\mathbf{L}z - \mathbf{K}) \begin{bmatrix} \mathbf{I} \\ \mathbf{P} \end{bmatrix} \mathbf{X}(z) = 0$$

where the matrices  $\mathbf{L}$  and  $\mathbf{K}$  are defined by this equation. An expression of the form  $\mathbf{L}z - \mathbf{K}$  is called a *pencil of matrices*. From the form of the equation, the only possibility of nonzero solutions  $\mathbf{X}(z)$  is for this pencil of matrices to be singular. The values of  $z$  at which the pencil is singular are called generalized eigenvalues, and vectors  $\hat{\mathbf{X}}$  for which  $(\mathbf{L}z - \mathbf{K})\hat{\mathbf{X}} = 0$  are called generalized eigenvectors. Thus from (13.126), the pencil of matrices

$$(13.127) \quad \begin{bmatrix} \mathbf{I} & \hat{\mathbf{S}} \\ 0 & \hat{\mathbf{A}}^T \end{bmatrix} z - \begin{bmatrix} \hat{\mathbf{A}} & 0 \\ -\hat{\mathbf{Q}} & \mathbf{I} \end{bmatrix} = \mathbf{L}z - \mathbf{K}$$

is to have eigenvectors  $\begin{bmatrix} \mathbf{I} \\ \mathbf{P} \end{bmatrix} \mathbf{X}(z_i)$  corresponding to eigenvalues  $z_i$ . For reasons identical to those of the time-continuous case, the sequence  $\{\mathbf{X}(t)\}$  must be asymptotically stable; therefore, the matrix  $\begin{bmatrix} \mathbf{I} \\ \mathbf{P} \end{bmatrix}$  must span the range of stable eigenvectors of the pencil, and  $\mathbf{P}$  can be found as in Example 16: find a basis  $\mathbf{V}_-$ , partitioned as in (13.91), for the range of the eigenvectors corresponding to asymptotically stable eigenvalues of the pencil, and solve Equation (13.92) for  $\mathbf{P}$ .

---

**Example 24**  
Eigenvalues of  
matrix pencils

When the entries of  $\mathbf{K}$  and  $\mathbf{L}$  are finite, eigenvalues of  $\mathbf{K} - \mathbf{L}z$  may be infinite, as when  $\mathbf{K} = \mathbf{1}$  and  $\mathbf{L} = 0$ ; or indeterminate, as when  $\mathbf{K} = \mathbf{L} = 0$ .

If the pencil is not square or if  $\det(\mathbf{L}z - \mathbf{K}) \equiv 0$  then the pencil is said to be singular. For square pencils, a procedure called the QZ algorithm uses orthogonal matrices  $\mathbf{Q}$  and  $\mathbf{Z}$  such that

$$\mathbf{Q}^T(\mathbf{K} - \mathbf{L}z)\mathbf{Z} = \bar{\mathbf{K}} - \bar{\mathbf{L}}z$$

where  $\bar{\mathbf{K}}$  and  $\bar{\mathbf{L}}$  are upper triangular with diagonal entries  $\alpha_i$  in  $\bar{\mathbf{K}}$  and  $\beta_i$  in  $\bar{\mathbf{L}}$ . The eigenvalues are given by  $\alpha_i/\beta_i$ , which is indeterminate if  $\alpha_i = \beta_i = 0$ , or infinite if  $\alpha_i \neq 0$  and  $\beta_i = 0$ .

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**Relation to  
continuous  
optimization**

Rather than developing the details of methods obtained from the pencil of matrices (13.127), it will be shown that any method that solves the continuous-time case can also be employed for the discrete-time optimization.

Consider the pencil eigenvector equation

$$(13.128) \quad (\mathbf{L}z - \mathbf{K}) \hat{\mathbf{X}} = 0$$

and apply the bilinear transformation

$$(13.129) \quad z = \frac{s+1}{s-1},$$

which maps values of  $s$  in the open left-hand plane to the interior of unit circle, to obtain

$$(13.130) \quad \left( (\mathbf{K} - \mathbf{L})s - (\mathbf{K} + \mathbf{L}) \right) \hat{\mathbf{X}} = 0.$$

Suppose for a moment that  $\mathbf{K} - \mathbf{L}$  is singular. Then either (13.130) has an eigenvalue at infinity (see Example 24) and (13.126) has an eigenvalue at  $z = 1$ , or if  $\mathbf{K} + \mathbf{L}$  is singular as well there is the possibility that (13.130) has arbitrary eigenvalues, as does (13.126). In either case the eigenvalues of (13.126) cannot be sorted into stable and unstable values, and the original discrete optimization is singular. Hence, singularity of  $\mathbf{K} - \mathbf{L}$  implies a singular optimization problem, which is assumed not to apply to the following.

Pre-multiply (13.126) by  $(\mathbf{K} - \mathbf{L})^{-1}$ , giving

$$(13.131) \quad \left( s\mathbf{I} - (\mathbf{K} - \mathbf{L})^{-1}(\mathbf{K} + \mathbf{L}) \right) \begin{bmatrix} \mathbf{I} \\ \mathbf{P} \end{bmatrix} \mathbf{X} = 0.$$

Let  $\mathbf{P}$  satisfy the discrete-time algebraic Riccati equation

$$(13.132) \quad \hat{\mathbf{A}}^T \mathbf{P} (\mathbf{I} + \hat{\mathbf{S}} \mathbf{P})^{-1} \hat{\mathbf{A}} - \mathbf{P} + \hat{\mathbf{Q}} = 0$$

and let  $\mathbf{V}$  satisfy the *discrete Lyapunov equation*

$$(13.133) \quad \mathbf{A}_{\text{CL}} \mathbf{V} \mathbf{A}_{\text{CL}}^T - \mathbf{V} + (\mathbf{I} + \hat{\mathbf{S}} \mathbf{P})^{-1} \hat{\mathbf{S}} = 0.$$

From the discussion in Section 3.4 of Chapter 8, a solution  $\mathbf{V}$  is guaranteed to exist when  $\lambda_i(\mathbf{A}_{\text{CL}}) \lambda_j(\mathbf{A}_{\text{CL}}) \neq 1$  for all eigenvalue pairs  $i, j$  as is true when  $\mathbf{A}_{\text{CL}}$  is asymptotically stable; that is when  $|\lambda_i(\mathbf{A}_{\text{CL}})| < 1$  for all  $i$ . Then  $\mathbf{L}$  and  $\mathbf{K}$  will be transformed, using a change of basis:

$$(13.134) \quad \begin{bmatrix} \mathbf{X} \\ \mathbf{\Gamma} \end{bmatrix} = \begin{bmatrix} \mathbf{I} \\ \mathbf{P} \end{bmatrix} \mathbf{X} = \begin{bmatrix} \mathbf{I} & 0 \\ \mathbf{P} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{I} & -\mathbf{V} \\ 0 & \mathbf{I} \end{bmatrix} \tilde{\mathbf{X}} = \begin{bmatrix} \mathbf{I} & -\mathbf{V} \\ \mathbf{P} & \mathbf{I} - \mathbf{P} \mathbf{V} \end{bmatrix} = \mathbf{N} \tilde{\mathbf{X}},$$

where  $\mathbf{N}$  is defined in this equation. This change of basis and premultiplication by a matrix  $\mathbf{W}$  changes (13.128) to  $\mathbf{W}(\mathbf{L}z - \mathbf{K}) \mathbf{N} \tilde{\mathbf{X}} = 0$ . The matrix  $\mathbf{W}$  is

$$(13.135) \quad \mathbf{W} = \begin{bmatrix} \mathbf{I} & \mathbf{A}_{\text{CL}} \mathbf{V} \\ 0 & \mathbf{I} \end{bmatrix} \begin{bmatrix} (\mathbf{I} + \hat{\mathbf{S}} \mathbf{P})^{-1} & 0 \\ -\mathbf{A}_{\text{CL}}^T \mathbf{P} & \mathbf{I} \end{bmatrix},$$

so that  $\mathbf{L}$  and  $\mathbf{K}$  become respectively

$$(13.136) \quad \mathbf{W} \mathbf{L} \mathbf{N} = \begin{bmatrix} \mathbf{I} & 0 \\ 0 & -\mathbf{A}_{\text{CL}}^T \end{bmatrix}, \quad \mathbf{W} \mathbf{K} \mathbf{N} = \begin{bmatrix} \mathbf{A}_{\text{CL}} & 0 \\ 0 & \mathbf{I} \end{bmatrix}.$$

Inspection of these two matrices shows that  $\mathbf{W}(\mathbf{L}z - \mathbf{K}) \mathbf{N}$  (and hence  $\mathbf{L}z - \mathbf{K}$ ) has eigenvalues  $\lambda_i(\mathbf{A}_{\text{CL}})$ , together with their inverses. The inverse of transformation (13.129) is

$$(13.137) \quad s = \frac{z + 1}{z - 1}.$$

Consequently, an eigenvalue  $z_i$  of  $\mathbf{L}z - \mathbf{K}$  corresponds to a value  $s_i$  given by this formula, whereas its inverse  $1/z_i$  corresponds to

$$(13.138) \quad \frac{1/z_i + 1}{1/z_i - 1} = -\frac{z_i + 1}{z_i - 1} = -s_i.$$

Therefore, the eigenvalues of (13.131) will occur symmetrically with respect to the imaginary axis of the  $s$ -plane.

**Summary** A constant matrix  $\mathbf{P}$  that solves a nonsingular discrete optimization problem of the form of (13.4) subject to constraint (13.5) can be obtained by constructing the matrices

$$(13.139) \quad \mathbf{L} = \begin{bmatrix} \mathbf{I} & \hat{\mathbf{S}} \\ 0 & \hat{\mathbf{A}}^T \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} \hat{\mathbf{A}} & 0 \\ -\hat{\mathbf{Q}} & \mathbf{I} \end{bmatrix}, \quad \mathbf{H} = (\mathbf{K} - \mathbf{L})^{-1}(\mathbf{K} + \mathbf{L})$$

and applying the methods of Section 2.6 to  $\mathbf{H}$ . The state-feedback matrix is given by (13.124).

**Example 25**  
Application to  
the system of  
Example 23

The above method applied to the system given in Example 23 produces matrices

$$\mathbf{L} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} 1 & 0 \\ -q & 1 \end{bmatrix}, \quad \mathbf{K} - \mathbf{L} = \begin{bmatrix} 0 & -1 \\ -q & 0 \end{bmatrix}, \quad \mathbf{K} + \mathbf{L} = \begin{bmatrix} 2 & 1 \\ -q & 2 \end{bmatrix},$$

so that the equivalent  $\mathbf{H}$  given by (13.139) is

$$\mathbf{H} = \begin{bmatrix} 1 & -2/q \\ -2 & -1 \end{bmatrix}.$$

This matrix has eigenvalues at  $s = \pm\sqrt{1 + 4/q}$ . An eigenvector corresponding to the left half-plane eigenvalue is

$$\mathbf{V}_- = \begin{bmatrix} 2 \\ q(1 + \sqrt{1 + 4/q}) \end{bmatrix} = \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix}.$$

Solving (13.92) gives  $p = (q/2)(1 + \sqrt{1 + 4/q})$  as in Example 23.

4

#### Further study

The optimal state-feedback control of LTI systems is a now-classical problem that has been studied for decades. Many books have been written on the subject. Two that continue to be well worth consulting are [37] and [57]. An excellent treatment is in [18]. Reference [60] is a more recent addition to the literature. Many modern books such as [7] devote a chapter or more to the subject. For a discussion of the numerical aspects of solving the ARE for constant  $\mathbf{P}$ , see [39].

5

#### Problems

- 1 Solve for the steady-state value  $\mathbf{P}(t_0)$  for  $t_f = \infty$  in Example 8 using the results of Section 2.5.

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- 2 Rework Examples 1, 9, 23, and 25 for parameter value  $q = 3$ .

---

- 3 A constant optimal state-feedback matrix is to be found for a system, using cost-function matrices  $\mathbf{Q} = q\mathbf{I}$  and  $\mathbf{R} = r\mathbf{I}$  in (13.2), where  $q$  and  $r$  are parameters. The design begins by solving for  $q_0 = r_0 = 1$ . For the choices of  $q$  and  $r$  given below, determine whether the time-constants of the closed-loop system will be faster, slower, or the same as for the initial solution.
  - (a)  $q = 100, r = 1$ ,    (b)  $q = 1, r = 100$ ,    (c)  $q = 100, r = 100$ .

---

- 4 Write the following quadratic forms in matrix notation  $\mathbf{X}^T \mathbf{A} \mathbf{X}$ .
  - (a)  $x_1^2 + 6x_1x_2 + 3x_2^2$ ,    (b)  $2x_1^2 - 4x_1x_2 + x_2^2$ ,    (c)  $x_1^2 - 2x_2^2 - 3x_3^2 + 6x_1x_3 - 6x_2x_3$ .

- 5 Determine whether each of the following matrices is positive-definite and positive-semidefinite:

$$(a) \mathbf{Q} = \begin{bmatrix} 7 & 10 \\ 2 & 4 \end{bmatrix}, \quad (b) \mathbf{Q} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 3 & 2 \\ 0 & 2 & 5 \end{bmatrix}, \quad (c) \mathbf{Q} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

- 6 Write the following quadratic forms as scalar functions of the entries  $x_1, x_2, \dots$  of  $\mathbf{X}$ .

$$(a) \mathbf{X}^T \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \mathbf{X}, \quad (b) \mathbf{X}^T \begin{bmatrix} 2 & 2 \\ -2 & 1 \end{bmatrix} \mathbf{X}, \quad (c) \mathbf{X}^T \begin{bmatrix} 1 & 0 & 1 \\ 0 & -2 & -3 \\ 0 & 0 & -3 \end{bmatrix} \mathbf{X}.$$

- 7 The input for the system given by  $\dot{x} = u$  is to be found by optimizing the cost function  $\phi = \int_{t_0}^{t_1} (x^2 + u^2) dt$ . Find  $u(t)$  over the interval  $[t_0, t_1)$  as a function of time and as a function of  $x(t)$  over this interval.

- 8 Write the Hamiltonian matrix  $\mathbf{H}$  for the system described in Problem 7 with  $t_1 \rightarrow \infty$ . What will be the eigenvalue of the closed-loop optimal system?

- 9 For the system with matrices given by

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{C} = [1, 0], \quad \mathbf{D} = 0,$$

and infinite interval  $[0, \infty)$ , solve for the optimal constant state-feedback matrix  $\mathbf{K}$  for cost function

$$\phi = \int_0^\infty (y^2 + \rho^2 u^2) dt,$$

where  $\rho$  is a parameter. How do the transients of the closed-loop system differ in the two cases that  $\rho$  is very small and very large?

- 10 Solve Problem 9 for the case where  $\mathbf{D}$  has been changed to  $\mathbf{D} = 1$ .

- 11 Find the constant state-feedback matrix that minimizes the cost function shown for the system with matrices shown:

$$\phi = \int_0^\infty \left( \mathbf{X}^T \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix} \mathbf{X} + u^2 \right) dt, \quad \mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & \sqrt{5} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

- 12 Find the constant state-feedback matrix that minimizes the cost function shown for the system with matrices shown:

$$\phi = \int_0^\infty (\mathbf{X}^T \mathbf{X} + u^2) dt, \quad \mathbf{A} = \begin{bmatrix} -2 & 1 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$



- 13** Find the constant state-feedback matrix that optimizes the system given by

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{C} = [1, 0], \quad \mathbf{D} = 0$$

over the interval  $[0, \infty)$  with respect to the cost function

$$\phi = \int_0^{\infty} (y^2 + 4u^2) dt.$$

- 14** The state feedback to control the roll angle of a rocket by altering the fin angle is to be designed. The rate of change of fin angle can be controlled. Denoting fin angle as  $x_1$ , roll velocity as  $x_2$ , and roll angle as  $x_3$ , the plant equations are

$$\dot{x}_1 = u, \quad \dot{x}_2 = (k/\tau)x_1 - (1/\tau)x_2, \quad \dot{x}_3 = x_2,$$

where  $\tau = 1$  s and  $k = 10$ . The cost function is

$$\phi = \int_0^{\infty} \left( \frac{x_1^2}{(\pi/12)^2} + \frac{x_3^2}{(\pi/180)^2} + \frac{u^2}{\pi^2} \right) dt.$$

Find the constant state-feedback matrix for this controller.

- 15** Find the constant state-feedback matrix that solves the steady-state LQ problem for the following data:

$$\mathbf{A} = \begin{bmatrix} 0 & 8 \\ -8 & 0 \end{bmatrix}, \quad \mathbf{B} = \mathbf{Q} = \mathbf{R} = \mathbf{I}.$$

- 16** Verify that a diagonal cost matrix  $\mathbf{Q}$  in Equation (13.2) with  $t_f = \infty$  can be found to produce closed-loop eigenvalues identical to the result of pole placement in Example 4 of Chapter 12. Proceed as follows: substitute  $\gamma^2 = g(m + M)/(M\ell)$  and  $-mg/M = g - \gamma^2\ell$  in  $\mathbf{A}$  of the model given in the example. Construct the Hamiltonian matrix  $\mathbf{H}$  for  $\mathbf{R} = 1$  and diagonal  $\mathbf{Q}$ . Using a symbolic-algebra program, calculate the characteristic polynomial  $\phi(s)$  of  $\mathbf{H}$  and solve for the diagonal elements of  $\mathbf{Q}$  such that the coefficients of  $\phi(s)$  equal those of the desired characteristic polynomial

$$\begin{aligned} \beta(s) &= (s^2 - \gamma^2)^2 (s^2 + 2\zeta\omega s + \omega^2)(s^2 - 2\zeta\omega s + \omega^2) \\ &= (s^2 - \gamma^2)^2 (s^4 + 2\omega^2\delta s^2 + \omega^4), \end{aligned}$$

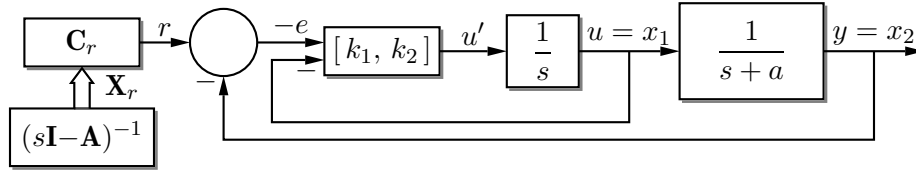
where  $\delta = 1 - 2\zeta^2$ .

- 17** A controller for a plant described by  $\mathbf{H}(s) = 1/(s+a)$  is required to produce zero steady-state output error for reference step signals of arbitrary size, unknown in advance. This requirement can be formulated as an infinite-interval optimization problem that assigns a positive cost to the error. The closed-loop system corresponds to Figure 12.3, with zero disturbance vector. However, the given plant has no pole at the origin; the controller must include an integrator to create a type 1 system capable of zero steady-state

step error. Putting an integrator in series with the plant results in state-space equations containing the matrices

$$\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 1 & -a \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{C} = [0, 1], \quad \mathbf{D} = [0],$$

as shown in Figure P13.17, where  $u'$  is the input of the composite plant.



**Fig. P13.17** Integral control in series with the plant

Because  $\mathbf{A}$  contains a pole at the origin, a step input  $r$  can be represented as the free response of the plant; that is

$$\dot{\mathbf{X}}_r = \mathbf{A}\mathbf{X}_r, \quad r = \mathbf{C}\mathbf{X}_r.$$

Then the system equations can be written using the state vector  $\mathbf{E} = \mathbf{X} - \mathbf{X}_r$  as

$$\frac{d}{dt}\mathbf{E} = \frac{d}{dt}\mathbf{X} - \frac{d}{dt}\mathbf{X}_r = \mathbf{A}\mathbf{X} + \mathbf{B}u' - \mathbf{A}\mathbf{X}_r = \mathbf{A}\mathbf{E} + \mathbf{B}u',$$

with the quantity

$$-e = y - r = \mathbf{C}\mathbf{E}$$

to go to zero in steady-state. The function to be minimized becomes

$$\phi = \int_0^{\infty} (\rho^2 e^2 + (u')^2) dt,$$

subject to the above equations, where  $\rho$  is a tuning parameter and the term in  $u'$  has been included to result in finite inputs and a nonsingular optimization problem.

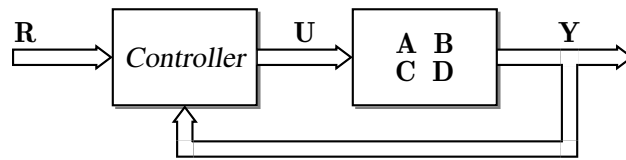
Let  $\alpha = 2$  and  $\rho = 1$ . First, solve for  $\mathbf{K} = [k_1, k_2]$  so that  $u' = -\mathbf{K}\mathbf{E} = -\mathbf{K}[e_1, e_2]^T$ . Then solve for  $e_1$  in terms of  $e = e_2$ , using the equation for  $\dot{\mathbf{E}}$  given above. Referring to Figure P13.17, write the expression for  $u$  in terms of  $e$  alone and, using this expression, redraw the diagram to show that the resulting controller with input  $e$  and output  $u$  is in PI (proportional plus integral) form.

# 14

## Output feedback

The powerful state-feedback design methods described in Chapters 12 and 13 suffer from an important defect: the plant state vector is not always accessible by direct physical measurement. Some entries of the state vector  $\mathbf{X}$  may have been defined simply to provide sufficient detail for adequately modeling plant behavior. Moreover, measuring the  $n$  variables in  $\mathbf{X}$  and designing the  $mn$  parameters in the state feedback matrix  $\mathbf{K}$  may be more complex than necessary for adequate control. The input vector  $\mathbf{U}$  and output vector  $\mathbf{Y}$ , however, typically have clear physical significance and accessibility.

This chapter investigates the simple configuration illustrated in Figure 14.1. The controller must be designed to generate the plant input  $\mathbf{U}(t)$  from the reference  $\mathbf{R}(t)$  and measured plant output  $\mathbf{Y}(t)$ . Complications may arise because of model inaccuracies, external disturbances acting on the plant, and noise in the measured values of  $\mathbf{Y}(t)$ .



**Fig. 14.1** Output-feedback control requires the controller to generate the control signal  $\mathbf{U}$  from the measured plant output  $\mathbf{Y}$  and the reference signal  $\mathbf{R}$ .

Although continuous-time notation is primarily employed in this chapter, much of the development applies equally in discrete time, and the observers to be discussed are often implemented as discrete-time systems.

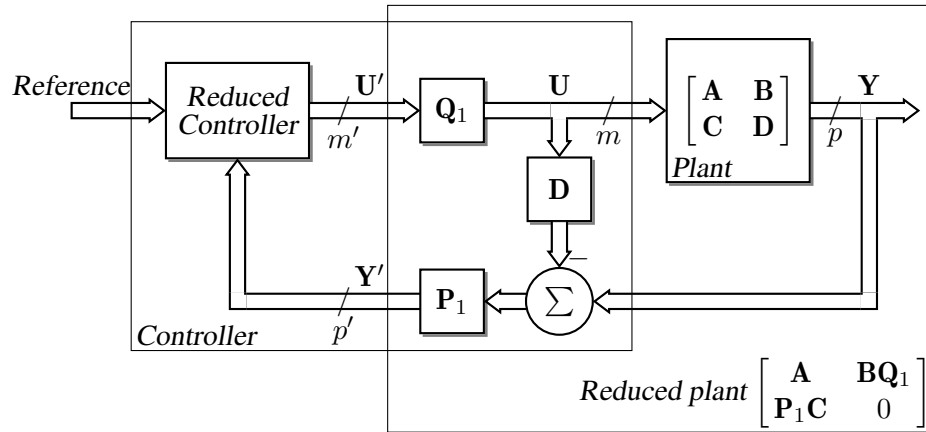
### Plant simplifications

Before considering the details of the controller design, it is instructive to inquire what may be assumed about the plant. We should expect to be able to define the closed-loop transient behaviour by generating  $\mathbf{U}(t)$  if the plant is completely controllable as described in Chapter 9. If the plant is also completely observable then measurement of  $\mathbf{Y}(t)$  should allow the determination of  $\mathbf{X}(t)$ , after which the techniques of Chapter 12 and Chapter 13 should be applicable.

If necessary to simplify controller design, the following may be assumed of the plant, as illustrated in Figure 14.2:

### Zero $\mathbf{D}$

The matrix  $\mathbf{D}$  is zero. If it were not, then a step at  $t = 0$  applied at one or more of the inputs corresponding to the entries of  $\mathbf{U}$  would result in an instantaneous change in  $\mathbf{Y}$  at  $t = 0+$ , which does not occur for physical systems exhibiting inertia, for example.



**Fig. 14.2** The controller cancels the effect of  $\mathbf{D}$  on the plant output, generates  $p'$  independent outputs in  $\mathbf{Y}'$ , and distributes the  $m'$  independent inputs over the plant input  $\mathbf{U}$ . The reduced controller can be designed for the resulting reduced plant with input  $\mathbf{U}'$ , output  $\mathbf{Y}'$ , and matrices shown.

Equivalently, nonzero  $\mathbf{D}$  implies a nonzero transfer matrix  $\mathbf{H}(s)$  in the limit as  $s \rightarrow \infty$ , hence an infinite bandwidth. However, the effect of a nonzero  $\mathbf{D}$  in the plant model can be cancelled within the controller by subtracting  $\mathbf{D}\mathbf{U}$  from the measured plant output  $\mathbf{Y}$ , as in Figure 14.2, provided  $\mathbf{D}$  is known and  $\mathbf{Y}$  can be accurately measured.

**Full-rank B** The matrix  $\mathbf{B}$  has full column rank  $m$ . If this is not true but  $\text{rank}(\mathbf{B}) = m' < m$ , then  $\mathbf{B}$  spans a subspace of dimension  $m'$  and there are, effectively, only  $m'$  independent inputs. Such a situation can occur when there is input redundancy. Find a basis  $\mathbf{B}\mathbf{Q}_1$  for the column range  $\mathcal{R}(\mathbf{B})$  of  $\mathbf{B}$  as described in Section 4.1 of Chapter 6 and let

$$(14.1) \quad \mathbf{U} = \mathbf{Q}_1 \mathbf{U}'.$$

Then the plant state equation becomes

$$(14.2) \quad \frac{d}{dt} \mathbf{X} = \mathbf{A}\mathbf{X} + (\mathbf{B}\mathbf{Q}_1) \mathbf{U}',$$

which describes a plant with full-rank matrix  $\mathbf{B}\mathbf{Q}_1$  and input  $m'$ -vector  $\mathbf{U}'$ . The matrix  $\mathbf{Q}_1$  can be chosen to scale plant input values as well as to distribute the  $m'$  independent inputs in  $\mathbf{U}'$  over the  $m$  entries of  $\mathbf{U}$ .

**Full-rank C** The matrix  $\mathbf{C}$  has full row rank  $p$ . Otherwise, if  $\text{rank}(\mathbf{C}) = p' < p$ , then the entries of  $\mathbf{Y}$  are linearly dependent and contain only  $p'$  independent variables. This situation can occur when there is redundancy of output sensors. Find a basis  $\mathbf{P}_1\mathbf{C}$  for the row range (a row compression) of  $\mathbf{C}$ . Then let

$$(14.3) \quad \mathbf{Y}' = \mathbf{P}_1 \mathbf{Y},$$

so that the plant output equation becomes

$$(14.4) \quad \mathbf{Y}' = \mathbf{P}_1 \mathbf{C} \mathbf{X} + \mathbf{P}_1 \mathbf{D} \mathbf{U} = (\mathbf{P}_1 \mathbf{C}) \mathbf{X} + \mathbf{P}_1 \mathbf{D} \mathbf{U},$$

which describes a plant with full-rank matrix  $\mathbf{P}_1\mathbf{C}$  and output vector  $\mathbf{Y}'$  of dimension  $p'$ . The rightmost term in this formula is zero if  $\mathbf{D}$  has been cancelled as in Figure 14.2.

Reducing the plant as described is not always necessary but can considerably simplify the discussion of design methods.

**Example 1**  
Reduced system

The system with matrices given below has nonzero  $\mathbf{D}$  matrix and neither  $\mathbf{B}$  nor  $\mathbf{C}$  has full rank. A state feedback matrix is to be obtained to place the poles at  $-2$  and  $-1 \pm j$ .

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 2 & 1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

The matrix  $\mathbf{Q}_1$  shown produces a basis  $\mathbf{BQ}_1$  for the column range of  $\mathbf{B}$ , which has rank 2, and similarly the matrix  $\mathbf{P}_1$  produces a row compression of  $\mathbf{C}$ :

$$\mathbf{Q}_1 = \begin{bmatrix} 1/4 \\ 1/2 \end{bmatrix}, \quad \mathbf{P}_1 = \begin{bmatrix} 1/2 & 1/4 & -1/2 \\ -1/2 & 1/4 & 1/2 \end{bmatrix}.$$

Neither  $\mathbf{P}_1$  nor  $\mathbf{Q}_1$  is unique, but the matrices shown have been chosen to utilize both inputs in  $\mathbf{U}$  and all measured variables in  $\mathbf{Y}$ , and to result in a gain of 1. The following reduced system equations can then be used for controller design, provided that  $\mathbf{D}$  is cancelled in the controller if output feedback is employed:

$$\frac{d}{dt}\mathbf{X} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix}\mathbf{X} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\mathbf{U}', \quad \mathbf{Y}' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\mathbf{X}.$$

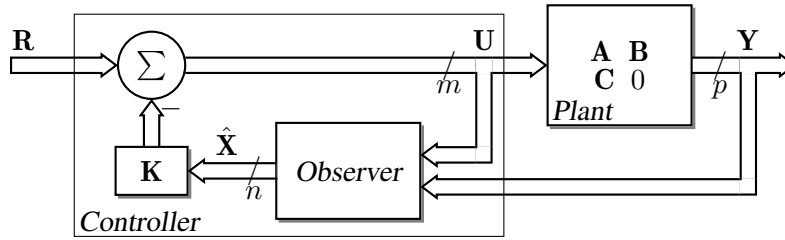
The desired closed-loop characteristic polynomial is  $s^3 + 4s^2 + 6s + 4$  and the plant is in control canonical form, so the state-feedback matrix for the reduced system is

$$\mathbf{K} = [4, 6, 4] - [-1, 1, -1] = [5, 5, 5].$$

Therefore, the input vector of the original system with reference input  $\mathbf{R}$  is

$$\mathbf{U} = \mathbf{Q}_1\mathbf{U}' = \mathbf{Q}_1(\mathbf{R} - \mathbf{KX}) = \begin{bmatrix} 1/4 \\ 1/2 \end{bmatrix}\mathbf{R} - \begin{bmatrix} 5/4 & 5/4 & 5/4 \\ 5/2 & 5/2 & 5/2 \end{bmatrix}\mathbf{X}.$$

One strategy for output feedback is to separate the controller design into two parts as shown in Figure 14.3: design a state-feedback matrix  $\mathbf{K}$  on the assumption that the state  $\mathbf{X}$  has been measured, and design a system to provide a vector  $\hat{\mathbf{X}}$  that suitably approximates  $\mathbf{X}$  from measurements of the plant input  $\mathbf{U}$  and output  $\mathbf{Y}$ . The subsystem that produces  $\hat{\mathbf{X}}$  from  $\mathbf{U}$  and  $\mathbf{Y}$  is called an *observer*. In the context of controller design,  $\mathbf{Y}$  must be measured but  $\mathbf{U}$  is generated by the controller itself. Nevertheless, the configuration shown in the figure is convenient for obtaining the observer equations. In general, the observer must contain dynamics in order to estimate the state  $\mathbf{X}$  of the plant.



**Fig. 14.3** The observer provides a vector  $\hat{\mathbf{X}}$  that is fed back through the state-feedback matrix  $\mathbf{K}$ .

### 1.1 The full-order observer

Assume that in Figure 14.3 the observer has the same order  $n$  as the plant. The observer state model can then be taken to be

$$(14.5) \quad \frac{d}{dt} \hat{\mathbf{X}} = \mathbf{A}_o \hat{\mathbf{X}} + [\mathbf{L}, \mathbf{M}] \begin{bmatrix} \mathbf{Y} \\ \mathbf{U} \end{bmatrix} = \mathbf{A}_o \hat{\mathbf{X}} + \mathbf{L}\mathbf{Y} + \mathbf{M}\mathbf{U},$$

where the output is  $\hat{\mathbf{X}}$ , the input vector contains the plant output  $\mathbf{Y}$  and input  $\mathbf{U}$ , and the matrices  $\mathbf{A}_o$ ,  $\mathbf{L}$ , and  $\mathbf{M}$  are to be determined. Since  $\hat{\mathbf{X}}$  is to approximate  $\mathbf{X}$ , it is convenient to define the observer error vector

$$(14.6) \quad \mathbf{E} = \mathbf{X} - \hat{\mathbf{X}}$$

and to require  $\mathbf{E}$  to approach 0. Writing the equation for the time-derivative of  $\mathbf{E}$  in terms of the observer and plant equations gives

$$(14.7) \quad \begin{aligned} \frac{d}{dt} \mathbf{E} &= \frac{d}{dt} \mathbf{X} - \frac{d}{dt} \hat{\mathbf{X}} \\ &= \mathbf{A}\mathbf{X} + \mathbf{B}\mathbf{U} - \mathbf{A}_o \hat{\mathbf{X}} - \mathbf{L}\mathbf{C}\mathbf{X} - \mathbf{M}\mathbf{U} \\ &= (\mathbf{A} - \mathbf{L}\mathbf{C})\mathbf{X} - \mathbf{A}_o \hat{\mathbf{X}} + (\mathbf{B} - \mathbf{M})\mathbf{U} \\ &= (\mathbf{A} - \mathbf{L}\mathbf{C})(\mathbf{X} - \hat{\mathbf{X}}) + (\mathbf{A} - \mathbf{L}\mathbf{C} - \mathbf{A}_o)\hat{\mathbf{X}} + (\mathbf{B} - \mathbf{M})\mathbf{U} \\ &= (\mathbf{A} - \mathbf{L}\mathbf{C})\mathbf{E} + (\mathbf{A} - \mathbf{L}\mathbf{C} - \mathbf{A}_o)\hat{\mathbf{X}} + (\mathbf{B} - \mathbf{M})\mathbf{U}. \end{aligned}$$

By inspection of this equation, we see that  $\mathbf{E}$  will approach zero if the rightmost two terms are zero and the matrix  $\mathbf{A} - \mathbf{L}\mathbf{C}$  is asymptotically stable. Therefore, let  $\mathbf{M}$  and  $\mathbf{A}_o$  be chosen as

$$(14.8a) \quad \mathbf{M} = \mathbf{B},$$

$$(14.8b) \quad \mathbf{A}_o = \mathbf{A} - \mathbf{L}\mathbf{C},$$

and choose  $\mathbf{L}$  so that  $\mathbf{A} - \mathbf{L}\mathbf{C}$  has the required eigenvalues. Then (14.7) becomes

$$(14.9) \quad \frac{d}{dt} \mathbf{E} = (\mathbf{A} - \mathbf{L}\mathbf{C})\mathbf{E}$$

and the limit of  $\mathbf{E}(t)$  is 0 as  $t \rightarrow \infty$ .

The eigenvalues of a matrix equal those of its transpose; therefore, the choice of  $\mathbf{L}$  to place eigenvalues of  $\mathbf{A} - \mathbf{L}\mathbf{C}$  can be performed by designing  $\mathbf{L}^T$  in  $\mathbf{A}^T - \mathbf{C}^T \mathbf{L}^T$ , using

the methods of Chapter 12. Arbitrary eigenvalue assignment requires  $\mathbf{A}^T$  and  $\mathbf{C}^T$  to be a controllable pair; that is,

$$(14.10) \quad \text{rank} [\mathbf{C}^T, \mathbf{A}^T \mathbf{C}^T, \dots, (\mathbf{A}^T)^{n-1} \mathbf{C}^T] = \text{rank} \begin{bmatrix} \mathbf{C} \\ \mathbf{C}\mathbf{A} \\ \vdots \\ \mathbf{C}\mathbf{A}^{n-1} \end{bmatrix} = n.$$

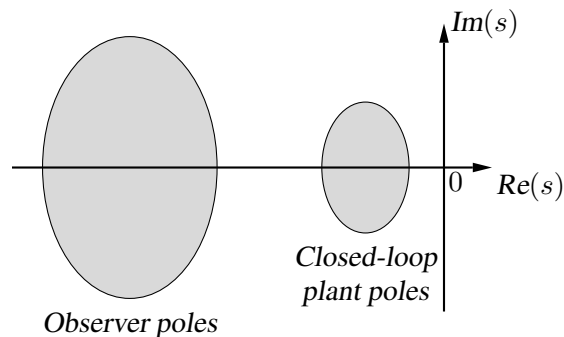
Therefore, the computation of  $\mathbf{L}$  reduces to a state-feedback computation such as discussed in Chapters 12 and 13. The eigenvalues can be placed arbitrarily provided the plant is completely observable.

### Open versus closed loop

Figure 14.3 was used as the motivation for developing the observer equations, but nowhere in the equations does the matrix  $\mathbf{K}$  appear, nor is the value of  $\mathbf{U}$  restricted. The generation of  $\mathbf{U}$  by a controller is not required. In theory then, an open-loop observer can be designed simply to measure  $\mathbf{U}$  and  $\mathbf{Y}$  and to generate  $\hat{\mathbf{X}}$  that asymptotically approaches  $\mathbf{X}$ , even if the plant is unstable. In practice however, unmodelled perturbations, measurement noise, and unmodelled nonlinearities may affect the error.

### Observer poles

In the context of feedback control illustrated by Figure 14.3, the closed-loop plant transient response to initial conditions will depend on the error  $\mathbf{E}$  between the observer-supplied state approximation  $\hat{\mathbf{X}}$  and the true value of  $\mathbf{X}$ . Therefore, a design rule of thumb is to choose  $\mathbf{L}$  to give observer poles well to the left (for time-continuous systems) of the closed-loop plant poles, so that observer time-constants are considerably faster than the closed-loop plant, as illustrated in Figure 14.4. However, a low-pass feedback-loop gain is often desirable, and observer time-constants that are too fast may result in a high feedback-loop bandwidth and consequent sensitivity to high-frequency noise or disturbances.



**Fig. 14.4** For analog systems, observer poles are normally placed to the left of the designed closed-loop system poles.

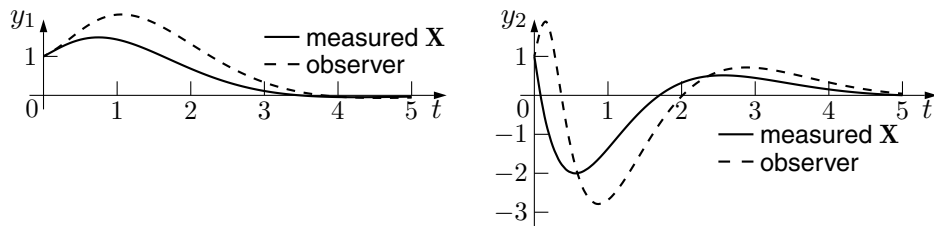
### Example 2 Observer for Example 1

A full-order observer will be designed for the reduced system of Example 1. Since the closed-loop plant has poles at  $-2$  and at  $-1 \pm j$ , observer poles at, say,  $-12$  will be chosen. A matrix that places all poles of  $\mathbf{A}_o = \mathbf{A} - \mathbf{L}\mathbf{C}$  at this value, and the matrix  $\mathbf{A}_o$

that results are shown:

$$\mathbf{L} = \begin{bmatrix} 13 & -11 \\ 12 & -131 \\ 0 & 24 \end{bmatrix}, \quad \mathbf{A}_o = \begin{bmatrix} -13 & 1 & 11 \\ -12 & 0 & 132 \\ 1 & -1 & -23 \end{bmatrix}.$$

The effect of the observer on the closed-loop free response to initial plant state  $\mathbf{X} = [1, 1, 1]^T$  is shown in Figure 14.5.



**Fig. 14.5** Free output responses for the reduced plant of Example 1 are shown. The solid lines are for the plant with state feedback and the dashed lines are for output feedback with full-order observer. The initial plant state is nonzero but the initial observer state is zero.

### Example 3 Observer for the inverse pendulum

In Example 11 of Chapter 13, a state-feedback LQ design was performed for specified numerical parameters of a linearized inverse pendulum.

Assuming that the cart position  $x = x_1$  and pendulum angle  $\theta = x_3$  only are measured, then  $\mathbf{D} = 0$  for the linearized system and the  $\mathbf{C}$  matrix is

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

The eigenvalues of  $\mathbf{A} - \mathbf{LC}$  of a full-order observer should be well to the left of those of  $\mathbf{A} - \mathbf{BK}$ . One way to find a matrix  $\mathbf{L}$  that produces suitable eigenvalues is to solve the LQ problem, replacing  $(\mathbf{A}, \mathbf{B})$  with  $(\mathbf{A}^T, \mathbf{C}^T)$ , the input weighting matrix  $\mathbf{R}$  with  $\mathbf{I}_2$ , and  $\mathbf{Q}$  with a diagonal matrix nonzero only in the second and fourth diagonal entries, so that the state variables  $v = x_2$  and  $\omega = x_4$  are weighted heavily. Setting these two diagonal entries to  $\mu = 10^5$ , for example, results in all four eigenvalues near  $-12.5 \pm j12$ , and the matrix

$$\mathbf{L} = \begin{bmatrix} 25 & 300 & 0 & 0 \\ 0 & 0 & 25 & 300 \end{bmatrix}^T \simeq \begin{bmatrix} 25.15 & 316.2 & -0.0400 & -0.0366 \\ -0.0400 & -1.996 & 25.62 & 328.2 \end{bmatrix}^T,$$

where  $\mathbf{L}$  is a simplified version of the computed LQ result on the right for the given value of  $\mu$ .

**Internal model** As developed above, the observer is a dynamical system described by the equation

$$(14.11) \quad \frac{d}{dt} \hat{\mathbf{X}} = (\mathbf{A} - \mathbf{LC}) \hat{\mathbf{X}} + \mathbf{LY} + \mathbf{BU} = \mathbf{A} \hat{\mathbf{X}} + \mathbf{BU} + \mathbf{L}(\mathbf{Y} - \mathbf{C} \hat{\mathbf{X}}).$$



This equation shows that the observer contains a model of the plant, augmented by an additional term that weights the error between the measured value of  $\mathbf{Y}$  and the predicted value  $\hat{\mathbf{Y}} = \mathbf{C}\hat{\mathbf{X}}$ . If the initial values  $\mathbf{X}(t_0)$  and  $\hat{\mathbf{X}}(t_0)$  are equal, then the output error  $\mathbf{Y} - \hat{\mathbf{Y}}$  in the response to  $\mathbf{U}(t)$  remains zero for all time. The rightmost term in this equation can be regarded as a correction for error in the predicted value  $\hat{\mathbf{X}}$  of  $\mathbf{X}$ .

**The separation principle**

It is instructive to write the state-space equations for the closed-loop system in Figure 14.3 for an observer constructed as described. A state vector can be chosen to contain the entries of  $\mathbf{X}$  together with those of  $\hat{\mathbf{X}}$ , but the state will be chosen instead as

$$(14.12) \quad \begin{bmatrix} \mathbf{X} \\ \mathbf{E} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & 0 \\ \mathbf{I} & -\mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ \hat{\mathbf{X}} \end{bmatrix}.$$

The result will be externally equivalent since these two state vectors are related by a nonsingular matrix in Equation (14.12). Including  $\mathbf{R}$  and  $\mathbf{E}$  in the plant equation gives

$$(14.13) \quad \begin{aligned} \frac{d}{dt}\mathbf{X} &= \mathbf{A}\mathbf{X} + \mathbf{B}(\mathbf{R} - \mathbf{K}\hat{\mathbf{X}}) = \mathbf{A}\mathbf{X} + \mathbf{B}\mathbf{R} - \mathbf{B}\mathbf{K}(\mathbf{X} - \mathbf{E}) \\ &= (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{X} + \mathbf{B}\mathbf{K}\mathbf{E} + \mathbf{B}\mathbf{R}, \end{aligned}$$

and combining this with Equation (14.9) gives the following model of the closed-loop system:

$$(14.14a) \quad \frac{d}{dt} \begin{bmatrix} \mathbf{X} \\ \mathbf{E} \end{bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{B}\mathbf{K} & \mathbf{B}\mathbf{K} \\ 0 & \mathbf{A} - \mathbf{L}\mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ \mathbf{E} \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix} \mathbf{R}$$

$$(14.14b) \quad \mathbf{Y} = [\mathbf{C}, 0] \begin{bmatrix} \mathbf{X} \\ \mathbf{E} \end{bmatrix}.$$

Inspection of these equations shows that the closed-loop eigenvalues are in two sets: the eigenvalues of  $\mathbf{A} - \mathbf{B}\mathbf{K}$  and the eigenvalues of  $\mathbf{A} - \mathbf{L}\mathbf{C}$ . This conclusion and its consequences are called the *separation principle*. The first set is obtained by calculating  $\mathbf{K}$  assuming state feedback, and the second set by calculating  $\mathbf{L}$  for the observer. Thus, if the plant is completely controllable and observable, the closed-loop poles can be placed arbitrarily in two independent design steps: state-feedback design and observer design.

**Closed-loop properties**

The response to the reference input  $\mathbf{R}$  and to disturbances will be investigated. Taking transforms and solving for  $\mathbf{Y}(s)$  as a function of  $\mathbf{R}(s)$  in (14.14) gives

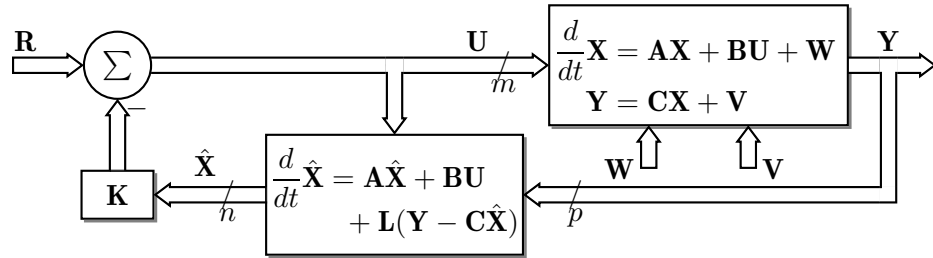
$$(14.15) \quad \mathbf{Y}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A} + \mathbf{B}\mathbf{K})^{-1}\mathbf{B}\mathbf{R}(s),$$

which is identical to the closed-loop response relation for state feedback. The observer has no effect. In fact, the system defined by Equation (14.14) is not completely controllable from input  $\mathbf{R}$ . However, an observer must provide an approximation of the plant state under non-ideal conditions. The plant equations will be modified to include a measurement noise vector  $\mathbf{V}(t)$  and a vector  $\mathbf{W}(t)$  to account for plant disturbances and modeling errors, as follows:

$$(14.16a) \quad \frac{d}{dt}\mathbf{X} = \mathbf{A}\mathbf{X} + \mathbf{B}\mathbf{U} + \mathbf{W}$$

$$(14.16b) \quad \mathbf{Y} = \mathbf{C}\mathbf{X} + \mathbf{V}.$$

The closed-loop result is shown in Figure 14.6. Including the noise and disturbance



**Fig. 14.6** The closed-loop system with observer, perturbation vector  $\mathbf{W}(t)$ , and measurement noise  $\mathbf{V}(t)$ .

vectors changes Equations (14.14) to

$$(14.17a) \quad \frac{d}{dt} \begin{bmatrix} \mathbf{X} \\ \mathbf{E} \end{bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{BK} & \mathbf{BK} \\ 0 & \mathbf{A} - \mathbf{LC} \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ \mathbf{E} \end{bmatrix} + \begin{bmatrix} \mathbf{B} & \mathbf{I} & 0 \\ 0 & \mathbf{I} & -\mathbf{L} \end{bmatrix} \begin{bmatrix} \mathbf{R} \\ \mathbf{W} \\ \mathbf{V} \end{bmatrix}$$

$$(14.17b) \quad \mathbf{Y} = [\mathbf{C}, 0] \begin{bmatrix} \mathbf{X} \\ \mathbf{E} \end{bmatrix} + [0, 0, \mathbf{I}] \begin{bmatrix} \mathbf{R} \\ \mathbf{W} \\ \mathbf{V} \end{bmatrix}.$$

As a function of perturbation  $\mathbf{W}$ , for example, the output transform  $\mathbf{Y}(s)$  obtained from these equations is

$$(14.18) \quad \mathbf{Y}(s) = \mathbf{C}(s\mathbf{I} - (\mathbf{A} - \mathbf{BK}))^{-1}(\mathbf{I} + \mathbf{BK}(s\mathbf{I} - (\mathbf{A} - \mathbf{LC}))^{-1})\mathbf{W}(s).$$

The poles of the transfer function defined by this equation are, in general, the poles of  $\mathbf{A} - \mathbf{BK}$  and of  $\mathbf{A} - \mathbf{LC}$ .

## 1.2 Reduced-order observers

The state-feedback matrices obtained in Chapter 12 and Chapter 13 required no dynamics in the controller, whereas the controller shown in Figure 14.3 and developed in Section 1.1 is a dynamical system of order  $n$  equal to the order of the plant model. However, assuming that  $\mathbf{C}$  has full row rank  $p$  (or the plant model has been modified as described at the beginning of this chapter), then knowledge of  $p$  components of  $\mathbf{X}$  is available directly from  $\mathbf{Y}$ , and only  $n - p$  variables require estimation at any time.

To find a reduced-order observer, first perform a column compression on the rank- $p$  plant matrix  $\mathbf{C}$  using nonsingular matrix  $\mathbf{S} = [\mathbf{S}_1, \mathbf{S}_2]$  such that

$$(14.19) \quad \mathbf{CS} = \mathbf{C}[\mathbf{S}_1, \mathbf{S}_2] = [\mathbf{I}_p, 0].$$

Then the change of basis  $\mathbf{X} = \mathbf{S}\mathbf{X}'$  transforms the plant equations to

$$(14.20a) \quad \frac{d}{dt}\mathbf{X}' = \mathbf{A}'\mathbf{X}' + \mathbf{B}'\mathbf{U} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \mathbf{X}' + \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix} \mathbf{U}$$

$$(14.20b) \quad \mathbf{Y} = \mathbf{C}'\mathbf{X}' = [\mathbf{I}_p, 0] \mathbf{X}',$$

where the matrices have been partitioned conformably; that is,  $\mathbf{A}_{11}$  is  $p \times p$  and  $\mathbf{B}_1$  has  $p$  rows. The output vector  $\mathbf{Y}$  is identical to the first  $p$  entries of the current state vector  $\mathbf{X}'$ .

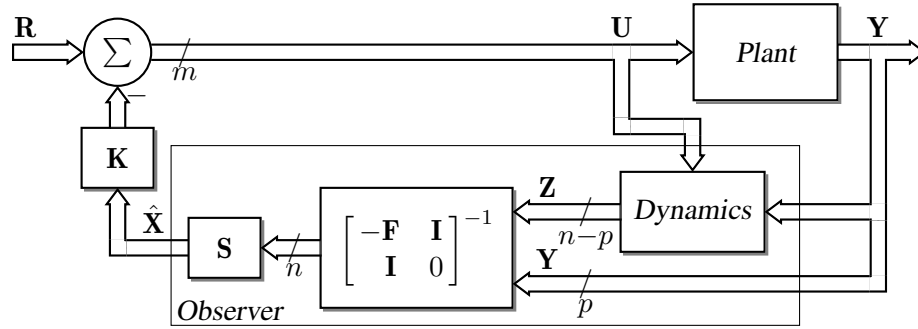
An observer will be designed with output  $\mathbf{Z}$  that converges to the last  $n - p$  components of  $\mathbf{X}'$  plus a linear combination of the first  $p$  components. That is, as  $t \rightarrow \infty$ ,

$$(14.21) \quad \begin{bmatrix} \mathbf{Z} \\ \mathbf{Y} \end{bmatrix} \rightarrow \begin{bmatrix} -\mathbf{F} & \mathbf{I}_{n-p} \\ \mathbf{I}_p & 0 \end{bmatrix} \mathbf{X}' = \begin{bmatrix} -\mathbf{F} & \mathbf{I}_{n-p} \\ \mathbf{I}_p & 0 \end{bmatrix} \mathbf{S}^{-1} \mathbf{X}$$

for some matrix  $\mathbf{F}$  to be found. Then  $\lim_{t \rightarrow \infty} \hat{\mathbf{X}} = \mathbf{X}$ , where

$$(14.22) \quad \hat{\mathbf{X}} = \mathbf{S} \begin{bmatrix} -\mathbf{F} & \mathbf{I}_{n-p} \\ \mathbf{I}_p & 0 \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{Z} \\ \mathbf{Y} \end{bmatrix} = \mathbf{S} \begin{bmatrix} 0 & \mathbf{I}_p \\ \mathbf{I}_{n-p} & \mathbf{F} \end{bmatrix} \begin{bmatrix} \mathbf{Z} \\ \mathbf{Y} \end{bmatrix} = \mathbf{S}_2 \mathbf{Z} + (\mathbf{S}_1 + \mathbf{S}_2 \mathbf{F}) \mathbf{Y}.$$

The resulting closed-loop configuration is shown in Figure 14.7.



**Fig. 14.7** Dynamics of order  $n - p$  are required in a reduced-order observer.

In order to obtain the required dynamics, let the  $(n-p)$ -vector  $\mathbf{Z}$  satisfy the following equation containing matrices to be found:

$$(14.23) \quad \frac{d}{dt} \mathbf{Z} = \mathbf{A}_o \mathbf{Z} + \mathbf{L} \mathbf{Y} + \mathbf{M} \mathbf{U}.$$

For simplicity of notation, write  $\mathbf{T}$  for  $[-\mathbf{F}, \mathbf{I}_{n-p}]$ . Since  $\mathbf{Z}$  is to approach  $\mathbf{T} \mathbf{X}'$ , define the error vector

$$(14.24) \quad \mathbf{E} = \mathbf{T} \mathbf{X}' - \mathbf{Z}$$

and calculate its rate of change as shown:

$$(14.25) \quad \begin{aligned} \frac{d}{dt} \mathbf{E} &= \frac{d}{dt} \mathbf{T} \mathbf{X}' - \frac{d}{dt} \mathbf{Z} \\ &= \mathbf{T} (\mathbf{A}' \mathbf{X}' + \mathbf{B}' \mathbf{U}) - (\mathbf{A}_o \mathbf{Z} + \mathbf{L} \mathbf{C}' \mathbf{X}' + \mathbf{M} \mathbf{U}) \\ &= \mathbf{A}_o (\mathbf{T} \mathbf{X}' - \mathbf{Z}) + (\mathbf{T} \mathbf{A}' - \mathbf{A}_o \mathbf{T} - \mathbf{L} \mathbf{C}') \mathbf{X}' + (\mathbf{T} \mathbf{B}' - \mathbf{M}) \mathbf{U}. \end{aligned}$$

The error will approach zero if  $\mathbf{A}_o$  has eigenvalues placed for asymptotic stability and if the rightmost two terms above are zero as implied by the following equations:

$$(14.26a) \quad \mathbf{T} \mathbf{B}' - \mathbf{M} = 0$$

$$(14.26b) \quad \mathbf{T} \mathbf{A}' - \mathbf{A}_o \mathbf{T} - \mathbf{L} \mathbf{C}' = 0.$$

When written in detail, the second of these two equations is

$$(14.27) \quad [-\mathbf{F}, \mathbf{I}_{n-p}] \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} - \mathbf{A}_o [-\mathbf{F}, \mathbf{I}_{n-p}] - \mathbf{L} [\mathbf{I}_p, 0] = 0,$$

of which the rightmost of the two blocks can be written to define  $\mathbf{A}_o$  as

$$(14.28) \quad \mathbf{A}_o = \mathbf{A}_{22} - \mathbf{F}\mathbf{A}_{12}.$$

The eigenvalues of this matrix can be placed by choosing  $\mathbf{F}$  as for Equation (14.9). Since observability is unaffected by similarity transformation,

$$(14.29) \quad \text{rank} \begin{bmatrix} \mathbf{C} \\ \mathbf{A} - \lambda\mathbf{I} \end{bmatrix} = \text{rank} \begin{bmatrix} \mathbf{C}' \\ \mathbf{A}' - \lambda\mathbf{I} \end{bmatrix} = \text{rank} \begin{bmatrix} \mathbf{I} & 0 \\ \mathbf{A}_{11} - \lambda\mathbf{I} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} - \lambda\mathbf{I} \end{bmatrix}.$$

If the plant is completely observable, then by the BPH test (Example 9 of Chapter 9), these matrices have full column rank for all  $\lambda \in \mathbb{C}$ . The rightmost matrix in (14.29) has full column rank only if the matrix  $\begin{bmatrix} \mathbf{A}_{12} \\ \mathbf{A}_{22} - \lambda\mathbf{I} \end{bmatrix}$  has full column rank. Therefore, complete observability of the plant implies that  $\mathbf{F}$  can be designed to give  $\mathbf{A}_o$  the required eigenvalues.

With  $\mathbf{F}$  and  $\mathbf{A}_o$  now known, the matrix  $\mathbf{L}$  is obtained by writing the left block of Equation (14.27) as

$$(14.30) \quad \mathbf{L} = [-\mathbf{F}, \mathbf{I}] \begin{bmatrix} \mathbf{A}_{11} \\ \mathbf{A}_{21} \end{bmatrix} + \mathbf{A}_o\mathbf{F}.$$

Finally, Equation (14.26a) gives

$$(14.31) \quad \mathbf{M} = [-\mathbf{F}, \mathbf{I}] \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix}.$$

The state-space equations of the observer are given by (14.23) together with (14.22).

### Closed-loop properties

Just as for the full-order observer, the value of  $\mathbf{K}$  does not enter into the equations and the observer will theoretically produce a valid open-loop estimate of  $\mathbf{X}$ . The separation principle continues to hold. To see this, write the state-estimate vector  $\hat{\mathbf{X}}$  as

$$(14.32) \quad \begin{aligned} \hat{\mathbf{X}} &= \mathbf{S} \begin{bmatrix} \mathbf{T} \\ \mathbf{C}' \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{Z} \\ \mathbf{Y} \end{bmatrix} = \mathbf{S} \begin{bmatrix} \mathbf{T} \\ \mathbf{C}' \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{T}\mathbf{X}' - \mathbf{E} \\ \mathbf{C}'\mathbf{X}' \end{bmatrix} = \mathbf{S} \begin{bmatrix} \mathbf{T} \\ \mathbf{C}' \end{bmatrix}^{-1} \left( \begin{bmatrix} \mathbf{T} \\ \mathbf{C}' \end{bmatrix} \mathbf{X}' - \begin{bmatrix} \mathbf{E} \\ 0 \end{bmatrix} \right) \\ &= \mathbf{S} \left( \mathbf{X}' - \begin{bmatrix} 0 & \mathbf{I}_p \\ \mathbf{I}_{n-p} & \mathbf{F} \end{bmatrix} \begin{bmatrix} \mathbf{E} \\ 0 \end{bmatrix} \right) = \mathbf{S}\mathbf{X}' - \mathbf{S} \begin{bmatrix} 0 \\ \mathbf{I}_{n-p} \end{bmatrix} \mathbf{E} = \mathbf{X} - \mathbf{S}_2\mathbf{E}. \end{aligned}$$

Then the plant input is  $\mathbf{U} = \mathbf{R} - \mathbf{K}(\mathbf{X} - \mathbf{S}_2\mathbf{E})$  and the closed-loop equations for the plant with observer are

$$(14.33a) \quad \frac{d}{dt} \begin{bmatrix} \mathbf{X} \\ \mathbf{E} \end{bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{BK} & \mathbf{BKS}_2 \\ 0 & \mathbf{A}_o \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ \mathbf{E} \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix} \mathbf{R}$$

$$(14.33b) \quad \mathbf{Y} = [\mathbf{C}, 0] \begin{bmatrix} \mathbf{X} \\ \mathbf{E} \end{bmatrix}.$$

The poles of this system are the eigenvalues of  $\mathbf{A} - \mathbf{BK}$ , corresponding to the closed-loop state-feedback system, together with the eigenvalues of  $\mathbf{A}_o$ , which are the poles of the reduced-order observer.

In contrast to the full-order observers, the bandwidth in the transfer from  $\mathbf{Y}$  to components of  $\hat{\mathbf{X}}$  is infinite, so high-frequency noise may be a design factor.

**Example 4**  
Reduced-order  
observer

A reduced-order observer for the reduced plant of Example 1 will have order 1 since there are three state variables and two independent outputs. A similarity matrix satisfying Equation (14.19) and the corresponding transformed matrices are given by

$$\mathbf{S} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{A}' = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{B}' = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{C}' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

A matrix  $\mathbf{F}$  that places the poles of  $\mathbf{A}_o$  in Equation (14.28) at  $-12$  as in Example 2 and matrices  $\mathbf{A}_o$ ,  $\mathbf{L}$ , and  $\mathbf{M}$  computed from (14.28), (14.30), and (14.31) respectively are shown:

$$\mathbf{F} = [-6, 6], \quad \mathbf{A}_o = -12, \quad \mathbf{L} = [-66, 79], \quad \mathbf{M} = 6.$$

**Example 5**  
Reduced-order  
observer for the  
inverse  
pendulum

For the inverse pendulum of Example 3 with numerical values given in Example 11 of Chapter 13, a similarity matrix  $\mathbf{S}$  that interchanges the second and third entries of the state vector produces the transformed matrix  $\mathbf{A}'$  shown, where  $\mathbf{A}_{22}$  is  $2 \times 2$ :

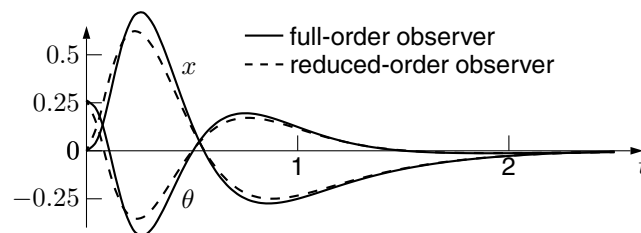
$$\mathbf{A}' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1.96 & 0 & 0 \\ 0 & 11.76 & 0 & 0 \end{bmatrix}$$

The matrix  $\mathbf{F} = \mu \mathbf{I}_2$  assigns a double eigenvalue at  $-\mu$  to  $\mathbf{A}_o = -\mu \mathbf{I}_2$ . Then for  $\mu = 12$ , the observer equations (14.23) and (14.22) are as shown:

$$\frac{d}{dt} \mathbf{Z} = \begin{bmatrix} -12 & 0 \\ 0 & -12 \end{bmatrix} \mathbf{Z} + \begin{bmatrix} -144 & -1.96 \\ 0 & -132.24 \end{bmatrix} \mathbf{Y} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \mathbf{U}$$

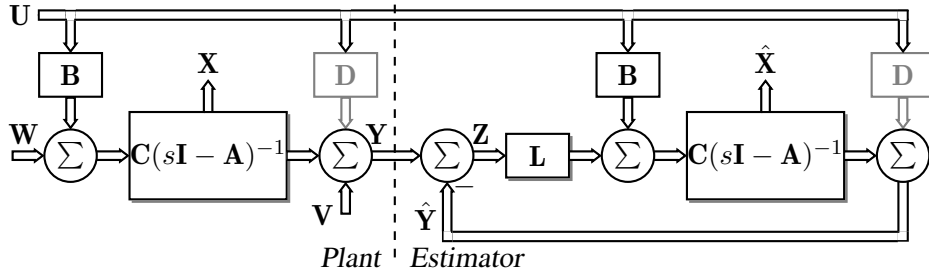
$$\hat{\mathbf{X}} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{Z} + \begin{bmatrix} 1 & 0 \\ 12 & 0 \\ 0 & 1 \\ 0 & 12 \end{bmatrix} \mathbf{Y} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \mathbf{U}.$$

The free closed-loop responses for the full-order observer of Example 3 and the reduced-order observer are displayed in Figure 14.8, using the gain  $\mathbf{K}$  and initial plant state given in Example 11 of Chapter 13. The initial observer state is set to zero in both cases, so these responses correspond to the effect of an impulse in the noise vector  $\mathbf{W}$  in Equations (14.16). For both observers, the responses are somewhat greater than for state feedback with matrix  $\mathbf{K}$ .



**Fig. 14.8** Closed-loop free responses of cart position  $x$ , angle  $\theta$  of the inverse pendulum for a full-order and reduced-order observer. The state-feedback matrix is the same as for Figure 13.3.

Consider an open-loop full-order observer of Section 1.1 without closing the loop on the plant through a state-feedback matrix. Then Figure 14.6 can be redrawn as in Figure 14.9, where plant and observer noise vectors  $\mathbf{W}$  and  $\mathbf{V}$  have been included as in Equations (14.16). Blocks containing  $\mathbf{D}$  matrices are shown in gray in the figure and do not affect the conclusions but these blocks are assumed to be zero in the plant model Equations (14.16). The observer model is given by Equation (14.11). An intuitive explanation of the Kalman filter (estimator) will be given in the following.



**Fig. 14.9** The steady-state Kalman filter is a full-order observer with gain matrix  $\mathbf{L}$  computed optimally according to the intensity of the noise vectors  $\mathbf{W}$  and  $\mathbf{V}$ . The input  $\mathbf{U}$  has no effect on the error  $\mathbf{Z}$  in the output estimate. The  $\mathbf{D}$  matrices shown in gray can be included without affecting the calculation of  $\mathbf{L}$ .

Figure 14.9 makes certain conclusions evident. First, assume that  $\mathbf{L} = 0$  and note that the transfer from  $\mathbf{U}$  to  $\mathbf{Y}$  is  $\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$  (when  $\mathbf{D}$  is included), which is also the transfer from  $\mathbf{U}$  to  $\hat{\mathbf{Y}}$ . Since  $\mathbf{Y}$  and  $\hat{\mathbf{Y}}$  are subtracted,  $\mathbf{Z}$  is independent of  $\mathbf{U}$ , and changing  $\mathbf{L}$  to nonzero values will not affect this independence. This intuitive result can be confirmed by writing the equations for  $\mathbf{Z}$  and solving as a function of  $\mathbf{U}$ . Consequently, the effect of  $\mathbf{U}$  on  $\mathbf{X}$  is identical to its effect on  $\hat{\mathbf{X}}$ , and the error  $\mathbf{X} - \hat{\mathbf{X}}$  is independent of  $\mathbf{U}$ . This error is therefore entirely a function of the noise vectors  $\mathbf{W}$  and  $\mathbf{V}$ , assuming as in the following that the initial states of the plant and estimator are identical or that the free plant response is asymptotic to zero and initial transients can be ignored.

Suppose that the measurement noise  $\mathbf{V}$  is identically zero. Increasing the gain  $\mathbf{L}$  to infinity will force  $\hat{\mathbf{X}}$  to follow  $\mathbf{X}$  exactly as input  $\mathbf{W}$  causes  $\mathbf{X}$  to vary. However, if  $\mathbf{V}$  is nonzero then the measured  $\mathbf{Y}$  contains noise and has to be averaged, requiring long estimator time constants and small values in  $\mathbf{L}$ . In the case that  $\mathbf{W}$  is zero, then the plant is driven only by the input  $\mathbf{U}$  and the estimator gain  $\mathbf{L}$  should be zero to ignore the plant output  $\mathbf{Y}$  entirely.

When neither  $\mathbf{W}$  nor  $\mathbf{V}$  is zero then the intensity of these two noise vectors must be weighed to obtain an optimal estimator gain  $\mathbf{L}$ .

### LQG design

The process known as LQG design makes specific assumptions about  $\mathbf{W}$  and  $\mathbf{V}$  and results in an optimal estimator gain matrix  $\mathbf{L}$ . Assume that  $\mathbf{W}$  and  $\mathbf{V}$  are independent zero-mean random noise vectors with Gaussian distribution and covariance matrices (measures of noise intensity)  $\hat{\mathbf{Q}}$  and  $\hat{\mathbf{R}}$  respectively. To obtain  $\mathbf{L}$  in steady state, solve the

Riccati equation

$$(14.34) \quad \mathbf{A}\tilde{\mathbf{P}} + \tilde{\mathbf{P}}\mathbf{A}^T - \tilde{\mathbf{P}}\mathbf{C}^T\tilde{\mathbf{R}}^{-1}\mathbf{C}\tilde{\mathbf{P}} + \tilde{\mathbf{Q}} = 0$$

(the dual of the ARE for the control problem), for symmetric, positive-definite  $\tilde{\mathbf{P}}$ , and calculate the steady-state gain matrix  $\mathbf{L}$  as

$$(14.35) \quad \mathbf{L} = \tilde{\mathbf{P}}\mathbf{C}^T\tilde{\mathbf{R}}^{-1}.$$

The Kalman filter is most often implemented in discrete time to provide open-loop information about dynamic processes such as navigation or the optimal “fusion” of measurements, often arriving at different times, from several sensors.

### 3 Further study

The introduction of the state-space model resulted in intense research related to observers that lasted for more than two decades. A classic publication is reference [42]. A range of observer structures and design criteria are often found in design textbooks such as [37] or [62].

Optimal state estimation in the form of the Kalman filter dates from the introduction of the state-space model, as described in [33]. The results are summarized in compendia such as [59].

### 4 Problems

- 1 Find a reduced model with zero  $\mathbf{D}$  matrix and full-rank  $\mathbf{B}$  and  $\mathbf{C}$  matrices for the system described by the following data:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Are the matrices in the reduced system unique? Are they unique if the matrices  $\mathbf{B}\mathbf{Q}_1$  and  $\mathbf{P}_1\mathbf{C}$  are constrained to have Euclidean norm of 1?

- 
- 2 Design a full-order observer with poles at  $-20$  to estimate the state of the system with matrices given in Problem 8 of Chapter 12.

- 
- 3 Design a reduced-order observer with poles at  $-20$  to estimate the state of the system with matrices given in Problem 8 of Chapter 12. Write the observer equations in state-space form.

- 
- 4 Design a reduced-order observer with poles at  $-15$  to estimate the state of the system with matrices given in Problem 10 of Chapter 12.

- 
- 5 Design a full-order observer with poles at  $-15$  to estimate the state of the system with matrices given in Problem 10 of Chapter 12.
- 
- 6 Write the state-space equations for Figure 14.9 with  $\mathbf{D} \neq 0$  using the entries of  $\mathbf{X}$  and  $\hat{\mathbf{X}}$  as state variables. Compare the resulting equations to Equations (14.11) and (14.16). How does the value of the vector  $\mathbf{Z}$  in the diagram differ for the two cases of  $\mathbf{D} \neq 0$  and  $\mathbf{D} = 0$ ?
- 
- 7 Suppose that an optimal observer weighting matrix  $\mathbf{L}$  has been designed for Figure 14.9 and that the filter poles are near  $-15$  for the system of Problem 5. What are the weights associated with the noise vector  $\mathbf{W}$  relative to the noise vector  $\mathbf{V}$  assuming a diagonal  $\tilde{\mathbf{Q}}$ ? In other words, if  $\tilde{\mathbf{R}}$  is 1, what should  $\tilde{\mathbf{Q}}$  be to obtain observer poles at  $-15$ ? What does the relative magnitudes of these weights imply about the relative mean sizes of the noise vectors?



# 15

## High gain feedback

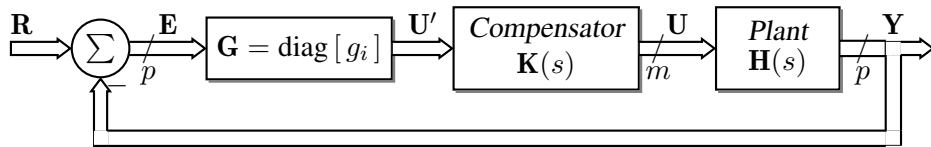
This chapter extends elementary classical results to multi-input, multi-output systems and describes methods to design controllers that have very simple structure when appropriate assumptions are met. Figure 15.1 shows a prototype of a multivariable control loop containing plant transfer matrix  $\mathbf{H}(s)$ , forward-path compensator  $\mathbf{K}(s)$ , and a diagonal gain block  $\mathbf{G} = \text{diag}[g_i]$ . This configuration is the multivariable generalization of the classical single-input, single-output control loop. Increasing the loop-gain parameters  $g_i$  reduces the error  $\mathbf{E}$ , thus improving performance, provided that closed-loop stability and robustness to perturbations are adequately maintained.

If the feedback path in Figure 15.1 is cut, the gain of the resulting open loop from one side to the cut to the other is  $\mathbf{HKG}$ . This quantity is called the *return ratio matrix* (or simply the *return ratio*) of the system and, in the diagram, it is also the gain of the forward path. The closed-loop transfer matrix from  $\mathbf{R}$  to  $\mathbf{Y}$  can be obtained by writing an implicit equation for the output  $\mathbf{Y}$  as

$$(15.1) \quad \mathbf{Y} = \mathbf{HKG}(\mathbf{R} - \mathbf{Y}),$$

and then solving for  $\mathbf{Y}$  to get

$$(15.2) \quad \mathbf{Y} = (\mathbf{I} + \mathbf{HKG})^{-1} \mathbf{HKG} \mathbf{R}.$$



**Fig. 15.1** The principle of high-gain feedback is to ensure that if the loop gains  $g_i$  become large and the output  $\mathbf{Y}$  remains finite, the error  $\mathbf{E}$  becomes small. The compensator must be designed to adjust plant structure and frequency response if necessary.

A basic observation about the migration of closed-loop poles of single-input, single-output systems under high-gain feedback with gain parameter  $g$  is that as  $g \rightarrow \infty$ , the closed-loop poles approach the open-loop zeros or infinity. This behavior holds

for multi-input, multi-output systems and explains why non-minimal phase systems (continuous-time systems with zeros in the right-half plane or discrete-time systems with zeros outside the unit circle) can be difficult to control. Transient performance specifications requiring high feedback gain may not be achievable, because the closed-loop system is guaranteed to become unstable as poles approach the zeros.

An informal development of the above result will be given. For simplicity, assume that the number  $m$  of plant inputs equals the number  $p$  of outputs and that the gains  $g_i$  in Figure 15.1 have identical value  $g$ . Let the plant transfer matrix in the left-factored form described in Section 1 of Chapter 10 be

$$(15.3) \quad \mathbf{H}(s) = \mathbf{A}_H(s)^{-1} \mathbf{B}_H(s),$$

where  $\mathbf{A}_H(s)$  and  $\mathbf{B}_H(s)$  are polynomial matrices such that the transmission poles are the values of  $s$  for which  $\mathbf{A}_H(s)$  has less than full rank, and the transmission zeros are the values of  $s$  for which  $\mathbf{B}_H(s)$  has less than full column rank.

Similarly, let the compensator be described by

$$(15.4) \quad \mathbf{K}(s) = \mathbf{A}_K(s)^{-1} \mathbf{B}_K(s).$$

Then the closed-loop system obeys the equations

$$(15.5) \quad \begin{bmatrix} -\mathbf{A}_H & \mathbf{B}_H & 0 \\ -g\mathbf{B}_K & -\mathbf{A}_K & g\mathbf{B}_K \end{bmatrix} \begin{bmatrix} \mathbf{Y} \\ \mathbf{U} \\ \mathbf{R} \end{bmatrix} = 0,$$

and its system poles are the values of  $s$  for which

$$(15.6) \quad \det \begin{bmatrix} -\mathbf{A}_H & \mathbf{B}_H \\ -g\mathbf{B}_K & -\mathbf{A}_K \end{bmatrix} = 0.$$

### Finite closed-loop poles

For  $g = 0$ , this determinant becomes  $\det \mathbf{A}_H \det \mathbf{A}_K$  and the system poles are the poles of the open-loop plant together with those of the compensator. Assume that  $\text{rank } \mathbf{B}_H = m$  except at a finite number of zeros (otherwise there would be less than  $m$  effective inputs) and perform elementary operations on the above matrix to obtain the following:

$$(15.7) \quad \begin{bmatrix} \mathbf{I} & -\mathbf{A}_H(g\mathbf{B}_K)^{-1} \\ 0 & \mathbf{I} \end{bmatrix} \begin{bmatrix} -\mathbf{A}_H & \mathbf{B}_H \\ -g\mathbf{B}_K & -\mathbf{A}_K \end{bmatrix} \begin{bmatrix} \mathbf{I} & -(g\mathbf{B}_K)^{-1} \mathbf{A}_K \\ 0 & \mathbf{I} \end{bmatrix} \\ = \begin{bmatrix} 0 & \mathbf{B}_H + \mathbf{A}_H(g\mathbf{B}_K)^{-1} \mathbf{A}_K \\ -g\mathbf{B}_K & 0 \end{bmatrix}.$$

As shown in Example 6 of Chapter 10, such operations may change the number of poles or zeros at  $s = \infty$ . For finite values of  $s$ , however, this matrix loses rank as  $g \rightarrow \infty$  at the values for which either  $\mathbf{B}_K$  or  $\mathbf{B}_H$  lose rank, since the upper-right block approaches  $\mathbf{B}_H$  as  $g$  becomes large. Thus, as  $g \rightarrow \infty$ , the finite closed-loop poles approach the zeros of the plant together with the zeros of the compensator.

### Asymptotes

The asymptotes of the infinite closed-loop poles depend on the structure of the plant and compensator parameters in (15.6) and are best described by means of examples. However, the use of arbitrarily large gain normally implies that the asymptotes should be designed to correspond to the negative real axis, since vertical asymptotes can result in

small damping ratio and other patterns can result in closed-loop instability. The desired asymptotes can be obtained as follows.

Choose  $\mathbf{A}_K$  and  $\mathbf{B}_K$  so that the open-loop gain  $\mathbf{H}\mathbf{K}$  approaches  $s^{-1}\mathbf{I}$  for large  $s$ . This requires the plant to have a right inverse at high frequency. Then the closed-loop system shown in Figure 15.1 is described approximately at high frequency by the equations

$$(15.8) \quad \begin{bmatrix} -s\mathbf{I} & \mathbf{I} & 0 \\ -g\mathbf{I} & -\mathbf{I} & g\mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{Y} \\ \mathbf{U}' \\ \mathbf{R} \end{bmatrix} = 0,$$

where  $\mathbf{U}'$  is the input to the compensator. The closed-loop system poles are the values of  $s$  for which nonzero solutions exist with  $\mathbf{R} \equiv 0$ ; that is, the values of  $s$  for which

$$(15.9) \quad \det \begin{bmatrix} -s\mathbf{I} & \mathbf{I} \\ -g\mathbf{I} & -\mathbf{I} \end{bmatrix} = 0.$$

This condition is true for  $s = -g$ , which is the desired asymptote for  $s$  as  $g \rightarrow \infty$ .

**Finite gain** In practice, the loop gain parameter  $g$  cannot be increased without bound but must be limited, for example, to produce a specified closed-loop system bandwidth. Nevertheless, the design technique is simple and useful: establish desired finite zeros and negative asymptotes, and turn up the gain “sufficiently.”

**First-order cutoff** Many physical plants are low-pass, which corresponds to  $\mathbf{D} = 0$  in a state-space model. Assume further that a given system has first-order cutoff; that is, that the magnitude of the entries of the transfer matrix change by  $-20$  dB per decade at high frequency. Expand the transfer matrix  $\mathbf{H}(s)$  in terms of its Markov matrices as

$$(15.10) \quad \begin{aligned} \mathbf{H}(s) &= \mathbf{H}_0 + s^{-1}\mathbf{H}_1 + s^{-2}\mathbf{H}_2 + \dots \\ &= \mathbf{D} + s^{-1}\mathbf{CB} + s^{-2}\mathbf{CAB} + \dots \end{aligned}$$

By assumption,  $\mathbf{H}(s)$  is approximately equal to the term  $s^{-1}\mathbf{CB}$  at high frequency, which implies that  $\mathbf{D} = 0$  and  $\mathbf{CB}$  is nonsingular. The required compensator gain is

$$(15.11) \quad \mathbf{K} = (\mathbf{CB})^{-1}.$$

Alternatively, let the plant left denominator  $\mathbf{A}_H$  be written as

$$(15.12) \quad \mathbf{A}_H(s) = \text{diag} [s^{n_i}] \mathbf{A}_0 + \text{diag} [s^{n_i-1}] \mathbf{A}_1 + \text{terms of lower degree in } s,$$

where  $\mathbf{A}_0$  is assumed to be nonsingular. Because the plant is low-pass, each row  $i$  of  $\mathbf{B}_H(s)$  contains entries of degree  $n_i - 1$  at most and this matrix can be written as

$$(15.13) \quad \mathbf{B}_H(s) = \text{diag} [s^{n_i-1}] \mathbf{B}_1 + \text{diag} [s^{n_i-2}] \mathbf{B}_2 + \text{terms of lower degree in } s.$$

Assume that  $\mathbf{B}_1$  is nonsingular. Then at high frequency,  $\mathbf{A}_H(s)$  and  $\mathbf{B}_H(s)$  can be approximated by their high-degree terms and the plant transfer matrix approaches

$$(15.14) \quad (\text{diag} [s^{n_i}] \mathbf{A}_0)^{-1} (\text{diag} [s^{n_i-1}] \mathbf{B}_1) = s^{-1} \mathbf{A}_0^{-1} \mathbf{B}_1,$$

which is nonsingular and changing in magnitude by  $-20$  dB per decade. Choosing the compensator to be

$$(15.15) \quad \mathbf{K} = (\mathbf{A}_0^{-1} \mathbf{B}_1)^{-1} = \mathbf{B}_1^{-1} \mathbf{A}_0$$

**Simple controller design**

produces the required high-gain asymptotes.

Suppose the plant has no finite zeros as is often the case, or they are in the desired region of the complex plane. If the plant exhibits first-order low-pass cutoff at high frequencies and is sufficiently well known to allow high gain feedback without danger of instability, then formula (15.11) or (15.15) provides a simple design method for  $\mathbf{K}$ , with which the loop gain  $g$  can be set to give desired performance.

**Example 1**  
Numerical example

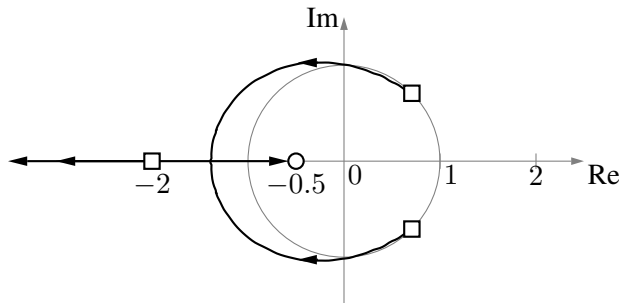
Consider the system described by matrices

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 2\sqrt{2}-1 & \sqrt{2}-2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}, \quad \mathbf{D} = 0.$$

This plant has poles at  $1/\sqrt{2} \pm j/\sqrt{2}$  and  $-2$ , one zero at  $-0.5$ , and exhibits first-order cutoff (that is,  $\mathbf{D} = 0$  and  $\det(\mathbf{CB}) \neq 0$ ) so, with high-gain feedback  $g(\mathbf{CB})^{-1}$ , the closed-loop poles will approach  $-0.5$  or infinity. The root loci are as shown in Figure 15.2 for the compensator matrix

$$\mathbf{K} = (\mathbf{CB})^{-1} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}^{-1} = \frac{1}{4} \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}.$$

Figure 15.2 shows that two poles migrate to  $-\infty$  and one moves to  $-0.5$  as  $g \rightarrow \infty$ .



**Fig. 15.2** The closed-loop root loci start at the open-loop pole locations (marked by squares) for  $g = 0$ . As  $g \rightarrow \infty$ , one pole approaches  $-0.5$  (marked by a small circle) and the others move left on the negative real axis.

**2****Integral control**

The steady-state error  $\mathbf{E}$  in Figure 15.1 will be zero for step reference signals or constant disturbances provided the gain in each of the  $m$  loops is infinity at zero frequency. That is, as  $s \rightarrow 0$ , the loop gain  $\mathbf{H}(s)\mathbf{K}(s)$  should have the form

$$(15.16) \quad \mathbf{H}(s)\mathbf{K}(s) \rightarrow \mathbf{M} \text{diag} [s^{-k_i}],$$

where  $\mathbf{M}$  is nonsingular and  $k_i > 0$ ,  $i = 1, \dots, m$ . The nonnegative integers  $k_i$  are the generalization of the system *type* parameter of single input, single output plants. If one or

more  $k_i = 0$  then the controller must introduce the required integral action. Introducing the proportional-plus-integral (PI) gain term

$$(15.17) \quad g_i = 1 + \frac{\rho_i}{s} = \frac{s + \rho_i}{s}$$

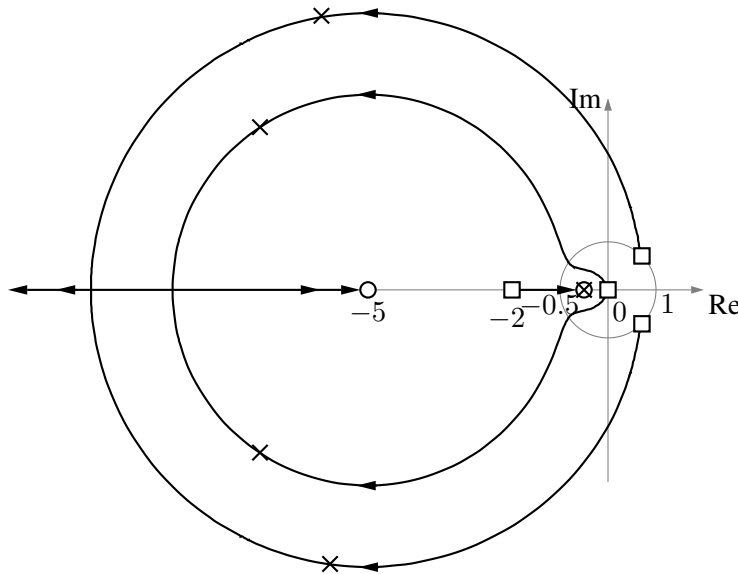
for each loop in which  $k_i = 0$  will provide the desired zero-frequency gain. If an integrator is introduced into all  $m$  loops, then the system order is increased by  $m$ , which is the additional number of open-loop poles at the origin. In addition,  $m$  open-loop zeros will have been added along the real axis at  $-\rho_i$ ,  $i = 1, \dots, m$ . Assume that the plant state-space matrices satisfy  $\mathbf{D} = 0$  and  $\det(\mathbf{CB}) \neq 0$  as previously. Then the PI compensator has transfer matrix

$$(15.18) \quad \mathbf{K}(s) = s^{-1} \text{diag}[s + \rho_i] (\mathbf{CB})^{-1}.$$

**Example 2**  
PI control for  
Example 1

Introducing integral action and two controller zeros at  $-5$  to the controller of Example 1 results in the root loci shown in Figure 15.3 for the compensator

$$\mathbf{K}(s) = \frac{s + 5}{s} (\mathbf{CB})^{-1} = \frac{s + 5}{4s} \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}.$$



**Fig. 15.3** The PI terms in the compensator introduce two poles at 0 and two zeros at  $-5$ . For large values of gain one locus approaches the zero at  $-0.5$ , two approach the zeros at  $-5$ , and the other two move left along the real axis. The x marks show the poles for  $g = 13$ , which might be chosen to give acceptable system transients in the absence of other specifications, although the slowest mode is near  $-0.5$ .

## 3

## Derivative action

In cases where the plant has a cutoff faster than first-order in one or more loops, then derivative action, or PID (proportional, integral, derivative) control may be appropriate. For example, suppose that in the plant representation (15.10), the matrices  $\mathbf{H}_0$  and  $\mathbf{H}_1$  are zero and  $\mathbf{H}_2$  is nonsingular, giving a second-order high-frequency cutoff. Then an expression for the plant inverse at high frequency is given by

$$(15.19) \quad \mathbf{H}(s)^{-1} = s^2 \mathbf{H}_2^{-1} - s \mathbf{H}_2^{-1} \mathbf{H}_3 \mathbf{H}_2^{-1} - (\mathbf{H}_2^{-1} \mathbf{H}_4 \mathbf{H}_2^{-1} - \mathbf{H}_2^{-1} \mathbf{H}_3 \mathbf{H}_2^{-1} \mathbf{H}_3 \mathbf{H}_2^{-1}) + \dots,$$

which can be verified by multiplying both sides of the above formula by  $\mathbf{H}(s) = s^{-2} \mathbf{H}_2 + s^{-3} \mathbf{H}_3 + s^{-4} \mathbf{H}_4 + \dots$  and comparing terms. For left-directed asymptotes, the product  $\mathbf{H}\mathbf{K}$  should be  $s^{-1} \mathbf{I}$  at high frequency as before, and the controller is

$$(15.20) \quad \begin{aligned} \mathbf{K}(s) &= s^{-1} \mathbf{H}(s)^{-1} \\ &= s \mathbf{H}_2^{-1} - \mathbf{H}_2^{-1} \mathbf{H}_3 \mathbf{H}_2^{-1} - s^{-1} (\mathbf{H}_2^{-1} \mathbf{H}_4 \mathbf{H}_2^{-1} - \mathbf{H}_2^{-1} \mathbf{H}_3 \mathbf{H}_2^{-1} \mathbf{H}_3 \mathbf{H}_2^{-1}) + \dots \end{aligned}$$

Retaining only the leftmost three terms shown gives a PID controller. The fact that three terms of the plant inverse are included in the controller means that  $\mathbf{H}\mathbf{K} \simeq s^{-1} \mathbf{I}$  over a range of high frequencies. However, the proportional and integral gain matrices may have to be adjusted if the zeros of  $\mathbf{K}(s)$  are not in appropriate places and very high gain is required. Considerable design freedom is available, as any nonsingular integral coefficient matrix will produce a decoupled system at steady state, for example.

**Controller  
loop  
structures**

It may be that different controller structures are required in different loops. Assume that the plant denominator  $\mathbf{A}_H(s)$  is given by (15.12) with  $\mathbf{A}_0$  a nonsingular upper-right echelon matrix, and that the numerator matrix  $\mathbf{B}_H(s)$  (15.13) is of the form

$$(15.21) \quad \mathbf{B}_H(s) = \text{diag} [s^{n_i - d_i}] (\mathbf{B}_1 + s^{-1} \mathbf{B}_2 + \text{terms of lower degree in } s),$$

where each  $d_i$  shown is a non-negative integer called the *relative degree* of row  $i$ ,  $i = 1, \dots, p$ . Assume that the plant has an inverse at high frequency so that  $\mathbf{B}_1$  is nonsingular. The details of the computation of these matrices are beyond the scope of the discussion and are omitted since the required structures are often available by inspection. For example, if all  $d_i = 1$  corresponding to first-order cutoff, then choosing  $\mathbf{A}_K = \mathbf{B}_1$  and  $\mathbf{B}_K = \mathbf{A}_0$  gives (15.15) as before.

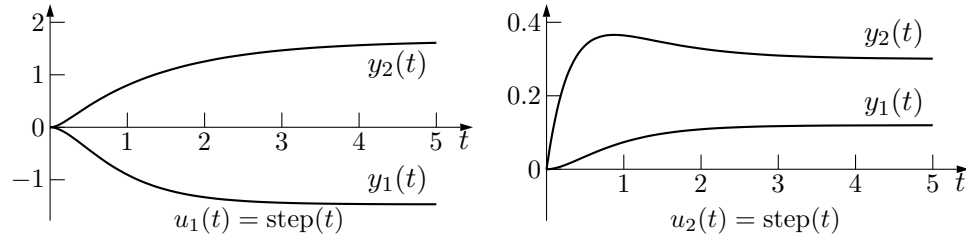
Inspection of (15.6) shows that if the rows of  $[\mathbf{B}_K(s), \mathbf{A}_K(s)]$  are row-wise proportional to those of  $[s^{-1} \mathbf{A}_H(s), \mathbf{B}_H(s)]$  and  $\mathbf{A}_K(s)$  is nonsingular as  $s \rightarrow \infty$ , then  $\mathbf{H}\mathbf{K} \rightarrow s^{-1} \mathbf{I}$  as required. Then choosing  $\mathbf{A}_K$  equal to  $\mathbf{B}_1$  and the rows of  $\mathbf{B}_K(s)$  of desired structure and proportional to the rows of  $\mathbf{A}_H(s)$  at high frequency gives the desired asymptotes.

**Example 3**  
PID action in  
one loop

The transfer matrix is shown below for a linearized three-phase generator driven by a motor [28], with rate of change of power angle  $y_1$ , output voltage deviation  $y_2$ , input torque deviation  $u_1$ , field voltage deviation  $u_2$ , and load perturbation disturbance  $u_3$ :

$$\begin{bmatrix} \frac{-7.018}{\phi(s)} & \frac{0.5729}{\phi(s)} & \frac{-6.445}{\phi(s)} \\ \frac{9.763s^2 + 41.61s + 34.3}{\gamma(s)} & \frac{1.497s^3 + 13.72s^2 + 17.78s + 6.237}{\gamma(s)} & \frac{1.583}{s + 0.8333} \end{bmatrix}.$$

The two denominator pole functions shown are  $\phi(s) = s^2 + 4.678s + 4.774$  and  $\gamma(s) = s^4 + 12.72s^3 + 45.49s^2 + 54.95s + 20.79$ . The open-loop step responses in Figure 15.4 show the non-diagonal nature of the transfer matrix and the steady-state step error. The control objectives are zero steady-state disturbance response, fast decoupled response to control inputs, and closed-loop pass-bands from  $r_1$  to  $y_1$  and  $r_2$  to  $y_2$  of approximately 40 rad/s, where  $r_1$  and  $r_2$  are reference signals for  $y_1$  and  $y_2$ , respectively.



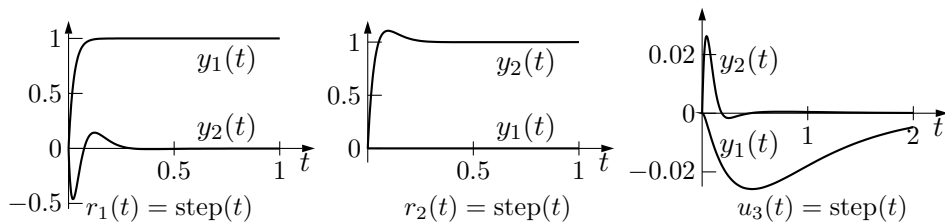
**Fig. 15.4** The open-loop step responses for the generator show an interaction between  $u_1$  and  $y_2$ , and between  $u_2$  and  $y_1$ .

The first row of the transfer matrix has relative degree 2 and therefore requires derivative action to produce left-directed high-gain asymptotes, whereas the second row has relative degree 1 and does not require derivative action. Integrators are required in both loops for zero steady-state position error and disturbance response.

Therefore, choose a PID controller to generate  $u_1$  and a PI controller for  $u_2$  to produce closed-loop step responses such as in Figure 15.5. The controller matrices are of the form

$$(15.22) \quad \mathbf{B}_K(s) = \begin{bmatrix} s + \alpha + \beta/s & 0 \\ 0 & 1 + \nu/s \end{bmatrix}, \quad \mathbf{A}_K(s) = \mathbf{B}_1 = \begin{bmatrix} -7.018 & 0.5729 \\ 0 & 1.497 \end{bmatrix},$$

where  $\mathbf{B}_1$  is the high-degree coefficient matrix, shown in (15.21), of the numerator of the plant without disturbance input. For a controller as close as possible to a system inverse at high frequencies, candidate controller parameters can be chosen from  $\phi(s)$  by inspection as  $\alpha = 4.678$ ,  $\beta = 4.774$ , and from  $\gamma(s)$  as  $\nu = 12.72$ . These values are suitable because they produce controller zeros in the left half of the complex plane. The loop gains must now be adjusted to meet the closed-loop bandwidth criterion; for this example, increasing both loop gains by a factor of approximately 40 produces the



**Fig. 15.5** The closed-loop compensated system is significantly decoupled. The figure shows faster step responses than for the open-loop plant, zero steady-state error, and zero steady-state disturbance response. The compensator is a plant inverse only at high frequencies, so there is a small transient interaction.

required bandwidth values and produces responses to step reference signals and disturbance shown in Figure 15.5.

**4**

#### **Further study**

References [4] and [24] provide excellent introductions to feedback control in general.

**5**

#### **Problems**



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# Appendix

The following sections contain solutions of odd-numbered problems.

## Chapter 11

## Solutions

1 To calculate  $\Phi(t, 0)$  assume initial conditions  $\xi_1 = \mathbf{e}_1$  and  $\xi_2 = \mathbf{e}_2$ .

(a) Choosing  $\xi_1 = \mathbf{e}_1$  results in  $\psi_1 = \mathbf{e}_1$ . For  $\xi_2 = \mathbf{e}_2$ , the solution is  $\psi_2 = \begin{bmatrix} t^2/2 \\ 1 \end{bmatrix}$ .

Therefore  $\Phi(t, 0) = \begin{bmatrix} 1 & t^2/2 \\ 0 & 1 \end{bmatrix}$ .

(b) Choosing  $\xi_1 = \mathbf{e}_1$  results in  $\psi_1 = \begin{bmatrix} e^{t^2/2} \\ 0 \end{bmatrix}$ . For  $\xi_2 = \mathbf{e}_2$ , first solve  $\dot{x}_2 = t x_2$  to get  $x_2(t) = e^{t^2/2}$ . Then

$$\begin{aligned} x_1(t) &= \Phi(t, 0)x_1(0) + \int_0^t \Phi(t, \tau) x_2(\tau) d\tau \\ &= \int_0^t e^{t^2/2} e^{-\tau^2/2} e^{\tau^2/2} d\tau = t e^{t^2/2}, \end{aligned}$$

where  $\Phi(t, \tau)$  is the state-transition matrix for  $x_1(t)$ . Thus

$$\Phi(t, 0) = \begin{bmatrix} e^{t^2/2} & e^{t^2/2} + t e^{t^2/2} \\ 0 & e^{t^2/2} \end{bmatrix}.$$

2 To calculate  $\Phi(t, 0)$  :

(a) The solution of  $\dot{x} = a(t)x$  is  $x(t) = e^{\int_{t_0}^t a(\tau) d\tau} x(t_0)$ , so  $\Phi(t, t_0) = e^{\int_{t_0}^t a(\tau) d\tau} = e^{\int_{t_0}^t \tau d\tau} = e^{(t^2 - t_0^2)/2}$ .

(b) Apply Equation (11.16) with  $\mathbf{A}_0 = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$ ,  $\mathbf{A}_1 = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$  to get

$$\Phi(t, 0) = e^{t\mathbf{A}_0} = \begin{bmatrix} 1 & 0 \\ 0 & e^{-t} \end{bmatrix}$$

for  $0 \leq t < 1$ , and

$$\Phi(t, 0) = e^{(t-1)\mathbf{A}_1} e^{1\mathbf{A}_0} = \begin{bmatrix} 1 & 0 \\ -(t-1) & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^{-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1-t & e^{-1} \end{bmatrix}$$

for  $t \geq 1$ .

- 3** The given  $\mathbf{A}(t)$  has the property  $\mathbf{A}(t)\mathbf{A}(\tau) = \mathbf{A}(\tau)\mathbf{A}(t)$ , so the state-transition matrix is

$$\Phi(t, 0) = e^{\int_0^t \begin{bmatrix} 0 & \tau \\ 0 & 0 \end{bmatrix} d\tau} = e^{\begin{bmatrix} 0 & t^2/2 \\ 0 & 0 \end{bmatrix}} = \begin{bmatrix} 1 & t^2/2 \\ 0 & 1 \end{bmatrix}.$$

Solving (a) for  $\mathbf{X}(t)$  gives

$$\begin{aligned} \mathbf{X}(t) &= \begin{bmatrix} 1 & t^2/2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \int_0^t \begin{bmatrix} 1 & t^2/2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -\tau^2/2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ \tau+1 \end{bmatrix} \text{step}(\tau) d\tau \\ &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & t^2/2 \\ 0 & 1 \end{bmatrix} \int_0^t \begin{bmatrix} -\tau^2(\tau+1)/2 \\ \tau+1 \end{bmatrix} d\tau \\ &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & t^2/2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{2}(\frac{t^4}{4} + \frac{t^3}{3}) \\ \frac{t^2}{2} + t \end{bmatrix} = \begin{bmatrix} 1 + \frac{t^4}{8} + \frac{t^3}{3} \\ \frac{t^2}{2} + t \end{bmatrix}. \end{aligned}$$

Now (b), substituting this in  $\mathbf{Y}(t) = \mathbf{C}(t)\mathbf{X}(t) + \mathbf{D}(t)\mathbf{U}(t)$  gives

$$\mathbf{Y}(t) = [1, -t^2/4] \begin{bmatrix} 1 + \frac{t^4}{8} + \frac{t^3}{3} \\ \frac{t^2}{2} + t \end{bmatrix} + 0 = 1 + \frac{t^3}{12}.$$

- 4** The matrix  $\mathbf{A}(t)$  has the property  $\mathbf{A}(t)\mathbf{A}(\tau) = \mathbf{A}(\tau)\mathbf{A}(t)$ , so the state-transition matrix is

$$\Phi(t, 0) = e^{\int_0^t \begin{bmatrix} 0 & \tau \\ 0 & 0 \end{bmatrix} d\tau} = e^{\begin{bmatrix} 0 & t^2/2 \\ 0 & 0 \end{bmatrix}} = \begin{bmatrix} 1 & t^2/2 \\ 0 & 1 \end{bmatrix}.$$

Solving (a) for  $\mathbf{X}(t)$  gives

$$\begin{aligned} \mathbf{X}(t) &= \begin{bmatrix} 1 & t^2/2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \int_0^t \begin{bmatrix} 1 & t^2/2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -\tau^2/2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tau \\ 0 \end{bmatrix} \text{step}(\tau) d\tau \\ &= \begin{bmatrix} -t^2/2 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 & t^2/2 \\ 0 & 1 \end{bmatrix} \int_0^t \begin{bmatrix} \tau \\ 0 \end{bmatrix} d\tau = \begin{bmatrix} -t^2/2 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 & t^2/2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} t^2/2 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ -1 \end{bmatrix}. \end{aligned}$$

For (b), substituting this in  $\mathbf{Y}(t) = \mathbf{C}(t)\mathbf{X}(t) + \mathbf{D}(t)\mathbf{U}(t)$  gives

$$\mathbf{Y}(t) = [t, 0] \begin{bmatrix} 0 \\ -1 \end{bmatrix} + 0 = 0.$$

- 5** To give  $\mathbf{A}'(t) = 0$  the transformation is  $\mathbf{S}(t) = \Psi(t)e^{-t0} = \Psi(t)\mathbf{I}$ , where  $\Psi(t)$  is a fundamental matrix. The state-transition matrix, which is a fundamental matrix, was found in the solution to Problem 1, so let  $\mathbf{S}(t) = \begin{bmatrix} 1 & t^2/2 \\ 0 & 1 \end{bmatrix}$ . This will produce the

$$\text{matrices } \mathbf{A}'(t) = \mathbf{S}^{-1}(\mathbf{A}\mathbf{S} - \dot{\mathbf{S}}) = 0,$$

$$\mathbf{B}'(t) = \mathbf{S}^{-1}\mathbf{B} = \begin{bmatrix} 1 & -t^2/2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ t+1 \end{bmatrix} = \begin{bmatrix} -t^2(t+1)/2 \\ t+1 \end{bmatrix},$$

$$\mathbf{C}'(t) = \mathbf{C}\mathbf{S} = [1, -t^2/4] \begin{bmatrix} 1 & t^2/2 \\ 0 & 1 \end{bmatrix} = [1, t^2/4],$$

and  $\mathbf{D}(t)$  is unchanged.

- 6** To give  $\mathbf{A}'(t) = 0$  the transformation is  $\mathbf{S}(t) = \Psi(t)e^{-t^0} = \Psi(t)\mathbf{I}$ , where  $\Psi(t)$  is a fundamental matrix. The state-transition matrix, which is a fundamental matrix, was found in the solution to Problem 1, so let

$$\mathbf{S}(t) = \begin{bmatrix} e^{t^2/2} & e^{t^2/2} + te^{t^2/2} \\ 0 & e^{t^2/2} \end{bmatrix}.$$

This will give the matrices  $\mathbf{A}'(t) = \mathbf{S}^{-1}(\mathbf{A}\mathbf{S} - \dot{\mathbf{S}}) = 0$ ,

$$\mathbf{B}'(t) = \mathbf{S}^{-1}\mathbf{B} = \begin{bmatrix} e^{-t^2/2} & \times \\ 0 & \times \end{bmatrix} \begin{bmatrix} t \\ 0 \end{bmatrix} = \begin{bmatrix} te^{-t^2/2} \\ 0 \end{bmatrix},$$

$$\mathbf{C}'(t) = \mathbf{C}\mathbf{S} = [t, 0] \begin{bmatrix} e^{t^2/2} & e^{t^2/2} + te^{t^2/2} \\ 0 & e^{t^2/2} \end{bmatrix} = [te^{t^2/2}, te^{t^2/2} + t^2e^{t^2/2}],$$

and  $\mathbf{D}(t)$  is unchanged.

- 7** First, the state transition matrix  $\Phi(t, 0)$  will be found. Write  $\mathbf{A}(t) = \begin{bmatrix} \alpha(t) & 1 \\ 0 & 0 \end{bmatrix}$  for simplicity, where  $\alpha(t) = \sin t$ , which has period  $T = 2\pi$ . Solving  $\frac{d}{dt}\mathbf{X} = \mathbf{A}(t)\mathbf{X}$  for initial condition  $\xi_1 = \mathbf{e}_1$  gives  $\psi_1 = \begin{bmatrix} \phi(t) \\ 0 \end{bmatrix}$ , where  $\phi(t) = e^{\int_0^t \alpha(\tau) d\tau} = e^{1-\cos t}$ . Initial condition  $\xi_2 = \mathbf{e}_2$  gives  $x_2(t) = 1$  and

$$x_1(t) = \phi(t) \cdot 0 + \int_0^t \phi(t)(\phi(\tau))^{-1} \cdot 1 d\tau = \phi(t)\gamma(t),$$

where  $\gamma(t) = \int_0^t (\phi(\tau))^{-1} d\tau$ . Thus

$$\Phi(t, 0) = \begin{bmatrix} \phi(t) & \phi(t)\gamma(t) \\ 0 & 1 \end{bmatrix}, \text{ and } \Phi(T, 0) = \begin{bmatrix} \phi(T) & \phi(T)\gamma(T) \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \gamma(T) \\ 0 & 1 \end{bmatrix}.$$

Since  $\Phi(T, 0)$  has an eigenvalue pair at  $-1$ , the logarithm is

$$\log_e \Phi(T, 0) = \alpha_0 \mathbf{I} + \alpha_1 \Phi(T, 0),$$

where  $\log_e 1 = \alpha_0 + \alpha_1(1)$  and  $\frac{d}{dz} \log_e z|_{z=1} = 1 = \frac{d}{dz}(\alpha_0 + \alpha_1 z)|_{z=1} = \alpha_1$ , so that

$$T\mathbf{Q} = \log_e \Phi(T, 0) = (-1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 1 \begin{bmatrix} 1 & \gamma(T) \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & \gamma(T) \\ 0 & 0 \end{bmatrix},$$

from which

$$\mathbf{Q} = \begin{bmatrix} 0 & \frac{\gamma(T)}{T} \\ 0 & 0 \end{bmatrix}, \text{ and } e^{t\mathbf{Q}} = \begin{bmatrix} 1 & \frac{t}{T}\gamma(T) \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{t}{T} \int_0^T e^{-(1-\cos \tau)} d\tau \\ 0 & 1 \end{bmatrix}.$$

Thus  $\Phi(t, 0) = \mathbf{S}(t)e^{t\mathbf{Q}}$  where

$$\begin{aligned} \mathbf{S}(t) &= \Phi(t, 0) e^{-t\mathbf{Q}} = \begin{bmatrix} \phi(t) & \phi(t)\gamma(t) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{t}{T}\gamma(T) \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} e^{1-\cos t} & e^{-\cos t} \left( \int_0^t e^{\cos \tau} d\tau - \frac{t}{2\pi} \int_0^{2\pi} e^{\cos \tau} d\tau \right) \\ 0 & 1 \end{bmatrix}, \end{aligned}$$

which is periodic, as can be seen by observing that  $\mathbf{S}(0) = \mathbf{S}(2\pi)$ .

Because  $\mathbf{Q}$  does not have eigenvalues exclusively in the left half-plane, it cannot be concluded that all free solutions are asymptotic to 0.

- 8** Find  $\Phi(t, 0)$  and  $\mathbf{S}(t) = \Phi(t, 0)e^{-t\mathbf{Q}}$ . Initial condition  $\mathbf{e}_1$  gives solution  $\Psi_1(t) = \begin{bmatrix} e^{-t} \\ 0 \end{bmatrix}$ . Initial condition  $\mathbf{e}_2$  gives solution  $x_2 = e^{-t}$  and

$$x_1 = e^{-t} \cdot 0 + \int_0^t e^{-t} e^{\tau} (1 + e^{\tau}) e^{-\tau} d\tau = e^{-t}(1 + t - e^{-t}),$$

so that the state-transition matrix is

$$\Phi(t, 0) = \begin{bmatrix} e^{-t} & e^{-t}(1 + t - e^{-t}) \\ 0 & e^{-t} \end{bmatrix}.$$

The desired matrix and the required exponent are

$$\mathbf{Q} = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}, \quad e^{-t\mathbf{Q}} = \begin{bmatrix} e^t & -te^t \\ 0 & e^{-t} \end{bmatrix},$$

so that the transformation matrix  $\mathbf{S}(t)$  is

$$\mathbf{S}(t) = \Phi(t, 0)e^{-t\mathbf{Q}} = \begin{bmatrix} e^{-t} & e^{-t}(1 + t - e^{-t}) \\ 0 & e^{-t} \end{bmatrix} \cdot \begin{bmatrix} e^t & -te^t \\ 0 & e^{-t} \end{bmatrix} = \begin{bmatrix} 1 & 1 - e^{-t} \\ 0 & 1 \end{bmatrix}.$$

- 9** Taking  $\mathbf{C}(t)$  and  $\mathbf{D}(t)\mathbf{U}(t)$  under the integral sign using the sifting property of an impulse as for Equation (2.32), and requiring  $\mathbf{D}(t)\mathbf{U}(t)$  to be continuous, we get

$$\mathbf{Y}(t) = \int_{t_0}^t \{\mathbf{C}(t)\Phi(t, \tau)\mathbf{B}(\tau) + \delta(t - \tau)\mathbf{D}(\tau)\} \mathbf{U}(\tau) d\tau.$$

Therefore the matrix  $\mathbf{H}(t, \tau)$  is

$$\mathbf{H}(t, \tau) = \begin{cases} \mathbf{C}(t)\Phi(t, \tau)\mathbf{B}(\tau) + \delta(t - \tau)\mathbf{D}(\tau) & t \geq \tau, \\ 0 & t < \tau. \end{cases}$$

For the given matrices, with  $\Phi(t, 0)$  calculated from Problem 1(a), the impulse-response matrix  $\mathbf{H}(t, t_0)$  is

$$\begin{aligned} \mathbf{H}(t, t_0) &= \mathbf{C}(t)\Phi(t, 0)\Phi(t_0, 0)^{-1}\mathbf{B}(t_0) + 0\delta(t - t_0) \\ &= [1, -t^2/4] \begin{bmatrix} 1 & t^2/2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & t_0^2/2 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ t_0 + 1 \end{bmatrix} \\ &= (t^2/4 - t_0^2/2)(t_0 + 1). \end{aligned}$$

- 10** The state-transition matrix  $\Phi(t, 0)$  as given in the solution of Problem 1(b) is

$$\Phi(t, 0) = \begin{bmatrix} e^{t^2/2} & e^{t^2/2} + te^{t^2/2} \\ 0 & e^{t^2/2} \end{bmatrix}.$$

Therefore

$$\begin{aligned} \mathbf{H}(t, t_0) &= \mathbf{C}(t)\Phi(t, 0)\Phi(t_0, 0)^{-1}\mathbf{B}(t_0) + 0\delta(t - t_0) \\ &= [1, -t^2/4] \begin{bmatrix} e^{t^2/2} & e^{t^2/2} + te^{t^2/2} \\ 0 & e^{t^2/2} \end{bmatrix} \begin{bmatrix} e^{t_0^2/2} & e^{t_0^2/2} + t_0 e^{t_0^2/2} \\ 0 & e^{t_0^2/2} \end{bmatrix}^{-1} \begin{bmatrix} t_0 \\ 0 \end{bmatrix} \\ &= e^{(t^2 - t_0^2)/2} t_0. \end{aligned}$$



11 The product  $\mathbf{A}(t)\mathbf{A}(\tau)$  is

$$\begin{bmatrix} a & e^{-t} \\ -e^{-t} & a \end{bmatrix} \begin{bmatrix} a & e^{-\tau} \\ -e^{-\tau} & a \end{bmatrix} = \begin{bmatrix} a^2 - e^{-t-\tau} & a(e^{-\tau} + e^{-t}) \\ -a(e^{-\tau} + e^{-t}) & a^2 - e^{-t-\tau} \end{bmatrix}$$

which equals the product  $\mathbf{A}(\tau)\mathbf{A}(t)$ . Then formula (11.18) can be applied.

## Chapter 12

## Solutions

1 Since the systems have one input variable, Equation (12.9) applies, as follows.

(a) The matrices are in control form, so  $\mathbf{S} = \mathbf{I}$ . By inspection,  $\hat{\mathbf{a}} = [0, 0, 0]$ . The coefficients of the desired polynomial are  $\hat{\mathbf{b}} = [125, 75, 15]$ . Therefore

$$\mathbf{K} = (\hat{\mathbf{b}} - \hat{\mathbf{a}})\mathbf{S}^{-1} = [125, 75, 15].$$

(b) For the given matrices,  $[\mathbf{A}, \mathbf{B}]$  is in lower left reduced row echelon form, so  $\mathbf{S}_0 = \mathbf{I}$ . From the leftmost column of  $\mathbf{A}$ , the vector of coefficients  $\hat{\mathbf{a}}$  is  $\hat{\mathbf{a}} = [-2, 0, 1]$ , so  $\mathbf{S}$  and  $\mathbf{K}$  are

$$\mathbf{S}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad \mathbf{K} = [127, 75, 14] \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} = [66, 61, 14].$$

2 The entries of  $\mathbf{K}$  are given by Equation (12.9). In this equation,  $\hat{\mathbf{a}} = 0$  so, in this case, the entries of  $\mathbf{K}$  are linear combinations of  $\hat{\mathbf{b}} = [\alpha^3, 3\alpha^2, 3\alpha]$ . Therefore, if  $\alpha > 5$ , the entries of  $\mathbf{K}$  will be larger, and if  $\alpha < 5$ , the entries of  $\mathbf{K}$  will be smaller than the corresponding values in the the solution of Problem 1

3 In closed-loop, the free response of the state is

$$\mathbf{X}_{\text{free}}(t) = e^{t(\mathbf{A}-\mathbf{BK})}\mathbf{X}(0),$$

and  $u(t) = -\mathbf{K}\mathbf{X}(t)$  for  $r = 0$ . To place both poles at the required location,  $\mathbf{K} = [\alpha^2, 2\alpha]$ . The Laplace transform of  $u(\cdot)$  is

$$u(s) = -\mathbf{K}(s\mathbf{I} - \mathbf{A} + \mathbf{K})^{-1}\mathbf{X}(0) = \frac{-\alpha^2 - 2\alpha s}{(s + \alpha)^2}$$

and, by the initial-value theorem,

$$u(0+) = \lim_{s \rightarrow \infty} su(s) = -2\alpha.$$

Therefore, the more the value of  $\alpha$  is increased to get fast transient response, the larger the control signal has to be relative to perturbations of the second entry of the state vector.

- 4 Taking the Laplace transform for a system of dimension  $n$ , the required state feedback contains the coefficients of the polynomial  $(s + \alpha)^n$ ; that is,  $\mathbf{K} = [\alpha^n, \dots, n\alpha]$ , so that the Laplace transform of  $u(\cdot)$  is

$$u(s) = -\mathbf{K}(s\mathbf{I} - \mathbf{A} + \mathbf{K})^{-1}\mathbf{X}(0) = \frac{-\alpha^n \dots - n\alpha s^{n-1}}{(s + \alpha)^n}.$$

By the initial-value theorem,

$$u(0+) = \lim_{s \rightarrow \infty} su(s) = -n\alpha,$$

which is  $-3\alpha$  for the matrices of Problem 1(a). In each case the initial value of  $u$  is proportional to  $\alpha$ .

- 5 In each case, the required closed-loop characteristic polynomial is

$$\det(s\mathbf{I} - (\mathbf{A} - \mathbf{BK})) = (s + 1)(s + 2) = s^2 + 3s + 2.$$

(a) The following are calculated in order:

$$\mathbf{A} - \mathbf{BK} = \begin{bmatrix} -k_{11} & -1 - k_{12} \\ 1 & 0 \end{bmatrix},$$

$$\det(s\mathbf{I} - (\mathbf{A} - \mathbf{BK})) = s^2 + k_{11}s + (1 + k_{12}), \quad k_{11} = 3, \quad k_{12} = 1,$$

$$\mathbf{H}(s) = [1, 1] \begin{bmatrix} s+3 & 2 \\ -1 & s \end{bmatrix}^{-1} = \left[ \frac{1}{s+2}, \frac{1}{s+2} \right].$$

(b) The following are calculated in order:

$$\mathbf{A} - \mathbf{BK} = \begin{bmatrix} 0 & -1 \\ 1 - k_{21} & -k_{22} \end{bmatrix},$$

$$\det(s\mathbf{I} - (\mathbf{A} - \mathbf{BK})) = s^2 + k_{22}s + (1 - k_{21}), \quad k_{21} = -1, \quad k_{22} = 3,$$

$$\mathbf{H}(s) = [1, 1] \begin{bmatrix} s & 1 \\ -2 & s+3 \end{bmatrix}^{-1} = \left[ \frac{s+5}{(s+1)(s+2)}, \frac{s-1}{(s+1)(s+2)} \right].$$

(c) The following are calculated in order:

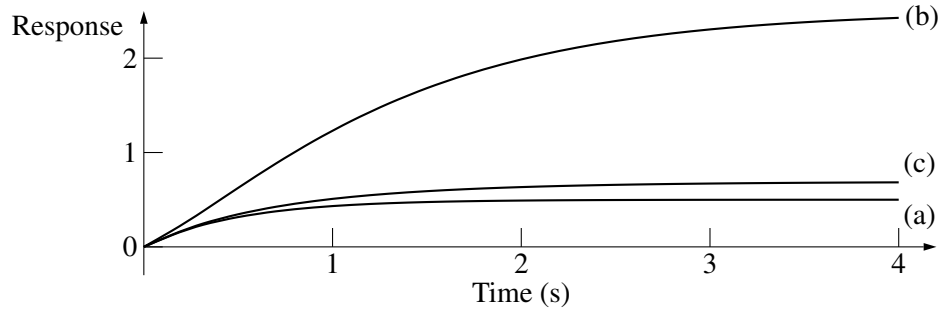
$$\mathbf{A} - \mathbf{BK} = \begin{bmatrix} -k_{11} & -1 \\ 1 & -k_{22} \end{bmatrix},$$

$$\det(s\mathbf{I} - (\mathbf{A} - \mathbf{BK})) = s^2 + (k_{11} + k_{22})s + (k_{11}k_{22} + 1),$$

$$k_{11} = \frac{3 + \sqrt{5}}{2}, \quad k_{22} = \frac{3 - \sqrt{5}}{2},$$

$$\begin{aligned} \mathbf{H}(s) &= [1, 1] \begin{bmatrix} s + (3 + \sqrt{5})/2 & 1 \\ -1 & s + (3 - \sqrt{5})/2 \end{bmatrix}^{-1} \\ &= \frac{1}{(s+1)(s+2)} [s + (5 - \sqrt{5})/2, s + (1 + \sqrt{5})/2]. \end{aligned}$$

The forced responses for the three cases are shown in Figure S12.5.



**Fig. S12.5** Forced response to a step at the first input

- 6** Since  $\mathbf{U}(t) = \mathbf{R}(t) - \mathbf{K}\mathbf{Y}(t)$  and  $\mathbf{D} = 0$ , the system of Problem 5 is described in closed loop by the equations

$$\dot{\mathbf{X}} = (\mathbf{A} - \mathbf{B}\mathbf{K}\mathbf{C})\mathbf{X} + \mathbf{B}\mathbf{U},$$

$$\mathbf{Y} = \mathbf{C}\mathbf{X},$$

where  $\mathbf{K}$  is  $2 \times 1$  and contains two parameters. The closed-loop characteristic polynomial is

$$\det(s\mathbf{I} - (\mathbf{A} - \mathbf{B}\mathbf{K}\mathbf{C})) = s^2 + (k_1 + k_2)s + (1 + k_1 - k_2).$$

Equating  $k + 1 + k_2$  to 3 and  $1 + k_1 - k_2$  to 2 as required gives  $k_1 = 2$  and  $k_2 = 1$ .

If  $\mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$  then  $\mathbf{K}$  is  $1 \times 1$  and contains only one parameter, so the two coefficients of the desired closed-loop polynomial cannot be obtained in general, and result in the inconsistent equations  $k = 3$  and  $1 - k = 2$  for the desired pole locations.

- 7** The transformation matrix  $\mathbf{S}$  required for finding  $\mathbf{K}$  is a unit matrix since the system as given is in the required control form. Since  $\mathbf{A} - \mathbf{B}\mathbf{K}$  is to have two poles at  $-3$ , the desired characteristic polynomial is  $\beta(s) = (s + 3)^2 = s^2 + 6s + 9$ , from which  $\mathbf{K}$  is given by

$$\mathbf{K} = (\hat{\mathbf{b}} - \hat{\mathbf{a}})\mathbf{S}^{-1} = ([9, 6] - [2, 1])\mathbf{I} = [11, 7].$$

The matrix  $\mathbf{A} - \mathbf{B}\mathbf{K}$  and its inverse are

$$\mathbf{A} - \mathbf{B}\mathbf{K} = \begin{bmatrix} 0 & 1 \\ -9 & -6 \end{bmatrix}, \quad (\mathbf{A} - \mathbf{B}\mathbf{K})^{-1} = \frac{1}{9} \begin{bmatrix} -6 & -1 \\ 9 & 0 \end{bmatrix},$$

and Equation (12.35) becomes

$$[3, 1] \frac{1}{9} \begin{bmatrix} -6 & -1 \\ 9 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} [\mathbf{G}_r, \mathbf{G}_d] = [3, 1] \frac{1}{9} \begin{bmatrix} -6 & -1 \\ 9 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & | & 1 \\ 2 & 1 & | & 0 \end{bmatrix},$$

which simplifies to

$$-\frac{1}{3} [\mathbf{G}_r, \mathbf{G}_d] = -[2/3, 4/3, 1],$$

from which,

$$\mathbf{G}_r = [1, 4], \quad \mathbf{G}_d = 3.$$

- 8** First,  $\mathbf{A}$  and  $\mathbf{B}$  are transformed to control form using the similarity transformation  $\mathbf{S}$  shown:

$$\mathbf{A}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -6 & -7 \end{bmatrix}, \quad \mathbf{B}' = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{S} = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}.$$

$$\begin{aligned} \mathbf{K} &= (\hat{\mathbf{b}} - \hat{\mathbf{a}}) \mathbf{S}^{-1} = ([60, 47, 12] - [3, 6, 7]) \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \\ &= [353, -130, -166]. \end{aligned}$$

- 9** From the given pole locations, the factors of the desired characteristic polynomial are  $s^2 + 6s + 18$ ,  $s^2 + 6s + 18$ , and  $(s + 4)(s^2 + 6s + 18) = s^3 + 10s^2 + 42s + 72$ .

- 10** The open-loop poles are the eigenvalues of the given  $\mathbf{A}$  and the zeros can be calculated as in Chapter 10, but these quantities are also available from the control form required for pole placement. The required similarity matrix  $\mathbf{S}$  and the resulting system matrices are

$$\mathbf{S} = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}, \quad \mathbf{A}' = \mathbf{S}^{-1} \mathbf{A} \mathbf{S} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -3 \end{bmatrix}, \quad \mathbf{C}' = \mathbf{C} \mathbf{S} = [5, 1, 0].$$

By inspection of these matrices, we see that the eigenvalues of  $\mathbf{A}$  are the roots of  $s^3 + 3s^2$ , equal to 0, 0, and  $-3$ . Similarly, the zeros are the roots of  $s + 5$ ; that is, there is a zero equal to  $-5$ . Modifying the bottom row of  $\mathbf{A}'$  by means of state feedback to produce a pole at  $-5$  will create a pole-zero cancellation at  $-5$ , since  $\mathbf{D} = 0$  and  $\mathbf{C}'$  will remain unchanged. Therefore, the closed-loop system will not be minimal. The state-feedback matrix is given by

$$\begin{aligned} \mathbf{K} &= (\hat{\mathbf{b}} - \hat{\mathbf{a}}) \mathbf{S}^{-1} = ([60, 47, 12] - [0, 0, 3]) \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \\ &= [364, -133, -171]. \end{aligned}$$

- 11** The entries of  $\mathbf{K}'$  are given by  $\mathbf{K}' = \hat{\mathbf{b}} - \hat{\mathbf{a}}$  and, in this example,  $\hat{\mathbf{a}} = 0$ . The desired characteristic polynomial is as shown, expanded by invoking the binomial theorem:

$$(s + 10)^n = \sum_{k=0}^n \binom{n}{k} s^k 10^{n-k}.$$

The largest entry in  $\mathbf{K}'$  is the largest coefficient in this polynomial and is equal to  $10^n$ . Consequently the control of systems of high dimension by one input signal may require high-gain feedback.

- 1 From Example 6, the matrices required in the ARE are

$$\hat{\mathbf{R}} = r + q = 2s, \quad \hat{\mathbf{A}} = -q/(q+r) = -1/2, \quad \hat{\mathbf{Q}} = q - q^2/(q+r) = s/2,$$

and the ARE is

$$\begin{aligned} 0 &= \mathbf{P}\mathbf{A} + \mathbf{A}^T\mathbf{P} - \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P} + \mathbf{Q} \\ &= p(-1/2) + (-1/2)p - p \cdot 1 \cdot (1/2s) \cdot 1 \cdot p + s/2, \end{aligned}$$

or

$$p^2 + 2sp - s^2 = 0,$$

the positive solution of which is  $p = s(\sqrt{2} - 1)$ .

- 2 Substituting the numerical values into the Equation (13.66) gives the ARE with unknown  $p$  as

$$0 = 2p - p^2 + q,$$

which has positive solution  $p = 1 + \sqrt{1+q}$ . Then the closed-loop matrix is

$$\mathbf{A}_{\text{CL}} = \mathbf{A} - \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P} = 1 - 1 - \sqrt{1+q},$$

which equals  $-2$  for  $q = 3$ . The Hamiltonian matrix is

$$\mathbf{H} = \begin{bmatrix} 1 & -1 \\ -q & -1 \end{bmatrix},$$

which has eigenvalues equal to  $\pm\sqrt{1+q}$  and, for  $q = 3$ , the left-plane eigenvalue equals  $-2$ .

- 3 Refer to Section 1:

- (a) Increasing  $q$  relative to  $r$  weights the state vector heavier, resulting in faster transients and larger initial inputs. Time-constants will be smaller (faster).
- (b) Increasing  $r$  relative to  $q$  will have the opposite effect to (a); that is, smaller initial input values and larger time-constants (slower convergence to the origin).
- (c) Since  $q$  and  $r$  have been increased by the same ratio, no change to transient behavior results.

- 4

$$\text{(a) } \mathbf{X}^T \begin{bmatrix} 1 & 3 \\ 3 & 3 \end{bmatrix} \mathbf{X}, \quad \text{(b) } \mathbf{X}^T \begin{bmatrix} 2 & -2 \\ -2 & 1 \end{bmatrix} \mathbf{X}, \quad \text{(c) } \mathbf{X}^T \begin{bmatrix} 1 & 0 & 3 \\ 0 & -2 & -3 \\ 3 & -3 & -3 \end{bmatrix} \mathbf{X}.$$

5 Refer to Example 3:

- (a) The symmetric part of the matrix is  $\mathbf{Q}_s = \begin{bmatrix} 7 & 6 \\ 6 & 4 \end{bmatrix}$ , the leading principal minors of which are 7 and  $-8$ , so the matrix is neither positive-definite nor positive semidefinite.
- (b) The second principal minor is  $\det \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = -1$  so the matrix is not positive-definite or positive semidefinite.
- (c) The 3, 3 element is zero so  $\mathbf{X}^T \mathbf{Q} \mathbf{X} = 0$  for any nonzero vector  $\mathbf{X}^T = [0, 0, x_3]$ , so the matrix cannot be positive definite. Its principal  $1 \times 1$  minors are 1, 3, 0, its  $2 \times 2$  minors are 2, 0,  $-1$ , and its  $3 \times 3$  minor is  $-1$ . Any minor less than zero (there are two in this case) indicates a matrix that is not positive semidefinite. Alternatively, one of the three computed eigenvalues is negative, giving the same conclusion.

- 6 (a)  $x_1^2 + 4x_1x_2 + 3x_2^2$ , (b)  $2x_1^2 + x_2^2$ ,  
 (c)  $x_1^2 + x_1x_3 - 2x_2^2 - 3x_1x_3 - 3x_2x_3 - 3x_3^2$ .

7 The required matrix data are  $\mathbf{A} = 0$ ,  $\mathbf{B} = 1$ ,  $\mathbf{Q} = 1$ ,  $\mathbf{R} = 1$ , and  $\mathbf{S} = 0$ , so Equation (13.49) reduces to

$$-\frac{dp}{dt} = p^2 - 1,$$

with final condition  $p(t_1) = 0$ . In this case, the equation is separable:

$$\begin{aligned} \frac{dp}{p^2 - 1} &= \frac{dp}{(p - 1)(p + 1)} = \left( \frac{1/2}{p - 1} + \frac{-1/2}{p + 1} \right) dp \\ &= \frac{1}{2} d \ln(p - 1) - \frac{1}{2} d \ln(p + 1) = dt \end{aligned}$$

and integrating from  $t$  to  $t_1$  gives

$$\begin{aligned} \frac{1}{2} \left( \ln(p(t_1) - 1) - \ln(p(t) - 1) - \ln(p(t_1) + 1) + \ln(p(t) + 1) \right) \\ = \frac{1}{2} \ln \left( \frac{p(t_1) - 1}{p(t) - 1} \cdot \frac{p(t) + 1}{p(t_1) + 1} \right) = t_1 - t. \end{aligned}$$

Since  $p(t_1) = 0$ , this reduces to

$$\frac{1}{2} \ln \frac{1 + p(t)}{1 - p(t)} = t_1 - t,$$

which can be solved for  $p(t)$  to give

$$p(t) = \frac{e^{2(t_1-t)} - 1}{e^{2(t_1-t)} + 1} = \frac{e^{t_1-t} - e^{-(t_1-t)}}{e^{t_1-t} + e^{-(t_1-t)}} = \tanh(t_1 - t),$$

and this function is positive for  $t < t_1$ . As a function of  $\mathbf{X}$ , the input  $u(t)$  is

$$u(t) = -\mathbf{R}^{-1} \mathbf{B}^T \mathbf{P}(t) \mathbf{X} = -\mathbf{K}(t) \mathbf{X} = -(\tanh(t_1 - t)) x(t).$$

The state  $x(t)$  is the solution of the differential equation

$$\dot{x} = -(\tanh(t_1 - t)) x(t),$$

with initial condition  $x(t_0)$ . The solution is

$$x(t) = x(t_0) e^{-\int_{t_0}^t \tanh(t_1 - \tau) d\tau},$$

so  $u(t)$  is

$$u(t) = -(\tanh(t_1 - t)) x(t_0) e^{-\int_{t_0}^t \tanh(t_1 - \tau) d\tau}.$$

**8** The Hamiltonian matrix for the matrices  $\mathbf{A} = 0$ ,  $\mathbf{B} = 1$ ,  $\mathbf{Q} = 1$ ,  $\mathbf{R} = 1$ , is

$$\mathbf{H} = \begin{bmatrix} \mathbf{A} & -\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T \\ -\mathbf{Q} & -\mathbf{A}^T \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix},$$

of which the eigenvalues are  $\pm 1$  so the eigenvalue of the optimal closed-loop system with constant feedback is  $-1$ .

**9** The integrand of the cost function is

$$y^2 + \rho^2 u^2 = \mathbf{X}^T \mathbf{C}^T \mathbf{C} \mathbf{X} + \rho^2 u^2,$$

so the ARE (13.66) is

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ - \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rho^{-2} [0, 1] \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} [1, 0] = 0.$$

Writing the top-left, top-right, and bottom-right entries of this matrix equation gives, respectively,

$$\begin{aligned} -\rho^{-2} p_{12}^2 + 1 &= 0, \\ p_{11} - \rho^{-2} p_{12} p_{22} &= 0, \\ 2p_{12} - \rho^{-2} p_{22}^2 &= 0. \end{aligned}$$

From the third equation,

$$p_{22} = \rho \sqrt{2p_{12}}$$

and, for real positive-definite  $\mathbf{P}$ , the positive square root must be taken and  $p_{12}$  must be positive. From the first equation,

$$p_{12} = \pm \rho$$

and the positive value must be taken. Then from the second equation,

$$p_{11} = \rho^{-2} p_{12} p_{22} = \sqrt{2\rho},$$

of which the positive square root is taken, and  $\mathbf{P}$  is

$$\mathbf{P} = \begin{bmatrix} \sqrt{2\rho} & \rho \\ \rho & \rho\sqrt{2\rho} \end{bmatrix}$$

The top-left element is positive and the second principal minor is

$$\det \mathbf{P} = \rho^2,$$

which is positive, so  $\mathbf{P}$  is positive-definite as required. The state feedback matrix is

$$\mathbf{K} = \mathbf{R}^{-1}\mathbf{B}^T\mathbf{P} = \rho^{-2} [0, 1] \begin{bmatrix} \sqrt{2\rho} & \rho \\ \rho & \rho\sqrt{2\rho} \end{bmatrix} = \left[ \frac{1}{\rho} \quad \frac{\sqrt{2}}{\sqrt{\rho}} \right].$$

Large values of  $\rho$  give small entries in  $\mathbf{K}$ , and vice-versa. Thus, large values of  $\rho$  result in small control signal  $u(t)$  and slow transients. Small values of  $\rho$  will give fast transients.

**10**

**11** The ARE is

$$\begin{bmatrix} 0 & 0 \\ 1 & \sqrt{5} \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & \sqrt{5} \end{bmatrix} - \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \\ + \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix} = 0.$$

Writing the top-left, top-right, and bottom-right entries of this matrix equation gives the three equations

$$\begin{aligned} -p_{12}^2 + 4 &= 0, \\ p_{11} + p_{12}\sqrt{5} - p_{12}p_{22} &= 0, \\ 2p_{12} + 2p_{22}\sqrt{5} - p_{22}^2 &= 0. \end{aligned}$$

Solving the third equation for  $p_{22}$  gives

$$p_{22} = \sqrt{5} \pm \sqrt{5 + 2p_{12}}$$

where, from the first equation,

$$p_{12} = \pm\sqrt{4} = \pm 2.$$

From the second equation,

$$p_{11} = p_{12}(-\sqrt{5} + p_{22}).$$

Choosing the positive square-root in each case gives

$$\mathbf{P} = \begin{bmatrix} 6 & 2 \\ 2 & 3 + \sqrt{5} \end{bmatrix},$$

which is positive-definite. Therefore, the state-feedback matrix is

$$\mathbf{K} = \mathbf{R}^{-1}\mathbf{B}^T\mathbf{P} = [2, 3 + \sqrt{5}].$$



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**12**


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- 13** Since the numerical values of all matrices are known, this problem can be solved numerically. Alternatively, the ARE reduces to the three equations shown in the entries of  $\mathbf{P}$ :

$$-p_{12}^2/4 + 1 = 0,$$

$$p_{11} - p_{12}p_{22}/4 = 0,$$

$$2p_{12} - p_{22}^2/4 = 0.$$

Solving the third equation for  $p_{22}$  gives

$$p_{22} = \pm 2\sqrt{2p_{12}}$$

which must be positive, so the positive square root is required and  $p_{12}$  must be positive. Therefore, from the first equation,

$$p_{12} = 2.$$

From the second equation,

$$p_{11} = 2$$

and, therefore,

$$\mathbf{P} = \begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix},$$

which is positive-definite. The feedback matrix is

$$\mathbf{K} = \mathbf{R}^{-1}\mathbf{B}^T\mathbf{P} = [1/2, 1].$$


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**14**


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- 15** This problem can be solved numerically. Alternatively, the ARE reduces to the three equations shown in the entries of  $\mathbf{P}$ :

$$-16p_{12} - p_{11}^2 - p_{12}^2 + 1 = 0,$$

$$-8p_{22} + 8p_{11} - p_{11}p_{12} - p_{12}p_{22} = 0,$$

$$16p_{12} - p_{12}^2 - p_{22}^2 + 1 = 0.$$

Solving for  $p_{12}$  in the first and third equation gives, respectively,

$$p_{12} = -8 \pm \sqrt{65 - p_{11}^2}$$

$$p_{12} = 8 \pm \sqrt{65 - p_{22}^2},$$

where  $p_{11}$  and  $p_{22}$  must be positive. From the second equation, which reduces to

$$p_{12} = \frac{8(p_{11} - p_{22})}{p_{11} + p_{22}}$$

and from the symmetry of the equations with respect to  $p_{11}$  and  $p_{22}$ , The values  $p_{12} = 0$ ,  $p_{11} = p_{22} = 1$  are seen to satisfy the equations and produce a positive-definite solution matrix  $\mathbf{P} = \mathbf{I}$ . Therefore, the state-feedback matrix is

$$\mathbf{K} = \mathbf{R}^{-1}\mathbf{B}^T\mathbf{P} = \mathbf{I}.$$

- 16 Constructing  $\mathbf{H}$  for  $\mathbf{R} = 1$  and  $\mathbf{Q} = \text{diag}[a, b, c, d]$  results in the characteristic polynomial  $\phi(s) = \det(s\mathbf{I} - \mathbf{H})$  shown:

$$\phi(s) = s^8 - \frac{(d + 2\gamma^2\ell^2M^2 + \ell^2b)s^6}{\ell^2M^2} + \frac{(c + 2b\ell g + \gamma^4\ell^2M^2 + \ell^2a)s^4}{\ell^2M^2} - \frac{g(bg + 2a\ell)s^2}{\ell^2M^2} + \frac{ag^2}{\ell^2M^2}.$$

The desired characteristic polynomial is

$$\beta(s) = s^8 + 2(\omega^2\delta - \gamma^2)s^6 + (\omega^4 - 4\gamma^2\omega^2\delta + \gamma^4)s^4 + 2\gamma^2\omega^2(\gamma^2\delta - \omega^2)s^2 + \gamma^4\omega^4.$$

Equating terms and solving gives

$$a = \gamma^4\omega^4 \frac{\ell^2M^2}{g^2},$$

$$b = 2\gamma^2\omega^2(g\omega^2 - g\gamma^2\delta - \ell\gamma^2\omega^2) \frac{\ell^2M^2}{g^3},$$

$$c = \omega^2(g^2\omega^2 - 4g^2\gamma^2\delta + 3\ell^2\gamma^4\omega^2 - 4g\gamma^2\omega^2\ell + 4g\gamma^4\delta\ell) \frac{\ell^2M^2}{g^2},$$

$$d = -2\omega^2(\delta g^3 + \gamma^2\ell^2g\omega^2 - \gamma^4\ell^2g\delta - \gamma^4\ell^3\omega^2) \frac{\ell^2M^2}{g^3}.$$

- 17 Solving the LQ problem with the given numerical parameters gives  $\mathbf{K} = [0.4495, 0.1010]$ . Then  $u' = -\mathbf{K}\mathbf{E} = -k_1e_1 - k_2e_2$ . But  $e_2 = e$  in Figure P13.17 and, from the second row of  $\dot{\mathbf{E}} = \mathbf{A}\mathbf{E} + \mathbf{B}u'$ , the first entry  $e_1$  of  $\mathbf{E}$  is obtained in terms of the second entry as  $e_1 = ae_2 + \dot{e}_2$  or, using the Laplace transform, as  $e_1 = ae_2 + se_2$ . Then the expression for  $u$  is

$$u = \frac{u'}{s} = \frac{-k_1se_2 - k_1ae_2 - k_2e_2}{s} = -\left(k_1 + \frac{k_1a + k_2}{s}\right)e_2,$$

which has PI form. The redrawn closed-loop diagram with input  $r$  is shown:

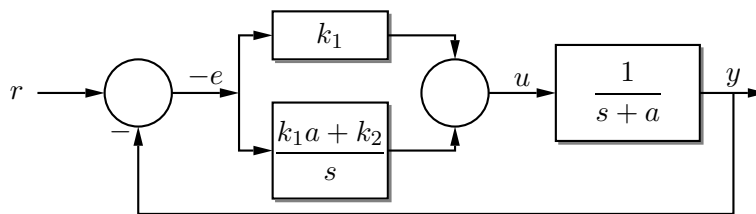


Fig. S13.17 The controller redrawn in PI form

Chapter 14

Solutions

- 1 Calculating a column compression of  $\mathbf{B}$  and a row compression of  $\mathbf{C}$  gives

$$\mathbf{B}\mathbf{Q}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \rho, \quad \mathbf{P}_1\mathbf{C} = [1, 1]\nu,$$

where  $\rho$  and  $\nu$  are arbitrary parameters, and consequently the result is not unique. Setting the Euclidean norms to 1 requires  $\rho = 1/\sqrt{5}$  and  $\nu = 1/\sqrt{2}$ , giving unique matrices.

- 2** The poles of  $\mathbf{A} - \mathbf{L}\mathbf{C}$  are those of  $\mathbf{A}^T - \mathbf{C}^T\mathbf{L}^T$  and techniques from Chapter 12 can be used to place the poles at  $-20$ . First,  $\mathbf{A}^T$  and  $\mathbf{C}^T$  are transformed to control form using the similarity transformation  $\mathbf{S}$  shown:

$$\mathbf{A}' = \mathbf{S}^{-1}\mathbf{A}^T\mathbf{S} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -6 & -7 \end{bmatrix}, \quad \mathbf{B}' = \mathbf{S}^{-1}\mathbf{C}^T = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

$$\mathbf{S} = \begin{bmatrix} 65 & 111 & 15 \\ -19 & -43 & -6 \\ -34 & -51 & -7 \end{bmatrix}.$$

Then  $\mathbf{L}^T$  is calculated as

$$\begin{aligned} \mathbf{L}^T &= (\hat{\mathbf{b}} - \hat{\mathbf{a}})\mathbf{S}^{-1} = ([8000, 1200, 60] - [3, 6, 7])\mathbf{S}^{-1} \\ &= [115.9, 852.8, -490.2]. \end{aligned}$$

- 3** A similarity matrix  $\mathbf{S}$  transforming  $\mathbf{C}$  to the form  $\mathbf{C}\mathbf{S} = [\mathbf{I}_p, 0]$  and the resulting transformed  $\mathbf{A}$  are given by

$$\mathbf{S} = \begin{bmatrix} 0.0484 & 0.3408 & 0.3976 \\ -0.0194 & 0.9373 & -0.0732 \\ -0.0226 & -0.0732 & 0.9146 \end{bmatrix}, \quad \mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \begin{bmatrix} 0.3548 & 1.254 & 0.6293 \\ -2.141 & -1.380 & -3.776 \\ -2.613 & -1.050 & -5.975 \end{bmatrix}.$$

Then the matrix  $\mathbf{F}$  that places the poles of  $\mathbf{A}_o$  at  $-20$  and the resulting observer matrices are given by

$$\mathbf{F} = \begin{bmatrix} 41.03 \\ -29.87 \end{bmatrix}, \quad \mathbf{A}_o = \begin{bmatrix} -52.82 & -29.60 \\ 36.40 & 12.82 \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} -120.10 \\ 93.62 \end{bmatrix},$$

$$\mathbf{L} = \begin{bmatrix} -163.44 \\ 238.15 \end{bmatrix}, \quad \mathbf{S}_1 + \mathbf{S}_2\mathbf{F} = \begin{bmatrix} 2.16 \\ 40.63 \\ -30.35 \end{bmatrix},$$

with  $\mathbf{S}_2$  given by the rightmost two columns of  $\mathbf{S}$  above. The observer equations in state-space form are given by Equations (14.23) and (14.22) with the matrices given above.

- 4** A similarity matrix  $\mathbf{S}$  transforming  $\mathbf{C}$  to the form  $\mathbf{C}\mathbf{S} = [\mathbf{I}_p, 0]$  and the resulting transformed  $\mathbf{A}$  are given by

$$\mathbf{S} = \begin{bmatrix} 0.0216 & 0.3525 & 0.3777 \\ -0.0089 & 0.9330 & -0.0717 \\ -0.0095 & -0.0717 & 0.9231 \end{bmatrix}, \quad \mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \begin{bmatrix} -0.1833 & 2.478 & -1.702 \\ 0.1537 & 3.308 & -3.892 \\ 0.2800 & 5.048 & -6.124 \end{bmatrix}.$$

Then the matrix  $\mathbf{F}$  that places the poles of  $\mathbf{A}_o$  at  $-15$  and the resulting observer matrices are given by

$$\mathbf{F} = \begin{bmatrix} 18.28 \\ 301.87 \end{bmatrix}, \quad \mathbf{A}_o = \begin{bmatrix} -537.6 & 367.6 \\ -743.1 & 507.6 \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} -651.8 \\ -901.6 \end{bmatrix},$$

$$\mathbf{L} = \begin{bmatrix} -7474 \\ -10454 \end{bmatrix}, \quad \mathbf{S}_1 + \mathbf{S}_2\mathbf{F} = \begin{bmatrix} 191 \\ 182 \\ 263 \end{bmatrix},$$

with  $\mathbf{S}_2$  given by the rightmost two columns of  $\mathbf{S}$  above.

- 5** The matrices  $\mathbf{A}^T$  and  $\mathbf{C}^T$  can be transformed to control form shown using the similarity transformation  $\mathbf{S}$  shown:

$$\mathbf{A}' = \mathbf{S}^{-1}\mathbf{A}^T\mathbf{S} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -3 \end{bmatrix}, \quad \mathbf{B}' = \mathbf{S}^{-1}\mathbf{C}^T = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

$$\mathbf{S} = \begin{bmatrix} -20 & 96 & 34 \\ 15 & -37 & -14 \\ 5 & -44 & -15 \end{bmatrix}.$$

Then  $\mathbf{L}^T$  is

$$\mathbf{L}^T = (\hat{\mathbf{b}} - \hat{\mathbf{a}})\mathbf{S}^{-1} = ([3375, 675, 45] - [0, 0, 3])\mathbf{S}^{-1} = [2401, 2341, 3254].$$

- 6** From the diagram, the plant equations are

$$\begin{aligned} \frac{d}{dt}\mathbf{X} &= \mathbf{A}\mathbf{X} + \mathbf{B}\mathbf{U} + \mathbf{W} \\ \mathbf{Y} &= \mathbf{C}\mathbf{X} + \mathbf{D}\mathbf{U} + \mathbf{V}, \end{aligned}$$

which differ from Equations (14.16) by including the term  $\mathbf{D}\mathbf{U}$  in the output equation. The observer equation is

$$\frac{d}{dt}\hat{\mathbf{X}} = \mathbf{A}\hat{\mathbf{X}} + \mathbf{B}\mathbf{U} + \mathbf{L}(\mathbf{Y} - \mathbf{C}\hat{\mathbf{X}} - \mathbf{D}\mathbf{U}),$$

which includes the term  $\mathbf{D}\mathbf{U}$  in the rightmost parentheses.

The vector  $\mathbf{Z}$  is

$$\mathbf{Z} = \mathbf{Y} - \hat{\mathbf{Y}} = \mathbf{C}\mathbf{X} + \mathbf{D}\mathbf{U} + \mathbf{V} - (\mathbf{C}\hat{\mathbf{X}} + \mathbf{D}\mathbf{U}) = \mathbf{C}(\mathbf{X} - \hat{\mathbf{X}}) + \mathbf{V},$$

which is independent of  $\mathbf{D}$  and therefore identical in the two cases.

- 7 Setting  $\tilde{\mathbf{R}}$  to 1,  $\tilde{\mathbf{Q}}$  to  $\text{diag}[q_i]$ ,  $i = 1, \dots, 3$  and constructing the Hamiltonian matrix  $\mathbf{H}$  of Equation (13.70), we can calculate its characteristic polynomial using a symbolic calculator program as

$$\det(s\mathbf{I} - \mathbf{H}) =$$

$$s^6 - (9q_1 + 9q_2 + 16q_3 + 9)s^4 + (3q_1 + 10q_2 + q_3)s^2 - (q_1 + q_2 + q_3).$$

By equating the coefficients of this polynomial to those of the desired polynomial

$$\det(s\mathbf{I} - \mathbf{H}) = (s^2 - 15^2)^3,$$

the diagonal entries of  $\mathbf{Q}$  are obtained as

$$q_1 = \frac{1\,718\,915\,256}{49} = 35.08 \times 10^6, \quad q_2 = -\frac{443\,169\,918}{49} = 9.044 \times 10^6,$$

$$q_3 = -\frac{102\,514\,959}{7} = 14.64 \times 10^6.$$

The relative weights in  $\tilde{\mathbf{Q}}$  are of the order of  $10^6$  times those of  $\tilde{\mathbf{R}}$ . These matrices are covariance matrices, that is, their diagonal entries are the mean of the square of the entries in  $\mathbf{W}$  and  $\mathbf{V}$ . Thus to obtain observer poles at  $-15$  for this plant, the mean magnitude of  $\mathbf{V}$  must be approximately  $10^{-3}$  of the entries of  $\mathbf{W}$ . In other words, the design of fast observer transients requires accurate measurement of the plant output.