ECE 204 *Numerical methods*
Sections 001
FINAL EXAMINATION

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1. **YOU MAY USE ANY PROPERTIES OF VECTOR SPACES, NORMS OR INNER PRODUCTS IN ANY QUESTION TO SIMPLIFY YOUR CALCULATIONS TO GET TO AN ANSWER.**
2. **The exam will be graded out of 50.**
3. Personal notes prepared by the student (hand-written, printed, or typeset).
4. Turn off all electronic media and store them under your desk.
5. If there is insufficient room, use the back of the last page.
6. You may ask only one question during the examination: "May I go to the washroom?"
7. Asking **any** other question **will** result in a deduction of 5 marks from the exam grade.
8. If you think a question is ambiguous, write down your assumptions and continue.
9. **Do not leave during first hour or after there are only 15 minutes left.**
10. Do not stand up until all exams have been picked up.
11. If a question only asks for an answer, you do not have to show your work to get full marks; however, if your answer is wrong and no rough work is presented to show your steps, no part marks will be awarded.
12. The questions are in the order of the course material.

1 [2] Multiply the two numbers stored as double-precision floating-point numbers

```
1 01111111111 10100000…0
1 10000000001 01000000…0
```

showing the multiplication in binary, write the resulting representation in binary and determine the corresponding number in decimal.

2 [1] Why does the approximation

$$\frac{\sin(x+h)-\sin(x-h)}{2h}$$

get worse as an approximation to the derivative as h gets very small while

$$\left(\sin(x+h)+\sin(x-h)\right)h$$

gets better as an approximation to $\int_{x-h}^{x+h}\sin(\xi)\,\mathrm{d}\xi$ as h gets very small.

3 [2] In the last question, assume a function f represents an actual signal, while $\tilde{f}$ represents a noisy signal where the noise has a mean of 0 and a standard deviation of ε. Explain why

$$\frac{\tilde{f}(x+h)-\tilde{f}(x-h)}{2h}$$

is an even worse approximation of the derivative. Additionally, explain why

$$\left(\tilde{f}(x+h)+\tilde{f}(x-h)\right)h$$

is never-the-less a reasonable approximation to $\int_{x-h}^{x+h}\sin(\xi)\,\mathrm{d}\xi$ despite the noise.

4 [1] MATLAB has a peculiar representation of polynomials, namely, an array with n entries representing a polynomial of degree $n - 1$ where the k^{th} entry (from 0 to $n - 1$) represents the coefficient of x^{n-k-1}. This makes adding two polynomials more difficult (e.g., the polynomial representing the sum `[0.3, 0.7, 0.2] + [0.5, 0.8]` is `[0.3, 1.2, 1.0]`), but justify this in light of the fact that Horner's rule is used for polynomial evaluation. (Think about the implementation.)

5 [2] Suppose you had one approximation of

$$a_0 = \frac{f(x+h) - f(x-h)}{2h}$$

which is $O(h^2)$ and you have a second better approximation

$$a_1 = \frac{f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right)}{2\frac{h}{2}}.$$

What formula combines these two approximations to come up with an even better approximation using the extrapolation techniques we used in the iterative method for approximating solutions to IVPs.

6 [2] Given the following,

$$f^{(1)}(x) = \frac{f(x+h) - f(x-h)}{2h} - \frac{1}{6} f^{(3)}(x) h^2 - \frac{1}{120} f^{(5)}(\xi) h^4$$

$$f^{(1)}(x) = \frac{f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right)}{2\frac{h}{2}} - \frac{1}{24} f^{(3)}(x) h^2 - \frac{1}{1920} f^{(5)}(\xi) h^4$$

show that the formula you found in Question 5 reduces the error from $O(h^2)$ to $O(h^4)$ and find the coefficient of the h^4 term. You don't have to find the exact value of the coefficient, but you must have something that, when plugged into a calculator, will evaluate to the correct coefficient.

7 [1] The following formula approximates the derivative:

$$f^{(1)}(x) = \frac{-f(x+2h)+8f(x+h)-8f(x-h)+f(x-2h)}{12h} + \frac{1}{30}f^{(5)}(\xi)h^4.$$

For what type of functions will this definitely provide an exact answer with zero error?

8 [4] Determine the error of the approximation

$$y^{(2)}(t) \approx \frac{y(t)-3y(t-2h)+2y(t-3h)}{3h^2}.$$

Recall that the error of $y^{(2)}(t) \approx \frac{y(t)-2y(t-2h)+y(t-3h)}{h^2}$ was already only O(h).

9 [2] Approximate the integral of $x^2(4-x)^2$ from x = 0 to x = 4 using the composite Simpson's rule with five function evaluations. You don't have to come up with a single rational number, but your number, when plugged into a calculator should evaluate to the result.

10 [1] Given the error in Question 9 is given by the formula $\frac{h^4}{180}(b-a)f^{(4)}(\xi)$, what is the error of the approximation in Question 9?

11 [3] Suppose you had a data-buffer that stored six (6) readings from a sensor. These readings are taken at equally-spaced time intervals. Describe three benefits of using a best-fitting quadratic when estimating extrapolating the value at the next point in time over finding the interpolating polynomial of degree five and evaluating that polynomial at the next point in time.

For the next two questions, we will substitute Heun's method with the mid-point method. Given the initial-value problem

$$y^{(1)}(t) = f(t, y(t))$$
$$y(t_0) = y_0$$

we approximate y_{k+1} with the formula:

$$s_0 = f(t_k, y_k)$$
$$s_1 = f\left(t_k + \frac{h}{2}, y_k + \frac{h}{2}s_0\right)$$
$$y_{k+1} = y_k + hs_1$$

The error of this method with one step is O(h^3).

12 [3] Approximate the solution to the initial-value problem

$$\begin{aligned} y^{(1)}(t) &= 1 + t - y(t) \\ y(0) &= 2 \end{aligned},$$

first using h = 2, and then a second time using h = 1 with two steps using the mid-point method.

13 [1] The coefficient of the error term of the mid-point method is smaller than the corresponding coefficient of Heun's method, so therefore it is reasonable to use Euler's method together with the mid-point method for an adaptive technique. Starting with $h = 1$, perform one step of the adaptive method and determine the scaling factor a. Given this value, determine whether we should continue approximating the next point at $t = 1 + 0.9sh$ or if we need to go back and approximate the function at $t = 0.9sh$. We are attempting to ultimately approximate $y(2)$ and our acceptable error is $\varepsilon_{\text{abs}} = 0.1$.

15 [6] Justify why the scaling factor a is found for the adaptive techniques when using one estimator that is $O(h^5)$ and another that is $O(h^6)$. You must give a full justification so that a student who has finished first-year can understand your reasoning; e.g., what are the assumptions, what are the constraints, etc. You must justify why values are used, as no marks will be given for just the value.

16 [1] Perform one step of 4th-order Runge Kutta with h = 2 for the following system of initial-value problems:

$$\begin{aligned} y^{(1)}(t) &= y(t)z(t) - 1 \\ z^{(1)}(t) &= y(t) - z(t) \\ y(0) &= 1 \\ z(0) &= 2 \end{aligned}$$

Your answer need not be evaluated to a single number, but you should be able to plug the numbers into a calculator to get the correct approximation.

17 [2] When using the shooting method for a boundary-value problem with a linear ordinary differential equation,

$$\begin{aligned} u^{(2)}(x) + a_1(x)u^{(1)}(x) + a_0(x)u(x) &= f(x) \\ u(a) &= 1 \\ u(b) &= 2 \end{aligned}$$

it is only necessary to find two solutions. How must we set up the problem so that we can use an initial-problem solver to find an approximation to the solution of the BVP? Justify your answer.

18 [2] In finding an approximation to a boundary-value problem with a linear ordinary differential equation as described in the above question with boundary values

$$u(0) = 1.3$$
$$u(1) = 4.0$$

you get the following four arrays of values:

[1.3, 1.3, 1.4, 1.5, 1.7, 2.0]
[0.0, 0.0, 0.1, 0.2, 0.3, 0.4]
[0.0, 0.2, 0.4, 0.6, 0.8, 1.0]
[1.0, 1.0, 1.0, 1.0, 1.0, 1.0]

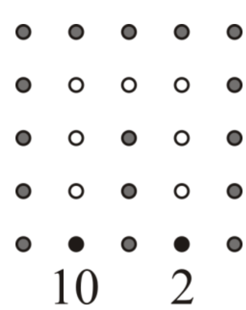
How would you combine these to get an approximation to solution the above boundary-value problem? Plot this solution given the corresponding *t* values [0.0, 0.3, 0.5, 0.7, 0.9, 1.0].

19 [2] Write the matrix that must be solved in order to find an approximation of the boundary-value problem

$$u^{(2)}(x) + u^{(1)}(x) + u(x) = \sin(x)$$
$$u(1) = 3$$
$$u(6) = 5$$

using six points using the finite-difference technique.

20 [3] Find the solution to Laplace's equation where you are given a room with a boundary wall. The temperature at one point (in black) is 10 degrees, while at another (also in black) it is 2 degrees. All other boundary points in gray are insulated. You must find the solution: no marks will be given for simply writing down the matrix, but full marks will be given for the correct solution.



21 [1] Explain how a system progresses into the future for the heat-conduction/diffusion equation if the initial state is a solution to Laplace's equation. Justify your answer.

22 [2] Approximate a solution to the following heat-conduction/diffusion equation using two intermediate points and with a step in time of $\Delta t = 1$ for two steps forward:

$$\frac{\partial}{\partial t}u(t,x) = 0.1\nabla^2 u(t,x)$$

The initial state is 0 everywhere, and where one boundary is at a temperature of 1 degree, while the other is at a temperature of 2 degrees. Use the explicit method.

22 [2] Approximate a solution to the following wave equation using two intermediate points and with a step in time of $\Delta t = 1$ for two steps forward:

$$\frac{\partial^2}{\partial t^2}u(t,x) = 0.1\nabla^2 u(t,x)$$

The initial state is 0 everywhere and it is also initially at rest, and where one boundary is held at 1, while the other is held at 2. Use the explicit method.

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