1 [3] List the six *tools* that we will use in this course for approximating solutions numerically.

2 [3] Describe the purpose of Horner's rule and show how it can be used to optimize the implementation of this polynomial that minimizes the relative error in approximating the sine function on the interval $[0, \pi/2]$.

$$-0.12982726700166469x^3 - 0.031041616418863258x^2 + 1.0034924244212799x$$

3 [4] Your system samples a quantity $y(t)$ once per 100 ms, and you would like to approximate the derivative using the standard three-point backward divided-difference method; however, occasionally, the last data-point is missing, so we must approximate the derivative using the last three available points:

$$y^{(1)}(t) \approx \frac{5y(t) - 9y(t-2h) + 4y(t-3h)}{6h}$$

You suspect the error is still $O(h^2)$, but determine the coefficient. Do you expect that coefficient of the error term to be larger or smaller in magnitude than the corresponding coefficient for the simper formula we saw in class:

$$y^{(1)}(t) \approx \frac{3y(t) - 4y(t-h) + y(t-2h)}{2h}?$$

3 [3] Your system samples a quantity $y(t)$ once per 100 ms, and you would like to approximate the second derivative using the standard three-point backward divided-difference method; however, occasionally, the last data-point is missing, so we must approximate the second derivative using the last three available points:

$$y^{(2)}(t) \approx \frac{y(t) - 3y(t-2h) + 2y(t-3h)}{3h^2}$$

You suspect the error must be $O(h)$, but determine the coefficient.

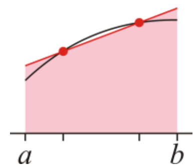
4 [3] Given that

$$f^{(1)}(x) = \frac{f(x+h) - f(x-h)}{2h} - \frac{1}{6} f^{(3)}(\xi) h^2,$$

show that this is correct by approximating the derivative of the function $f(x) = x^4 - 3x^3$ at the point $x = 2$ with $h = 1$. Be sure to describe why the error bounds are those you specify.

5 [3] Determine the error of approximating the integral $\int_a^b f(x)\,dx \approx f(a)(b-a)$ by integrating both sides of the Taylor series $f(a) = f(x) + f^{(1)}(\xi)(a-x)$.

6 [3] Suppose you realize that rather than using the end-points for the trapezoidal rule, yielding the approximation $\frac{f(a)+f(b)}{2}(b-a)$, you decide instead to use base the trapezoid on the two points 25% and 75% along the interval [a, b], as shown in this image:



What is the approximation of the integral for this modification to the trapezoidal rule?

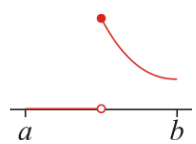
Do you expect this to give a better or worse approximation of the integral when compared with the trapezoidal rule? Justify your answer.

7 [2] Suppose that $\tau_1, \tau_2, \tau_3, \tau_4$ are four points on the interval [a, b]. Simply the following expression using the intermediate-value theorem, as used in class:

$$\frac{1}{16}y^{(3)}(\tau_1)h^2 + \frac{1}{8}y^{(3)}(\tau_2)h^2 + \frac{1}{8}y^{(3)}(\tau_3)h^2 + \frac{1}{16}y^{(3)}(\tau_4)h^2.$$

You may assume the third-derivative is continuous.

8 [2] Why is the error analysis used in class for approximating integrals useless when trying to approximate a function with a discontinuity? For example, consider the function shown here and trying to approximate the integral from a to b.



9 [2] Suppose that you have a thermometer that measures the temperature once every 100 seconds, and you would like to estimate the average temperature throughout the day given by the formula $\frac{1}{86400}\int_0^{86400} T(t)\,dt$. By the end of the day, you have 865 readings (there are 86400 s/day). Explain why, if the error for the composite Simpson's rule is $-f^{(4)}(\xi)\frac{b-a}{180}h^4$, what are the known quantities and why is this likely still a very good estimator of the average temperature.

10 [2] Your personal aircraft determines its position once per second to get its position via the Global Positioning System (GPS) and an altimeter yielding coordinates (x_0, y_0, z_0), (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), … . Explain how you can use the formula $0.8y_n + 0.5\,y_{n-1} + 0.2y_{n-2} - 0.1y_{n-3} - 0.4y_{n-4}$ that estimates the expected position one second into the future (at time t_{n+1}) to determine an estimator for the position of the aircraft at time t_{n+1}.

11 [1] Justify, in your own words, why finding the slope of the least-squares best-fitting line is a better approximation of the derivative when there is noisy data than finding the derivative of an interpolating polynomial.

12 [2] Suppose you have two resistors in series, both with a labeled resistance of 100 Ω, but each has an error described by a standard deviation of 10 Ω. What is the best estimator of the resistance across the two resistors, and what is the best estimator of the standard deviation of this the resistance?

13 [3] Starting with the Taylor series, assume that you are a distance h from a root r. What is your distance to the root after one iteration of Newton's method.

14 [2] You have the function $f(x) = x^2 - 0.04$. You are aware that a root exists on the interval [0, 1]. Apply one step of the constrained secant method to find a better approximation of the root, and explain which end-point you will update and why.

14 [2] You have the function $f(x) = x^2 - 0.04$. Apply one step of Newton's method starting with the approximation $x_0 = 1$ to find a better approximation of the root.

15 [4] You want to find a simultaneous root of the two non-linear equations

$$x^2 - 2y - 1 = 0$$
$$3x - y^2 - 1 = 0$$

Your first approximation of a root is the point $(x, y) = (2, 2)$. Find the next approximation after one step of applying Newton's method in two dimensions. Please note, the corresponding system of linear equations has a relatively simple solution.