ECE 204 *Numerical methods*

Sections 001, 002

MIDTERM EXAMINATION

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Rooms: E7-4043, 4053, 4417, 5343

1. **You may rip off the last page as soon as you sit down.**
2. There are 34 marks.
3. No aides.
4. Turn off all electronic media and store them under your desk.
5. You may ask only one question during the examination: “May I go to the washroom?”
6. Asking **any** other question **will** result in a deduction of 5 marks from the exam grade.
7. If you think a question is ambiguous, write down your assumptions and continue.
8. **Do not leave during first half hour or after there are only 15 minutes left.**
9. Do not stand up until all exams have been picked up.
10. If a question only asks for an answer, you do not have to show your work to get full marks; however, if your answer is wrong and no rough work is presented to show your steps, no part marks will be awarded.
11. The questions are in the order of the course material.

1 [2] Describe fixed-point iteration: What is the problem we are trying to solve, how is the solution being approximated, and under what circumstances does fixed-point iteration find an approximation to the solution this problem?

2 [2] Sum the following two double-precision floating-point numbers and write the result in the same format:

```
a3d4000000000000     23d6000000000000
```

3 [2] Multiply the following two floating-point numbers and write the result in the same format:

```
4000000000000000     3ffd000000000000
```

5 [4] Demonstrate, using 3rd-order Taylor series, that the error of this approximation of the derivative

$$f^{(1)}(x) \approx \frac{f(x+h) - f(x-h)}{2h}$$

is $O(h^2)$.

6 [2] Suppose you know not only $y(t_k)$ and $y(t_{k+1})$, but we also know $y^{(1)}(t_k)$ and $y^{(1)}(t_{k+1})$ and $y^{(2)}(t_k)$ and $y^{(2)}(t_{k+1})$, and assuming that these are accurate, what would be the approach to estimating the value of $y(t)$ for some value of $t_k < t < t_{k+1}$? You do not have to derive anything, give instructions as to how someone could set up the most appropriate system of linear equations that must be solved to find this approximation.

7 [3] Explain what tools we used and how to get the approximation

$$\int_{t-h}^{t} y(t)\,\mathrm{d}t \approx \frac{1}{12}\left(-y(t-2h)+8y(t-h)+5y(t)\right)h.$$

You do not actually have to derive the formula, but rather, explain the steps so as someone can follow those steps to find the above formula.

8 [1] Simpson's rule for approximating an integral is:

$$\int_{a}^{a+2h} f(x)\,dx \approx \frac{f(a)+4f(a+h)+f(a+2h)}{6}(2h)$$

where the error is proportional to $f^{(4)}(x)$. Suppose you were trying to approximate the integral of $\int_{a}^{a+2h} |f(x)|\,dx$ by using the formula

$$\frac{|f(a)|+4|f(a+h)|+|f(a+2h)|}{6}(2h).$$

Why would this approximation not be very good if there was a single root of f (that is, a root of multiplicity one) on the interval $[a, a + 2h]$?

9 [3] Given a very noisy power signal (watts) where the last five readings are 9 W (most recent), 8 W, 8 W, 9 W and 10 W (least recent) taken at 100 ms apart, and you know that the concavity is essentially zero on this interval, what is a reasonable approximation of the next reading one time step into the future?

10 [2] Two successive readings of a voltage are 0.8 and –0.2, and these readings were taken at times $t = 1.2$ and $t = 1.3$. Assuming continuity and differentiability, what is the best possible approximation of the time that the voltage was zero, and which technique are you using?

11 [4] Demonstrate that if the error for an approximation x_0 of a root r is $r - x_0$, show that after one iteration of Newton's method, the error $r - x_1$ is now proportional to $(x_0 - r)^2$ assuming that the second derivative is bounded between the approximation x_0 and the root r. Recall that

$$f(r) = f(x) + f^{(1)}(x)(r - x) + \tfrac{1}{2} f^{(2)}(x)(r - x)^2.$$

12 [6] Suppose you have a sensor that is being sampled periodically with period h. Your embedded system is only recording the two most recent values. Suppose you have a mechanism for determining when a reading from the sensor is faulty. In this case, it is better to use a best estimate for the current sensor reading.

Suppose that `y[1]` was the most recent sensor reading (one time stem into the past). What would be the error of using that reading as an estimator for the most current reading, recalling that

$$y(t - h) = y(t) - y^{(1)}(\tau)h.$$

Suppose that `y[1]` is the most recent reading and `y[0]` is the second most-recent reading. What would be a best estimator of the most the current sensor reading? Recall that

$$y(t - h) = y(t) - y^{(1)}(t)h + \tfrac{1}{2}y^{(2)}(\tau)h^2$$
$$y(t - 2h) = y(t) - 2y^{(1)}(t)h + 2y^{(2)}(\tau)h^2.$$

All of this requires nothing more than a judicious application of secondary school arithmetic.

13 [3] Use one step of Newton's method to find the next approximation of the solution to the two nonlinear equations

$$x^2 + x + y = 0.1 \text{ and } x - y^2 = -0.2$$

if your first approximation to a solution is $(x, y) = (0, 0)$.

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Floating-point representations

±EEMNNN ±M.NNN × 10^{EE-49}

seeeeeeeeeeebbbbbb…b $(-1)^s$ 1.bbbbbb…b × $10^{eeeeeeeeeee-01111111111}$

where $01111111111_2 = 1023$.

Fixed-point theorem: Solving $x = f(x)$, choose x_0 and let $x_k \leftarrow f(x_{k-1})$.

Gaussian elimination with partial pivoting is the Gaussian elimination algorithm but always swapping appropriate rows so that the largest entry is in the row that will be used to eliminate that term in all subsequent rows.

$$f(x+h) = \left(\sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(x) h^k\right) + \frac{1}{(n+1)!} f^{(n+1)}(\xi) h^{n+1} \quad \text{where } x < \xi < x+h.$$

$$f(x) = \left(\sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(x_0)(x-x_0)^k\right) + \frac{1}{(n+1)!} f^{(n+1)}(\xi) h^{n+1} \quad \text{where } x_0 < \xi < x.$$

Averaging noisy values with zero bias mitigates the effect, while differentiating noisy values magnifies the effect.

```
double horner( double a[], unsigned int degree; double x ) {
    double result{a[0]};
    for ( std::size_t k{1}; k <= degree; ++k ) {
        result += result*x + a[k];
    }
    return 0;
}
```

Formula of interest:

$$f^{(1)}(x) = \frac{f(x+h) - f(x-h)}{2h} - \frac{1}{6} f^{(3)}(\xi) h^2 \qquad y^{(1)}(t) = \frac{y(t) - y(t-h)}{h} + \frac{1}{2} y^{(2)}(\tau_-) h$$

$$y^{(1)}(t) = \frac{3y(t) - 4y(t-h) + y(t-2h)}{2h} + \frac{1}{3} y^{(3)}(\tau) h^2$$

$$f^{(2)}(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} - \frac{1}{12} f^{(4)}(\xi) h^2 \qquad y^{(2)}(t) = \frac{y(t) - 2y(t-h) + y(t-2h)}{h^2} + y^{(3)}(\tau) h$$

$$\int_a^b f(x)\,dx = \left(\frac{1}{2} f(a) + \frac{1}{2} f(b)\right)(b-a) - \frac{1}{12} f^{(2)}(\xi)(b-a)^3$$

$$\int_a^b f(x)\,dx = \frac{1}{6}(f_0 + 4f_1 + f_2)(b-a) - \frac{1}{2880} f^{(4)}(\xi)(b-a)^5$$

$$\int_a^b f(x)\,dx = \frac{1}{8}(f_0 + 3f_1 + 3f_2 + f_3)(b-a) - \frac{1}{6480} f^{(4)}(\xi)(b-a)^5$$

$$\int_a^b f(x)\,dx = \frac{1}{2}\left(f(a) + 2\left(\sum_{k=1}^{n-1} f(a+kh)\right) + f(b)\right) h - f^{(2)}(\xi) \frac{b-a}{12} h^2$$

$$\int_a^b f(x)\,dx = \frac{1}{3}\left(f_0 + 4\sum_{k=1}^{\frac{n}{2}} f_{2k-1} + 2\sum_{k=1}^{\frac{n}{2}-1} f_{2k} + f_n \right) h - f^{(4)}(\xi) \frac{b-a}{180} h^4$$

$$\int_a^b f(x)\,dx = \frac{3}{8}\left(f(a) + 3\left(\sum_{k=1}^{\frac{n}{3}} f(a+(3k-2)h)\right) + 3\left(\sum_{k=1}^{\frac{n}{3}} f(a+(3k-1)h)\right) + 2\left(\sum_{k=1}^{\frac{n}{3}-1} f(a+3kh)\right) + f(b)\right) h - f^{(4)}(\xi) \frac{b-a}{80} h^4$$

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Goal	Estimation
Estimate $y(t_n)$	$0.6\, y_n + 0.4\, y_{n-1} + 0.2\, y_{n-2} - 0.2\, y_{n-4}$
Estimate $y(t_n + h)$	$0.8\, y_n + 0.5\, y_{n-1} + 0.2\, y_{n-2} - 0.1\, y_{n-3} - 0.4\, y_{n-4}$
Estimate the rate of change of y over time	$\frac{0.2y_n + 0.1y_{n-1} - 0.1y_{n-3} - 0.2y_{n-4}}{h}$
Estimate the integral $\int_{t_n-4h}^{t_n} y(t)\,\mathrm{d}t$	$(4h)(0.2y_n + 0.2y_{n-1} + 0.2y_{n-2} + 0.2y_{n-3} + 0.2y_{n-4})$
Estimate the integral $\int_{t_n-h}^{t_n} y(t)\,\mathrm{d}t$	$h(0.5y_n + 0.35y_{n-1} + 0.2y_{n-2} + 0.05y_{n-3} - 0.1y_{n-4})$

Goal	Estimation
Estimate $y(t_n)$	$\frac{1}{35}(31y_n + 9y_{n-1} - 3y_{n-2} - 5y_{n-3} + 3y_{n-4})$
Estimate $y(t_n + h)$	$1.8y_n - 0.8y_{n-2} - 0.6y_{n-3} + 0.6y_{n-4}$
Estimate the rate of change of y over time at time t_n	$\frac{54y_n - 13y_{n-1} - 40y_{n-2} - 27y_{n-3} + 26y_{n-4}}{70h}$
Estimate the acceleration of y over time at time t_n	$\frac{2y_n - y_{n-1} - 2y_{n-2} - y_{n-3} + 2y_{n-4}}{7h^2}$
Estimate the integral $\int_{t_n-4h}^{t_n} y(t)\,\mathrm{d}t$	$(4h)\frac{11y_n + 26y_{n-1} + 31y_{n-2} + 26y_{n-3} + 11y_{n-4}}{105}$
Estimate the integral $\int_{t_n-h}^{t_n} y(t)\,\mathrm{d}t$	$h\frac{230y_n + 137y_{n-1} + 64y_{n-2} + 11y_{n-3} - 22y_{n-4}}{420}$

Method	Requirements	Iteration step	Rate of convergence	Is convergence guaranteed?
Bisection	An interval $[a, b]$ with $f(a)$ having the opposite sign of $f(b)$	Let $c \leftarrow \frac{a+b}{2}$ and update whichever endpoint has the same sign as $f(c)$.	$O(h)$	Yes
Bracketed secant	An interval $[a, b]$ with $f(a)$ having the opposite sign of $f(b)$	Let $c \leftarrow \frac{af(b) - bf(a)}{f(b) - f(a)}$ and update whichever endpoint has the same sign as $f(c)$.	$O(h)$	Yes
Secant	Two initial approximations x_0 and x_1 with $\lvert f(x_0)\rvert > \lvert f(x_1)\rvert$	Let $x_k \leftarrow \frac{x_{k-2} f(x_{k-1}) - x_{k-1} f(x_{k-2})}{f(x_{k-1}) - f(x_{k-2})}$.	$O(h^{\phi})$	No
Newton's	An initial approximation x_0	Let $x_k \leftarrow x_{k-1} - \frac{f(x_{k-1})}{f^{(1)}(x_{k-1})}$.	$O(h^2)$	No

Given a function $f(x, y)$ and an approximation to a root (x_k, y_k), we can solve

$$\begin{pmatrix} \frac{\partial}{\partial x} f(x_k, y_k) & \frac{\partial}{\partial y} f(x_k, y_k) \\ \frac{\partial}{\partial x} g(x_k, y_k) & \frac{\partial}{\partial y} g(x_k, y_k) \end{pmatrix} \begin{pmatrix} \Delta x_k \\ \Delta y_k \end{pmatrix} = \begin{pmatrix} -f(x_k, y_k) \\ -g(x_k, y_k) \end{pmatrix}$$

and then let $x_{k+1} \leftarrow x_k + \Delta x_k$, $y_{k+1} \leftarrow y_k + \Delta y_k$.