ECE 204 Examples of Newton's method

Douglas Wilhelm Harder

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1 Introduction

In this document, we look at four examples of Newton's method in two, three and four dimensions. Three of the four are polynomials, while the third example consists of a system of trigonometric functions.

2 Example 1

The solutions to the equation to $x^2+y^2 = 10$ form a circle, while the solutions to the equation $x^2-y^2 = 1$ form a hyperbola. Finding solutions to these equations is equivalent to finding simultaneous roots of the two bivariate functions

$$x^2 + y^2 - 10 \text{ and } x^2 - y^2 - 1.$$

Plotting these two functions of two variables, we see them in Figure 1. The solutions to these functions equalling zero are where the functions intersect the gray plane $z = 0$.

Looking at this graph from above, as shown in Figure 2, we see that there appear to be four points where both functions are zero.

There are four simultaneous solutions to these two root-finding problems, as the circle (the solutions to the first equation) intersects the hyperbola (the solutions to the second equation) at four points. Thus, let us define

$$\mathbf{f}(\mathbf{u}) = \begin{pmatrix} u_1^2 + u_2^2 - 10 \\ u_1^2 - u_2^2 - 1 \end{pmatrix}$$

and calculate the Jacobian:

$$J(\mathbf{f})(\mathbf{u}) = \begin{pmatrix} 2u_1 & 2u_2 \\ 2u_1 & -2u_2 \end{pmatrix}.$$

Now, we note that $\mathbf{u}_0 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ evaluates to $\mathbf{f}(\mathbf{u}_0) = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$, so we are not really too close to a root, as the value of both functions are not very close to zero. We can, however, apply one step of Newton's method, so solve:

$$J(\mathbf{f})(\mathbf{u}_0)\Delta\mathbf{u}_0 = -\mathbf{f}(\mathbf{u}_0),$$

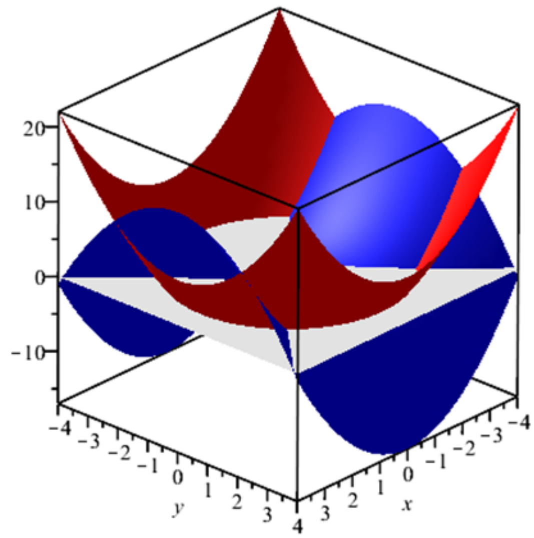

Figure 1: The function $x^2 + y^2 - 10$ in red and $x^2 - y^2 - 1$ in blue with the plane $z = 0$ in gray.

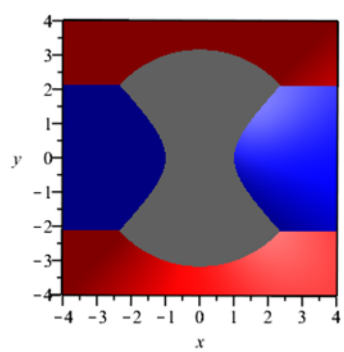

Figure 2: The functions in Figure 1 shown from above.

so substituting the values into the Jacobian, we have

$$\begin{pmatrix} 4 & 4 \\ 4 & -4 \end{pmatrix} \Delta\mathbf{u}_0 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

The solution to this system of linear equations is

$$\Delta\mathbf{u}_0 = \begin{pmatrix} 0.375 \\ 0.125 \end{pmatrix}$$

, and thus our next approximation is $\mathbf{u}_1 \leftarrow \mathbf{u}_0 + \Delta\mathbf{u} = \begin{pmatrix} 2.375 \\ 2.125 \end{pmatrix}$.

Note that $\mathbf{f}(\mathbf{u}_1) = \begin{pmatrix} 0.15625 \\ 0.125 \end{pmatrix}$. We can now iterate again, with

$$J(\mathbf{f})(\mathbf{u}_1)\Delta\mathbf{u}_1 = -\mathbf{f}(\mathbf{u}_1)$$

or

$$\begin{pmatrix} 4.75 & 4.25 \\ 4.75 & -4.25 \end{pmatrix} \Delta\mathbf{u}_1 = \begin{pmatrix} -0.15625 \\ -0.125 \end{pmatrix}$$

so $\Delta\mathbf{u}_1 = \begin{pmatrix} -0.029605263 \\ -0.003676471 \end{pmatrix}$, and thus $\mathbf{u}_2 \leftarrow \mathbf{u}_1 + \Delta\mathbf{u}_1 = \begin{pmatrix} 2.345394737 \\ 2.121323529 \end{pmatrix}$. Again, we note that this is $\mathbf{f}(\mathbf{u}_2) = \begin{pmatrix} 0.000889988 \\ 0.000862955 \end{pmatrix}$, so once again, we are closer to a simultaneous root. Iterating again, we have

$$J(\mathbf{f})(\mathbf{u}_2)\Delta\mathbf{u}_2 = -\mathbf{f}(\mathbf{u}_2)$$

or

$$\begin{pmatrix} 4.6907894737 & 4.2426470588 \\ 4.6907894737 & -4.2426470588 \end{pmatrix} \Delta\mathbf{u}_2 = \begin{pmatrix} -0.000889988 \\ -0.000862955 \end{pmatrix}$$

, so $\Delta\mathbf{u}_2 = \begin{pmatrix} -0.000186849 \\ -0.000003186 \end{pmatrix}$, and thus $\mathbf{u}_3 \leftarrow \mathbf{u}_2 + \Delta\mathbf{u}_2 = \begin{pmatrix} 2.345207887 \\ 2.121320344 \end{pmatrix}$. Again, we note that this is $\mathbf{f}(\mathbf{u}_3) = \begin{pmatrix} 0.000000035 \\ 0.000000035 \end{pmatrix}$, so again, we appear to be much closer to a simultaneous root. The actual solution, to 20 significant digits, is $\mathbf{u} = \begin{pmatrix} 2.3452078799117147773 \\ 2.1213203435596425732 \end{pmatrix}$, so after only three iterations, we have approximately nine significant digits of accuracy.

3 Example

The solutions to the equation to $2x^2 + 2xy - x + 3y + 5y^2 = 10$ form an ellipse, while the solutions to the equation $3x^2 - xy - 3x + 2y - 2y^2 = 1$, again, form a hyperbola. Finding solutions to these equations is equivalent to finding roots of the two bivariate functions $2x^2+2xy-x+3y+5y^2-10$ and $3x^2-xy-3x+2y-$

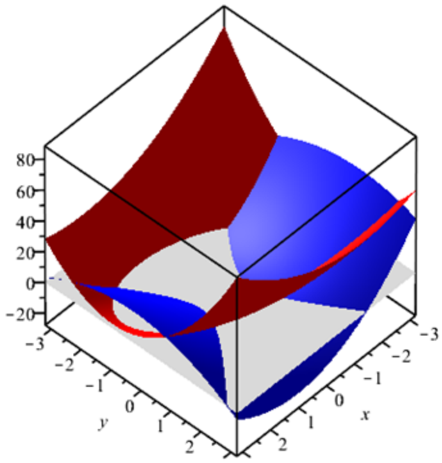

Figure 3: The function $2x^2 + 2xy - x + 3y + 5y^2 - 10$ in red and $3x^2 - xy - 3x + 2y - 2y^2 - 1$ in blue with the plane $z = 0$ in gray.

$2y^2 - 1$. Plotting these two functions of two variables, we see them in Figure 3. The solutions to these functions equalling zero are where the functions intersect the gray plane $z = 0$.

Looking at this graph from above, as shown in Figure 4, we see that there appear to be four points where both functions are zero.

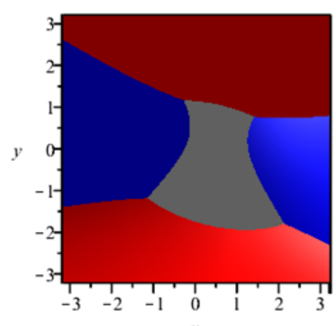

Figure 4: The functions in Figure 3 shown from above.

We note that there appears to be a root close to $x = 2$ and $y = -2$, so we will start at this point. There are four simultaneous solutions to these two root-finding problems, as the ellipse (the solutions to the first equation) intersects the hyperbola (the solutions to the second equation) at four points. Thus, let us define

$$\mathbf{f}(\mathbf{u}) = \begin{pmatrix} 2u_1^2 + 2u_1u_2 - u_1 + 3u_2 + 5u_2^2 - 10 \\ 3u_1^2 - u_1u_2 - 3u_1 + 2u_2 - 2u_2^2 - 1 \end{pmatrix}$$

and calculate the Jacobian:

$$J(\mathbf{f})(\mathbf{u}) = \begin{pmatrix} 4u_2 + 2u_2 - 1 & 2u_1 + 3 + 10u_2 \\ 6u_1 - u_2 - 3 & -u_1 + 2 - 4u_2 \end{pmatrix}.$$

Now, we note that $\mathbf{u}_0 = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$ evaluates to $\mathbf{f}(\mathbf{u}_0) = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$, so we are not really too close to a root, as the value of both functions are not very close to zero. We can, however, apply one step of Newton's method, so solve:

$$J(\mathbf{f})(\mathbf{u}_0)\Delta\mathbf{u}_0 = -\mathbf{f}(\mathbf{u}_0),$$

so substituting the values into the Jacobian, we have

$$\begin{pmatrix} 3 & -13 \\ 11 & 8 \end{pmatrix} \Delta\mathbf{u}_0 = \begin{pmatrix} -2 \\ 3 \end{pmatrix}.$$

The solution to this system of linear equations is $\Delta\mathbf{u}_0 = \begin{pmatrix} 0.137724551 \\ 0.185628743 \end{pmatrix}$, and thus our next approximation is $\mathbf{u}_1 \leftarrow \mathbf{u}_0 + \Delta\mathbf{u}_0 = \begin{pmatrix} 2.137724551 \\ -1.814371257 \end{pmatrix}$. Note that $\mathbf{f}(\mathbf{u}_1) = \begin{pmatrix} 0.261357524 \\ -0.037577540 \end{pmatrix}$. We can now iterate again, with

$$J(\mathbf{f})(\mathbf{u}_1)\Delta\mathbf{u}_1 = -\mathbf{f}(\mathbf{u}_1) or \begin{pmatrix} 3.922155689 & -10.868263473 \\ 11.640718563 & 7.119760479 \end{pmatrix} \Delta\mathbf{u}_1 = \begin{pmatrix} -0.261357524 \\ 0.037577540 \end{pmatrix},$$

so $\Delta\mathbf{u}_1 = \begin{pmatrix} -0.009404350 \\ 0.020653916 \end{pmatrix}$, and thus $\mathbf{u}_2 \leftarrow \mathbf{u}_1 + \Delta\mathbf{u}_1 = \begin{pmatrix} 2.128320201 \\ -1.793717342 \end{pmatrix}$. Again, we note that this is $\mathbf{f}(\mathbf{u}_2) = \begin{pmatrix} 0.001921331 \\ -0.000393606 \end{pmatrix}$, so once again, we are closer to a simultaneous root. Iterating again, we have

$$J(\mathbf{f})(\mathbf{u}_2)\Delta\mathbf{u}_2 = -\mathbf{f}(\mathbf{u}_2) \text{ or}$$
$$\begin{pmatrix} 3.925846119 & -10.680533015 \\ 11.563638546 & 7.046549166 \end{pmatrix} \Delta\mathbf{u}_2 = \begin{pmatrix} -0.001921331 \\ 0.000393606 \end{pmatrix},$$

so $\Delta\mathbf{u}_2 = \begin{pmatrix} -0.000061751 \\ 0.000157193 \end{pmatrix}$, and thus $\mathbf{u}_3 \leftarrow \mathbf{u}_2 + \Delta\mathbf{u}_2 = \begin{pmatrix} 2.128258450 \\ -1.793560148 \end{pmatrix}$. Again, we note that this is $\mathbf{f}(\mathbf{u}_3) = \begin{pmatrix} 0.000000112 \\ -0.000000028 \end{pmatrix}$, so again, we appear to be much closer to a simultaneous root. The actual solution, to 20 significant digits, is $u = \begin{pmatrix} 2.1282584467227969135 \\ -1.7935601391539985358 \end{pmatrix}$, so after only three iterations, we have approximately nine significant digits of accuracy.

4 A trigonometric example in three dimensions

Suppose we want to find a solution to the problem

$$\begin{aligned}\cos(xy+2z) &= 0.5\\ \cos(xz+3y) &= -0.25\\ \cos(yz+4x) &= 0.7\end{aligned}$$

Rewrite these as a root-finding problem, so finding a zero of

$$\mathbf{f}(\mathbf{u}) = \begin{pmatrix} \cos(u_1u_2+2u_3) - 0.5) \\ \cos(u_1u_3+3u_2) + 0.25 \\ \cos(u_2u_3+4u_1) - 0.7) \end{pmatrix}$$

and calculate the Jacobian:

$$J(\mathbf{f})(\mathbf{u}) = \begin{pmatrix} -\sin(u_1u_2+2u_3)u_2 & -\sin(u_1u_2+2u_3)u_1 & -2\sin(u_1u_2+2u_3) \\ -\sin(u_1u_3+3u_2)u_3) & -3\sin(u_1u_3+3u_2) & -\sin(u_1u_3+3u_2)u_1) \\ -4\sin(u_2u_3+4u_1) & -\sin(u_2u_3+4u_1)u_3 & -\sin(u_2u_3+4u_1)u_2) \end{pmatrix}.$$

Now, we note that $\mathbf{u}_0 = \begin{pmatrix} 6.0 \\ -3.3 \\ 0.1 \end{pmatrix}$ evaluates to $\mathbf{f}(\mathbf{u}_0) = \begin{pmatrix} -0.091917938 \\ -0.225536928 \\ -0.005262343 \end{pmatrix}$, so we are close to a root, as the value of three functions are all less than 0.25 in absolute value. We can, however, apply one step of Newton's method, so solve:

$$J(\mathbf{f})(\mathbf{u}_0)\Delta\mathbf{u}_0 = -\mathbf{f}(\mathbf{u}_0),$$

so substituting the values into the Jacobian, we have

$$\begin{pmatrix} -3.012719327 & 5.477671504 & 1.825890501 \\ 0.087969576 & -2.639087280 & -5.278174560 \\ 2.877052903 & -0.071926323 & -2.373568645 \end{pmatrix} \Delta\mathbf{u}_0 = \begin{pmatrix} 0.091917938 \\ 0.225536928 \\ 0.005262343 \end{pmatrix}.$$

The solution to this system of linear equations is $\Delta\mathbf{u}_0 = \begin{pmatrix} -0.038606344 \\ 0.012005801 \\ -0.049376440 \end{pmatrix}$, and thus our next approximation is $\mathbf{u}_1 \leftarrow \mathbf{u}_0 + \Delta\mathbf{u}_0 = \begin{pmatrix} 5.961393656 \\ -3.287994199 \\ -0.149376440 \end{pmatrix}$. Note that $\mathbf{f}(\mathbf{u}_1) = \begin{pmatrix} -0.002623891 \\ 0.011229114 \\ -0.000442121 \end{pmatrix}$ and the actual solution, to approximately 20 significant digits, is $\mathbf{u} = \begin{pmatrix} 5.9628757701217867163 \\ -3.2871989138030924414 \\ -0.1477973590241207728 \end{pmatrix}$, so with one step, we have progressed towards the solution.

5 A polynomial example in four dimensions

The following is a system of four equations

$$\begin{aligned} 3w^2 - 4x + 4y - 2z &= -0.8 \\ -5w + 4x^2 - 3y - 5z &= -18.1 \\ 6w - 7x + 2y^2 + 7z &= 38.1 \\ -3w - 3x + 2y + 5z^2 &= 76.8 \end{aligned}$$

Thus, our function is

$$\mathbf{f}(\mathbf{u}) = \begin{pmatrix} 3u_1^2 - 4u_2 + 4u_3 - 2u_4 + 0.8 \\ -5u_1 + 4u_2^2 - 3u_3 - 5u_4 + 18.1 \\ 6u_1 - 7u_2 + 2u_3^2 + 7u_4 - 38.1 \\ -3u_1 - 3u_2 + 2u_3 - 5u_4^2 - 76.8 \end{pmatrix}$$

which has the Jacobian

$$\mathbf{f}(\mathbf{u}) = \begin{pmatrix} 6u_1 & -4 & 4 & -2 \\ -5 & 8u_2 & -3 & 5 \\ 6 & -7 & 4u_3 & 7 \\ -3 & -3 & 2 & 10u_4 \end{pmatrix}$$

There happens to be solution close to $\mathbf{u}_0 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$, and $\mathbf{f}(\mathbf{u}_0) = \begin{pmatrix} -0.2 \\ 0.1 \\ -0.1 \\ 0.2 \end{pmatrix}$.

Thus, we need to iterate:

$$J(\mathbf{f})(\mathbf{u}_0)\Delta\mathbf{u}_0 = -\mathbf{f}(\mathbf{u}_0)$$

so substituting the values into the Jacobian, we have

$$\begin{pmatrix} 6 & -4 & 4 & -2 \\ -5 & 16 & -3 & -5 \\ 6 & -7 & 12 & 7 \\ -3 & -3 & 2 & 40 \end{pmatrix} \Delta\mathbf{u}_0 = \begin{pmatrix} 0.2 \\ -0.1 \\ 0.1 \\ -0.2 \end{pmatrix}$$

so a solution to this is $\Delta\mathbf{u}_0 = \begin{pmatrix} 0.04274 \\ 0.00502 \\ -0.00956 \\ -0.00094 \end{pmatrix}$ and thus $\mathbf{u}_1 \leftarrow \mathbf{u}_0 + \Delta\mathbf{u} =$ $\begin{pmatrix} 1.04274 \\ 2.00502 \\ 2.99044 \\ 3.99906 \end{pmatrix}$ and we note that $\mathbf{f}(\mathbf{u}_1) = \begin{pmatrix} 0.0054801228 \\ 0.0001008016 \\ 0.0001827872 \\ 0.000004418 \end{pmatrix}$, which you can see is much closer to a simultaneous root of all four equations. The correct solution to ten decimal digits of precision is $\mathbf{u} = \begin{pmatrix} 1.041089216 \\ 2.004562329 \\ 2.991096698 \\ 3.998868871 \end{pmatrix}$.