Buffer Occupancy Asymptotics in Networks with Heterogeneous Long-tailed Inputs

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Abstract

In this paper, we consider a network in which sessions with long-tailed session lengths arrive as Poisson processes. In particular, we assume that a session of type n transmits r_n cells per unit time and lasts for a random time τ_n with a generalized Pareto distribution given by $\mathbb{P}(\tau_n > x) \sim \alpha_n x^{-(1+\beta_n)}$ for large x, where $\alpha_n, \beta_n > 0$. The network is assumed to be loop-free with respect to source-destination routes. We characterize the order asymptotics of the complementary buffer occupancy distribution at each node in terms of the input characteristics of the sessions. In particular, we show that the distributions obey a power law whose exponent can be calculated via solving a fixed point and deterministic knapsack problem. The paper concludes with some canonical examples.

1 Introduction

Long range dependence and self similarity of network traffic have been demonstrated consistently in many studies, starting with the work of Leland *et. al.* [1]. There are many explanations for the presence of long-range dependence in network traffic. One explanation is that such effects arise due to the presence of sessions (traffic activity) with long duration. A detailed statistical analysis of such sessions suggests that they are well modelled by ON-OFF type of processes where the ON periods, of random duration, have a tail distribution

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which decays according to a long tailed distribution. Different definitions are used in the literature for long and heavy tails. Here we assume that a r.v. X is long-tailed if $\mathbb{P}\{X > x\} \sim const.\ x^{-\beta}$ with $\beta > 0$. If $\beta \in (0, 2]$, then X is said to heavy-tailed and thus it has infinite variance. When many independent such sessions arrive randomly, the aggregated input process is long range dependent.

The performance of networks and their ability to offer Quality of Service (QoS) depend on accurately capturing the parametric dependence of the QoS measures such as the delay or loss distributions. While calculating these distributions exactly is intractable, the asymptotics of the tail distribution are much more manageable. The stringent QoS requirements in terms of the tail of the delay distribution or the packet loss probabilities means that the asymptotic regime is quite appropriate to analyze.

Numerous studies have shown that the presence of self-similar traffic forces us to change the way the buffer dimensioning is done in that much more buffering is necessary to achieve similar buffer overflow characteristics than with conventional or short-range dependent traffic. Indeed, the way buffer overflows occur can be very different. This is essentially due to the way that large excursions of the buffer workload take place. For conventional traffic models, the tail of the stationary workload distribution is exponential while for long range dependent traffic, it has an asymptotic long-tail or sub-exponential decay.

There are many results available for buffer asymptotics of a single node with long-tailed and sub-exponential inputs. These have been obtained under different hypotheses. These hypotheses relate to conditions on the session length distribution decay rates, session transmission rates and models for session arrivals. All results deal with stationary queues where the average rate is less than capacity. The vast majority of results for FIFO systems are for source transmission rates being identical, referred to as the homogeneous case. These can be found in [2–7], for example. An excellent survey for the homogeneous case when sources are identical can be found in [8]. It has however been shown in [9] that assuming the same transmission rates can lead to erroneous conclusions on the asymptotic behavior of the tail distribution of the buffer length when the inputs have long-tailed session lengths. This was obtained for the so-called $M/G/\infty$ model where sessions arrive according to a Poisson process, transmit at different rates and have differing long-tailed random session holding times. Similar results, but for a fixed number of ON-OFF sources have also been obtained in [10].

The extension of above results to a general network is quite difficult since the traffic loses its simple parametric structure after passing through its entrance node. Networks with exponentially distributed service times were considered in [11] and [12]. In [13], the author considers the large deviations problem for feedforward networks in heavy traffic. The papers were basically concerned with the computation of the rate functions associated with the tail distributions of the buffer occupancy. This is a difficult variational problem. In general, when the input and output rate functions in queues have so-called “linear geodesics” an end-to-end analysis is feasible by an iterative procedure. In [14,15] it is shown that in queues with many inputs the linear geodesic property does not hold and thus calculating the rate functions of the outputs is difficult.

There are very few results available for networks with long-tailed service times. In [16], the authors consider the (max, plus) setting and derive asymptotics of the distributions of total response times for networks which include tandem queues. In [17], a feedforward network is analyzed when a node with subexponential service time has upstream nodes with lighter tailed service times.

In this paper, we consider a network with fixed loop-free routing in which sessions arrive independently as Poisson processes and transmit during their session duration at a fixed rate. The session durations are assumed to be long-tailed, having an asymptotic power law or generalized Pareto decay. The sessions are assumed to be heterogeneous, i.e., have different session distributions and differing transmission rates. This is gives rise to an $M/G/\infty$ type of model for the inputs. Although the path for a single end-to-end route is loop free, the interaction between the flows in the network means that the independence between flows does not hold within the network. We obtain the O-asymptotics (i.e. we identify the asymptotic power law decay) for the buffer occupancies at each node. The utility of this result is that we can identify the sources that are most problematic and also obtain rough estimates for the loss and delay distributions.

For the single node case, exact asymptotics of buffer occupancy and loss in obtained in [9] also showed that these asymptotics are not only governed by the tail distribution of sources but also depend on their rates. As we will show, this dependence is valid also in a general loop-free network. Due to the correlations of traffic inside the network, transmission rates and average loads are modified during large buffer exceedance times. Furthermore, since we do not assume that the network is feedforward, these “modified” rates are obtained as the fixed point solution of a network equation. It is difficult to improve these results (e.g. obtain exact asymptotics) since the Poisson arrival structure and independence of sessions are lost for the traffic inside the network.

The organization of this paper is as follows: In Section 2, we formulate the model and present the preliminaries. Section 3 contains the main result with the proofs. An example of a two node network is considered in Section 4. In

Section 5 we give a discussion of results and concluding remarks.

2 Model and Preliminaries

We consider a discrete time fluid FIFO model where traffic arrivals and services take place in slots indexed by $t \in \mathbb{Z}$ with the convention that arrivals take place at the beginning of a slot and services are completed at the end of the slot. This can be seen as corresponding to a situation in continuous time where arriving traffic coming in $(i, i+1]$ are all served at time $i+1$. We refer to t as the time instant in the discrete-time model. There is a finite set $\mathcal{N}$ of traffic types (classes) with $N = card(\mathcal{N})$ which are differentiated according to their transmission rates and their session lengths and different classes are assumed to be mutually independent. Session requests for type $n \in \mathcal{N}$ arrive randomly according to a Poisson process with rate λ_n. Let θ_t^n be the number of sessions of class n arriving at time t. We assume that θ_t^n are i.i.d. and

$$\mathbb{P}\left\{\theta_t^n = k\right\} = e^{-\lambda_n}\frac{\lambda_n^k}{k!}$$

A session of class n then transmits at the rate r_n for a duration τ_n which is assumed to have a long-tailed distribution. Let $\tau_{t,j}^n$ denote the session length of the j'th session of class n arriving at time t. The r.v.'s $\tau_{t,j}^n$ are assumed to be i.i.d. and satisfy

$$\mathbb{P}\left\{\tau_{t,j}^n \geq z\right\} = \mathbb{P}\left\{\tau_n \geq z\right\} \sim \alpha_n z^{-(1+\beta_n)}$$

where $\alpha_n, \beta_n > 0$ and $A(x) \sim B(x)$ means that $\lim_{x\to\infty}\frac{A(x)}{B(x)} = 1$. Also $A(x) \preceq (\succeq) B(x)$ means $A(x) \leq (\geq) \sim B(x)$, i.e., the inequalities are in an asymptotic sense. This model is a generalization of the $M/G/\infty$ model proposed by Cox [18], which we refer to as the $(M/G/\infty)^N$ model and is depicted in Fig. 1.

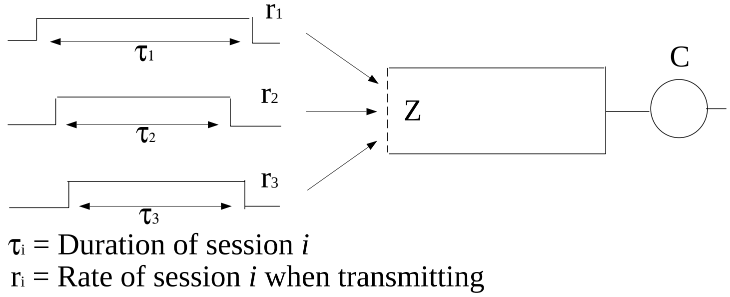

Fig. 1. Model of arriving sessions

The network is composed of M nodes (see Fig. 2 below). It is assumed that the packets from the sessions are admitted into an infinite buffer and the buffer is served at a rate of C_m per unit time for node $m = 1, \ldots, M$. Type $n \in \mathcal{N}$ traffic has a fixed route without any loops and its path is represented by the vector $\pi^n = \left[\pi_1^n, \ldots, \pi_{l_n}^n\right]$ where $\pi_i^n \in \{1, \ldots, M\}$. Hence type n traffic traverses the nodes by entering the network at node π_1^n and leaving after node $\pi_{l_n}^n$ and $\pi_i^n \neq \pi_j^n$ for $i \neq j$. For each node m, define the set of traffic types which pass through node m by $\mathcal{N}_m = \{n : \ \pi_i^n = m, \ 1 \leq i \leq l_n\}$.

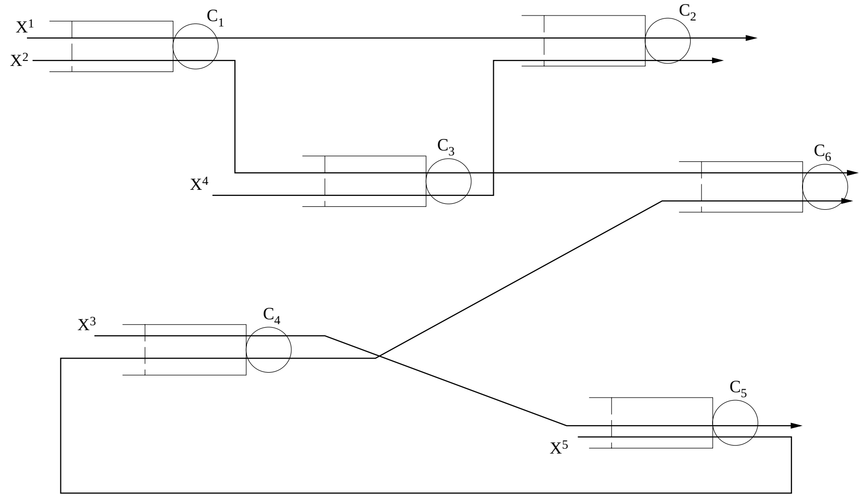

Fig. 2. A typical network considered in this paper

Let X_t^n be the input process of class n at time t. Then,

$$X_t^n = \sum_{i=t}^{-\infty} \sum_{j=1}^{\theta_i^n} r_n \mathbf{1}_{\{\tau_{i,j}^n \geq t-i\}}.$$

Average (mean) load of class n is $\rho_n \doteq \lambda_n r_n \mathbb{E}[\tau_n]$ and it is assumed that

$$\sum_{n \in \mathcal{N}_m} \rho_n < C_m$$

and hence all the queues in the network are stable since the network is loop-free for all classes of flows. Under this condition, by Loynes' theorem [19], there exists a stationary version of the workload at node m, denoted by W_t^m.

We will divide an input process X_t^n into two processes $X_t^{n,L}, X_t^{n,H}$ as follows: The process $X_t^{n,L}$ is formed by active sessions which have session lengths at most εz (will be referred as "short" sessions in the context) and given by

$$X_t^{n,L} = \sum_{i=t}^{t-\varepsilon z} \sum_{j=1}^{\theta_i^n} r_n \mathbf{1}_{\{\varepsilon z \geq \tau_{i,j}^n \geq t-i\}} \tag{1}$$

The process $X_t^{n,H}$ is composed of sessions with lengths greater than εz (will be referred as "long" sessions in the sequel) and given by

$$X_t^{n,H} = \sum_{i=t}^{-\infty} \sum_{j=1}^{\theta_i^n} r_n \mathbf{1}_{\{\tau_{i,j}^n > \varepsilon z, \tau_{i,j}^n > t-i\}} \tag{2}$$

As it can be seen from the definition, the processes $X_t^{n,L}, X_t^{n,H}$ are mutually independent and $X_t^n = X_t^{n,L} + X_t^{n,H}$. The superscript '$H$' indicates that the input is strictly "long". We will refer to $X^{n,L}$ and $X^{n,H}$ as short and long processes in the context. As $z \to \infty$, we have $\mathbb{E}X_t^{n,L} \to \mathbb{E}X_t^n$ and $\mathbb{E}X_t^{n,H} \to 0$. Despite the fact that $\mathbb{E}X_t^{n,H} \to 0$ as $z \to \infty$, the process $X_t^{n,H}$ will contribute significantly to the large buffer occupancy probability.

We also define $X_{k,t}^n$ ($Y_{k,t}^n$) to be the input (output) rate of class m traffic at node k and time t. For any process Z with rate Z_t at time t, the cumulative process in the time interval $(t_1, t_2]$ will be denoted by $Z(t_1, t_2)$, i.e., $Z(t_1, t_2) = \sum_{i=t_1+1}^{t_2} Z_t$.

We also need the following definitions for the statement of the main result. Let $J \in \mathbb{Z}_+^N$. In the sequel, J will correspond to the combination of long (longer than εz) sessions, i.e., there are J_n long sessions of class n. If the transmission rates of active long sessions at a time exceeds the difference of total capacity and average load, some scaling would occur. This happens because the service capacity will be shared among all these active sessions according to the FIFO scheduling. In particular, mean load of short sessions (which goes to the average workload as $z \to \infty$) and the transmission rates of long sessions would change at the output. To determine the scaling effect, we first define $p^n(m) = m'$ for $n \in \mathcal{N}_m$ if $\pi_i^n = m$ and $\pi_{i-1}^n = m'$; i.e. $\pi^n(m)$ is the upstream node of node m for class n. Take $p^n(m) = 0$ if $m = \pi_1^n$. We define the transmission rate and mean of class n traffic entering node m as follows:

$$r_m^{n,J} = \begin{cases} 0 & n \notin \mathcal{N}_m \\ r_n & p^n(m) = 0 \\ r_{p^n(m)}^{n,J} S_{p^n(m)}^J & otherwise \end{cases} \tag{3}$$

and

$$\rho_m^{n,J} = \begin{cases} 0 & n \notin \mathcal{N}_m \\ \rho_n & p^n(m) = 0 \\ \rho_{p^n(m)}^{n,J} S_{p^n(m)}^J & otherwise \end{cases} \tag{4}$$

where

$$S_m^J = \frac{C_m}{\max\{C_m, \sum_{n \in \mathcal{N}_m} J_n r_m^{n,J} + \rho_m^{n,J}\}} \tag{5}$$

Here $r_m^{n,J}$ ($\rho_m^{n,J}$) represent the transmission rate and average load of class n traffic as it enters node m. Note that the output traffic is no longer $M/G/\infty$ and the meaning of these definitions in relation to the ones of external flows should be interpreted heuristically. The function S_m^J measures the scaling of sources for the output transmission when input rates of long sessions exceed the difference between output capacity and total average loads at node m. Note that S_m^J is equal to the identity function if this does not happen.

Lemma 1 *For every $J \in \mathbb{Z}_+^N$, equations (3) and (4) are well defined, i.e., there exist unique solutions $r_m^{n,J}$ and $\rho_m^{n,J}$ satisfying these equations. Furthermore, for fixed J, $r_m^{n,J}$ and $\rho_m^{n,J}$ are continuous in parameters r_n and ρ_n.*

We now define the following for future usage:

$$\begin{aligned}
&R^J = \textstyle\sum_{n \in \mathcal{N}} J_n r_n, \quad R_m^J = \sum_{n \in \mathcal{N}_m} J_n r_m^{n,J} \\
&\rho = \textstyle\sum_{n \in \mathcal{N}} \rho_n, \qquad\quad \rho_m^J = \sum_{n \in \mathcal{N}_m} \rho_m^{n,J} \\
&\kappa_J = \textstyle\sum_{n \in \mathcal{N}} J_n \beta_n, \quad J_0^m = \arg\min_J \{\kappa_J : R_m^J > C_m - \rho_m^{n,J}\} \\
&\beta_{min} = \min_{n \in \mathcal{N}} \beta_n, \; r_{max} = \max_{n \in \mathcal{N}} r_n
\end{aligned}$$

The results below, taken from [9], provide bounds for deviations of a short process from its average rate.

Lemma 2 *For any $n \in \mathcal{N}$ and sufficiently small $\delta_1 > 0$, $\delta_2 > 0$ and sufficiently large z*

$$\mathbb{P}\left\{\inf_{t \geq 0}\{X^{n,L}(-t,0) - (\rho_m - \delta_1)t\} < -\delta_2 z\right\} \leq e^{-O(z)}$$

Lemma 3 *For any $n \in \mathcal{N}$ and $\delta_2^0 > 0, \delta_1 > 0, c_1 > 0$ and $0 < \varepsilon \min\{\frac{\delta_2^0}{2(\rho_m+\delta_1)}, \min_i \frac{\beta_i^*}{2c_1 r_i}\}$ where $\beta_i^* = \beta_i, 0 < \beta_i < 1$ and $\beta_i^* = \frac{1}{2}, \beta \geq 1$, if z is sufficiently large, then uniformly over all $\delta_2 > \delta_2^0$*

$$\mathbb{P}\left\{\sup_{t \geq 0}\{X^{n,L}(-t,0) > t(\rho_m + \delta_1)\} > \delta_2 z\right\} \leq z^{1 - c_1 \bar{\delta}_2}$$

where $\bar{\delta}_2 \doteq \delta_2 - (\rho_m + \delta_1)\varepsilon > 0$.

3 Main Result and Proofs

Our goal is to obtain the O-asymptotics for the probability that the buffer content at node m exceeds a large value z. We will use $f(z) = O(g(z))$ to mean $0 < \liminf_{z\to\infty} f(z)/g(z) < \limsup_{z\to\infty} f(z)/g(z) < \infty$. The buffer of node m will reach a large value z if there are enough long sessions which contribute to this. During this time, short sessions show average behavior, i.e., produce traffic around their mean load. Thus, short processes can be replaced with constant processes in the asymptotic regime without causing a significant change at large occupancy levels of buffers.

We first review the single node result ($M = 1$) from [9] where exact buffer asymptotics was obtained. A form of their result, sufficient for our purposes in this paper, is

$$\mathbb{P}(W_0^1 > z) = \sup_{t>0}(\sum_{n\in\mathcal{N}_1} X^n(0,t) - C_1 t) = O(z^{-\kappa_{J_0^1}}) \tag{6}$$

It is also shown that the most likely large buffer occupancy is generated by a busy period when J_0^1 long sessions are active during this busy period. If there is only a single type of flow ($card(\mathcal{N}) = 1$), then $J_0^1 = \lceil (C_1 - \rho_1)/r_1 \rceil$. Hence, unless $r_1 > C_1 - \rho_1$, the buffer length decay rate, $\kappa_{J_0^1}$, depends not only on the decay rates β_n's but also on the transmission rates r_n's.

For the general network case, we will first find an upper bound for the buffer asymptotics. To this end, a short process of type n as it enters the network will be replaced with a constant process $\rho_n + \delta$ ($\rho_n - \delta$) for small enough $\delta > 0$ if $n \in \mathcal{N}_m$ ($n \notin \mathcal{N}_m$). Then we will show that the busy period where buffer reaches z is created by combination of long sessions (J) which satisfy $R_m^J > C_m - \rho_m^{n,J}$. Note that the scaling defined before Lemma 1 will take place at a node if long sessions have a total rate more than the difference of output capacity and average input load at this node. This scaling causes a long session becoming longer at the output with a smaller transmission rate. Therefore we will consider a process which is obtained from the long process by shifting a session by the buffer occupancy at their arrival times and lengthening it by an amount proportional to the total traffic arriving during the lifetime of this session. The shifting is done to determine when the considered long session will be present at the downstream nodes. The scaling will allow us to bound the time when the service of this session finishes at the considered node. It will be shown that the probability that this new process has J active sessions at $t = 0$ is $O(z^{-\kappa_J})$. Furthermore, the case $R_J^m > C_m - \rho_J^m$ will dominate the other combinations in probability when the buffer occupancy at node m is greater than z. The optimal (asymptotically most likely) such combination was defined as J_0^m corresponding to a probability of $O(z^{-\kappa_{J_0^m}})$ and this will give the upper bound.

The lower bound part is relatively easier. We assume that the J_0^m long sessions arrive in $(-kz, k(1-\alpha)z)$ where $0 < \alpha < 1$ is small and $k > 0$ is later chosen big enough. This event has a probability of $O(z^{-\kappa_{J_0^m}})$. From the upper bound part, the probability that a buffer has a level greater than $\delta_1 z$ at time $(-kz-1)$ is $z^{-\gamma}$ for some $\gamma > 0$. For small enough δ_1, the contribution of buffer contents at time $(-kz-1)$ to the buffer content of node m at $t = 0$ will be less than $\delta_2 z$ for small $\delta_2 > 0$. Short sessions are again replaced with constant processes but this time with $\rho_n - \delta$ if $n \in \mathcal{N}_m$ and $\rho_n + \delta$ if $n \notin \mathcal{N}_m$. Then we show that this combination would produce enough traffic to make the buffer at node m bigger than z.

Now we state the main result of the paper:

Theorem 4 *(Buffer asymptotics in networks with fixed routes)*

Assume $R_m^J \neq C_m - \rho_m^J$ for all J, m and J_0^m is unique. Let W_t^m be the stationary workload of node m at time t. Then as $z \to \infty$,

$$\mathbb{P}(W_0^m > z) = O(z^{-\kappa_{J_0^m}})$$

PROOF.

Upper bound:

The main idea is to show that if buffer occupancy at time $t = 0$ for node m reaches a large value of z, then there must have been enough number of long sessions contributing to this. Consider a long session of type n at time t with length $\tau_{t,j}^n > \varepsilon z$ for $j = 1, \ldots, \theta_t^n$. Define $S_{t,j}^{n,0} = t$ and if node m_2 is successor of node m_1 for type n flow, define $S_{t,j}^{n,m_2} = S_{t,j}^{n,m_1} + W_{S_{t,j}^{n,m_1}}^{m_2} / C_{m_2}$. If m_2 has no predecessor, set $m_1 = 0$. Note that a long session of type n arriving to the network at time t starts being served at node m at time $S_{t,j}^{n,m}$. Consider a fictional queue with infinite buffer and service capacity ε_1 which serves the long sessions arriving at node m. Let $\bar{S}_{t,j}^{n,m}$ be the time when the service of the long session considered ends at this fictional queue. Also define

$$\xi_n^m = \sum_{t=-1}^{-\infty} \sum_{j=1}^{\theta_t^n} \mathbb{I}_{A_{t,j}^{n,m}}, \quad A_{t,j}^{n,m} = \{S_{t,j}^{n,m} < 0, \bar{S}_{t,j}^{n,m} > 0, \tau_{t,j}^n > \varepsilon z\}$$

Let $\xi^m = (\xi_n^m)$. We claim that

$$\mathbb{P}(\xi^m = J) = O(z^{-\kappa_J}) \tag{7}$$

Then we will write

$$\mathbb{P}(W_0^m > z) \leq \mathbb{P}(R_m^{\xi^m} > C_m - \rho_m^{\xi^m}) + \mathbb{P}(W_0^m > z, R_m^{\xi^m} < C_m - \rho_m^{\xi^m})$$

For the first term of right side,

$$\mathbb{P}(R_m^{\xi^m} > C_m - \rho_m^{\xi^m}) = \sum_{J \in \mathbb{Z}_+^N} \mathbb{P}(\xi^m = J, R_m^J > C_m - \rho_m^J)$$

¿From equation (7) and definition of J_0^m,

$$\mathbb{P}(R_m^{\xi^m} > C_m - \rho_m^{\xi^m}) = O(z^{-\kappa_{J_0^m}})$$

We will then show that

$$\mathbb{P}(W_0^m > z, R_m^{\xi^m} < C_m - \rho_m^{\xi^m}) = o(z^{-\kappa_{J_0^m}}) \tag{8}$$

and this will complete the proof of the upper bound.

Let us prove claim (7) now. Define $X_t \doteq \sum_{n \in \mathcal{N}} X_t^n$. First, for any $d > 0$, there exists $K > 0$ such that

$$\mathbb{P}(X(t, t+\tau) > K\tau) \leq o(\tau^{-d}) \tag{9}$$

as $\tau \to \infty$. Indeed, consider a queue with capacity $K - 1 > \rho$ with input X_t and let W_t' be its stationary buffer content. Then by using (6), we get

$$\mathbb{P}(X(t, t+\tau) > K\tau) = \mathbb{P}(X(-\tau, 0) > K\tau) \leq \mathbb{P}(W_0' > \tau) = O(\tau^{-\kappa_{J_0}})$$

where $J_0 = \arg\min_J\{\kappa_J : R^J > K - 1 - \rho\}$. Define $|J| = \sum_{n \in \mathcal{N}} J_n$. But $\kappa_{J_0}/\beta_{min} > |J_0|$ and $|J_0| > \frac{K-1-\rho}{r_{max}}$. Thus we can choose K such that $\kappa_{J_0} > d$, proving (9).

Now define

$$\tilde{A}_{t,j}^{n,m} = \{S_{t,j}^{n,m} < 0, \bar{S}_{t,j}^{n,m} > 0, \tau_{t,j}^n > \varepsilon z, \bar{S}_{t,j}^{n,m} - S_{t,j}^{n,\pi^n(1)} < (cK)^M(\tau_{t,j}^n)\}$$

where $c > 1/C_k$ for all k and $c > 1/\varepsilon_1$. Here cK will be the scaling (lengthening) factor that a long session will experience while going through a node where K is determined by the amount of traffic arriving during its transmission. Also define $\tilde{\xi}_n^m = \sum_{t=-1}^{-\infty} \sum_{j=1}^{\theta_t^n} 1\!\!1_{\tilde{A}_{t,j}^{n,m}}$, By using (9), we conclude

$$\mathbb{P}(\xi^m = J) \sim \mathbb{P}(\tilde{\xi}^m = J) \tag{10}$$

Let $D = (cK)^M + 1$. Define $W_t = (W_t^1, \dots, W_t^M)$. Then

$$\mathbb{P}(\tilde{\xi}^m = J) = \sum_{x \in \mathbb{R}_+^{M|J|}} \sum_{-\mathbf{t} \in Z_+^{|J|}} \mathbb{P}(W_{t_i} = x_i, c\,|x_i| + D\tau_{\mathbf{t}_i}^{n_i} > -\mathbf{t}_i, \tau_{\mathbf{t}_i}^{n_i} > \varepsilon z,$$

$$i = 1, \dots, |J|, n_i \in \mathcal{N}, \sum_i \mathbb{I}_{\{n_i = n\}} = J_n)$$

Above, $x = (x_1, \dots, x_{|J|})$ where $x_i \in \mathbb{R}_+^M$. Now note that W_{t_i} and $\tau_{t_i}^{n_i}$ are independent since W_{t_i} is determined by arrivals before t_i. It is easy to see that

$$\sum_{-t \in Z_+} \mathbb{P}(x + D\tau_t^n > -t, \tau_t^n > \varepsilon z) = O(z^{-\beta_n})$$

Furthermore since W_t is stationary, $\mathbb{P}(W_{t_i} = x_i) = \mathbb{P}(W_{\tilde{t}_i} = x_i)$ where $\tilde{t}_i = t_i - \min_i t_i$. Therefore,

$$\mathbb{P}(\tilde{\xi}^m = J) = \sum_{x \in \mathbb{R}_+^{M|J|}} \sum_{-\mathbf{t} \in Z_+^{|J|}} \mathbb{P}(W_{\tilde{t}_i} = x_i) O(z^{-\kappa_J}) = O(z^{-\kappa_J})$$

Now we will prove equality (8). Consider the system where all the buffers are empty at time $-T$ and let $W_{T,t} = (W_{T,t}^1, \dots, W_{T,t}^M)$ be the buffer occupancy (workload) at time t for this system. It is known that $W_{T,t} \to W_t$ a.s. as $T \to \infty$. Therefore we will first show that equality (8) holds when W_0^m is replaced by $W_{T,0}^m$ and then take the limit $T \to \infty$.

Let us investigate the total arrival traffic to node m in the time interval $(-T, 0)$ assuming that all the queues are empty at $-T$ and $R_m^{\xi^m} < C_m - \rho_m^{\xi^m}$. Choose $\varepsilon_1 > 0$ such that $2\varepsilon_1 > C_m - \max\{R_m^J + \rho_m^J \ : \ R_m^J + \rho_m^J < C_m\}$. If a long session is not in $A_{t,j}^{n,m}$, then it had been served before $t = 0$ in the fictional queue defined above. Consider another queue with capacity $C_m - \varepsilon_1$ which serves all of the remaining traffic. Note that the buffer content of this queue at $t = 0$ is bigger than $W_{T,0}^m$ provided that $R_m^{\xi^m} < C_m - \rho_m^{\xi^m}$.

Now we assume that the long sessions in $A_{t,j}^{n,m}$ are active in $(-T, 0)$ all the time and the remaining long sessions of other classes are removed. Note that this does not decrease the buffer content at node m. More generally, replacing flows accessing node m by pathwise larger ones and other flows by smaller ones does not decrease the buffer content at node m. This can be seen by a sample path argument. If all the queues are empty at $t = -1$, then from the arguments in Lemma 1, the claim will hold at $t = 0$ and an induction on t will complete the proof. Under the above assumption, let $\bar{W}_{T,0}^m$ be the buffer content of the queue with capacity $C_m - \varepsilon_1$. Then,

$$\mathbb{P}(W_{T,0}^m > z, R_m^{\xi^m} < C_m - \rho_m^{\xi^m}) \le \mathbb{P}(\bar{W}_{T,0}^m > z, R_m^{\xi^m} < C_m - \rho_m^{\xi^m})$$

Define $\mathcal{D}_m = \{R_m^{\xi^m} < C_m - \rho_m^{\xi^m}, \xi^m$ sessions are active in $(-T, 0)\}$. Also let $X_m \doteq \sum_{n \in \mathcal{N}_m} X_m^n$ be the total cumulative input to node m. Now assume that

$$\mathbb{P}(\sup_{0 \le t \le T} X_k^n(-T,-t) - (r_k^{n,\xi^m} + \rho_k^{n,\xi^m} + \varepsilon)(T-t) > \varepsilon_2 z, \mathcal{D}_m) \preceq z^{-d} \quad (11)$$

and

$$\mathbb{P}(\inf_{0 \le t \le T} X_k^n(-T,-t) - (r_k^{n,\xi^m} + \rho_k^{n,\xi^m} - \varepsilon)(T-t) < -\varepsilon_2 z, \mathcal{D}_m) \preceq z^{-d} (12)$$

for any $d > 0$. Note that this is true for $k = \pi_1^n$. Indeed, if $\mathcal{D}_m$ holds, then the long processes are constant $(= r_k^{n,\xi^m})$ in $(-T, 0)$ for type n flow. Furthermore, from Lemmas 2 and 3, a short process $X^{n,L}$ differs from the constant processes $\rho_n - \delta$ and $\rho_n + \delta$ with probability $o(z^{-\bar{d}})$. for any $\delta > 0$. Here $\bar{d}$ can be chosen arbitrarily big. Then,

$$\mathbb{P}(\sup_{0 \le t_1 \le T} Y_k^n(-T,t) - (\frac{r_m^{n,\xi^m} + \rho_m^{n,\xi^m}}{\max(C_k, R_m^{\xi^m} + \rho^{\xi^m})} + \varepsilon_1)(T-t) > \varepsilon_2 z, \mathcal{D}_m)$$

$$\le \mathbb{P}(\sup_{0 \le t \le T} X_k^n(-T,-t) - (r_k^{n,\xi^m} + \rho_k^{n,\xi^m} + \varepsilon_3)(T-t) > \varepsilon_4 z, \mathcal{D}_m) +$$

$$\sum_{j \in \mathcal{N}_k, j \ne n} \mathbb{P}(\inf_{0 \le t \le T} X_k^j(-T,-t) - (r_k^{j,\xi^m} + \rho_k^{j,\xi^m} - \varepsilon_3)(T-t) - \varepsilon_4 z, \mathcal{D}_m)$$

$$\le o(z^{-d})$$

We will choose $\varepsilon_3 = \varepsilon_1 / card(\mathcal{N}_k)$ and $\varepsilon_4 = (\varepsilon_2 - M\delta_1)/card(\mathcal{N}_k)$. Above we used the fact that the output of node k in the interval $(-T, -t]$ is equal to the arrival in an interval $(-T, t']$. Note that the time being discrete is not a problem here since there is an $\varepsilon_2 z$ term and z is taken to be large. Same argument can be used to show that

$$\mathbb{P}(\inf_{0 \le t \le T} Y_k^n(-T,-t) - (\frac{r_m^{n,\xi^m} + \rho_m^{n,\xi^m}}{\max(C_k, R_m^{\xi^m} + \rho_m^{\xi^m})} - \varepsilon_1)(T-t) < -\varepsilon_2 z, \mathcal{D}_m)$$

$$\le o(z^{-d})$$

¿From Lemma 1, the equations (11) and (12) should hold for all n, k. This can be more formally shown by considering the values

$$\bar{\alpha}(n,k) \doteq$$

$$\lim_{z \to \infty} \frac{1}{\log z} \log \left\{ \mathbb{P}(\sup_{0 \le t \le T} X_k^n(-T,-t) - (r_k^{n,\xi^m} + \rho_k^{n,\xi^m} + \varepsilon_1)(T-t) > \varepsilon_2 z, \mathcal{D}_m) \right\}$$

and defining similarly $\underline{\alpha}(n,k)$ when sup is replaced with inf. These expressions evaluate to ∞ when $k = \pi_1^n$. Above arguments show that it is also true for all n, k at the outputs assuming it for the inputs. Thus $\bar{\alpha}(n,k) = \underline{\alpha}(n,k) = \infty$ is a fixed point solution of these relations as given in Lemma 1. Since there can only be one solution, we conclude that the equation (11) is valid for all n, k. Thus for the input to node m, we have

$$\begin{aligned}
&\mathbb{P}(\sup_{0<t\le T} X_m(-t,0) - (R_m^{\xi^m} + \rho_m^{\xi^m} + \varepsilon_1)t > \varepsilon_2 z, \mathcal{D}_m) \\
&\le \mathbb{P}(X_m(-T,0) - (R_m^{\xi^m} + \rho_m^{\xi^m} + \varepsilon_1)T > 0.5\varepsilon_2 z, \mathcal{D}_m) \\
&\quad + \mathbb{P}(\inf_{0<t\le T} X_m(-T,t) - (R_m^{\xi^m} + \rho_m^{\xi^m} + \varepsilon_1)(T-t) < -0.5\varepsilon_2 z, \mathcal{D}_m) \\
&\preceq o(z^{-d})
\end{aligned}$$

for any $d > 0$. Remember that ρ_m^{n,J_0^m} and r_m^{n,J_0^m} are continuous functions of $\rho_k,\ r_k$'s. Then,

$$\mathbb{P}(\bar{W}_{T,0}^m > z, R_m^{\xi^m} < C_m - \rho_m^{\xi^m}, T_0 < T)$$

$$\le \mathbb{P}(\sup_{0<t\le T} X_m(-t,0) - (R_m^{\xi^m} + \rho_m^{\xi^m} + \varepsilon_1)t > \varepsilon_2 z, \mathcal{D}_m) \preceq o(z^{-d})$$

and finally by taking $T \to \infty$, we get

$$\mathbb{P}(W_0^m > z, R_m^{\xi^m} < C_m - \rho_m^{\xi^m}, T_0 < T) \le o(z^{-\kappa_{J_0^m}}).$$

<u>*Lower bound:*</u>

Let $A^J(x,\alpha)$ be the event that exactly J long sessions start in $(-x, -(1-\alpha)x]$ and still active at $t = 0$ where $0 < \alpha < 1$. First note that, for every class n, the number of such sessions is a Poisson r.v. with parameter $\sum_{t=-x+1}^{-(1-\alpha)x} \lambda_n \mathbb{P}(\tau_n > -t)$ which is $O(x^{-\beta_n})$. ¿From the independence of classes, it is easy to see that $\mathbb{P}(A^J(x,\alpha)) \sim O(x^{-\kappa_J})$. We will now show that

$$\mathbb{P}(A^{J_0^m}(x,\alpha)) \preceq \mathbb{P}(W_0^m > z)$$

for some $\alpha > 0$ and $x = bz$ with $b > 0$. First, from the upper bound proof,

$$\mathbb{P}(W_{-x}^k > \delta_1 z) \preceq O(z^{-\gamma}) \tag{13}$$

for some $\gamma > 0$ and any $k = 1, \ldots, M$. Note that $\{W_{-x}^k > \delta_1 z\}$ and $A^{J_0^m}(x,\alpha)$ are independent because W_{-x}^k is determined by sessions which arrived before

$-x+1$. Now define the following two events:

$$B_1 = \{\text{there are active long sessions at time } t = -x\}$$
$$B_2 = \{\text{a long session other than } J_0^m \text{ arrives between } t = -x \text{ and } t = 0\}$$

Arrival process of long sessions is also Poisson with rate $\lambda_m \mathbb{P}(\tau_m > \epsilon z)$. Since the arrivals at different times are independent, it is easy to see that

$$\mathbb{P}(B_1 | A^{J_0^m}(x,\alpha)) \leq O(z^{-\beta_{min}}) \text{ and } \mathbb{P}(B_2 | A^{J_0^m}(x,\alpha)) \leq O(xz^{-\beta_{min}-1}).$$

These and equation (13) give

$$\mathbb{P}(A^{J_0^m}(x,\alpha)) \sim \mathbb{P}(A^{J_0^m}(x,\alpha), \bar{B}_1, \bar{B}_2, W_{-x}^k < \delta_1 z, \forall k = 1, \ldots, M) \qquad (14)$$

where $\bar{B}_1, \bar{B}_2$ are complements of B_1, B_2. In other words, we can assume that all the buffers are almost empty at $t = -x$ regarding their contribution to a buffer level exceeding z at $t = 0$ and the only active long sessions during $(-x, 0)$ are the ones of J_0^m. Let

$$\mathcal{A}_m = \{A^{J_0^m}(x,\alpha), \bar{B}_1, \bar{B}_2, W_{-x}^k < \delta_1 z, \forall k = 1, \ldots, M\}$$

Let $T = (1-\alpha)x$. Now assume that

$$\mathbb{P}(\sup_{0 \leq t_1 \leq t_2 \leq T} X_k^n(-t_2, -t_1) - (r_k^{n,J_0^m} + \rho_k^{n,J_0^m} + \varepsilon)(t_2 - t_1) > \varepsilon_2 z, \mathcal{A}_m) \preceq z^{-d}$$

and

$$\mathbb{P}(\inf_{0 \leq t_1 \leq t_2 \leq T} X_k^n(-t_2, -t_1) - (r_k^{n,J_0^m} + \rho_k^{n,J_0^m} - \varepsilon)(t_2 - t_1) < -\varepsilon_2 z, \mathcal{A}_m) \preceq z^{-d}$$

for any $d > 0, \varepsilon, \varepsilon_2 > 0$. Note that this is true when $k = \pi_1^n$. Indeed, if $\mathcal{A}_m$ holds, then the long processes are constant ($= r_k^{n,J_0^m}$) in $(-T, 0)$ for type n flow. Furthermore, from Lemmas 2 and 3, a short process $X^{n,L}$ differs from the constant processes $\rho_n - \delta$ and $\rho_n + \delta$ with probability $o(z^{-\bar{d}})$. for any $\delta > 0$. Here $\bar{d}$ can be chosen arbitrarily big and since $T = (1-\alpha)bz$, we can take $d < \bar{d} - 1$. Then,

$$\mathbb{P}(\sup_{0 \leq t_1 \leq t_2 \leq T} Y_k^n(-t_2, -t_1) - (\frac{r_m^{n,J_0^m} + \rho_m^{n,J_0^m}}{\max(C_k, R_m^{J_0^m} + \rho_m^{J_0^m})} + \varepsilon_1)(t_2 - t_1) > \varepsilon_2 z, \mathcal{A}_m)$$
$$\leq \mathbb{P}(\sup_{0 \leq t_1 \leq t_2 \leq T} X_k^n(-t_2, -t_1) - (r_k^{n,J_0^m} + \rho_k^{n,J_0^m} + \varepsilon_3)(t_2 - t_1) > \varepsilon_4 z, \mathcal{A}_m) +$$

$$\sum_{j\in\mathcal{N}_k, j\neq n} \mathbb{P}(\inf_{0\leq t_1\leq t_2\leq T} X_k^j(-t_2,-t_1) - (r_k^{j,J_0^m} + \rho_k^{j,J_0^m} - \varepsilon_3)(t_2-t_1) - \varepsilon_4 z, \mathcal{A}_m)$$
$$\leq o(z^{-d})$$

We will choose $\varepsilon_3 = \varepsilon_1/card(\mathcal{N}_k)$ and $\varepsilon_4 = (\varepsilon_2 - M\delta_1)/card(\mathcal{N}_k)$. Above, we used the fact that the output of node k in the interval $(-t_2,-t_1]$ is equal to the arrival in an interval $(-t_2', t_1']$. Note that the error induced here due to the discreteness of time is not a problem since there is an $\varepsilon_2 z$ term and z is taken to be large. Same argument can be used to show that

$$\mathbb{P}(\inf_{0\leq t_1\leq t_2\leq T} Y_k^n(-t_2,-t_1) - (\frac{r_m^{n,J_0^m} + \rho_m^{n,J_0^m}}{\max(C_k + R_m^{J_0^m} + \rho_m^{J_0^m})} - \varepsilon_1)(t_2-t_1) < -\varepsilon_2 z, \mathcal{A}_m)$$
$$\leq o(z^{-d})$$

Then by using Lemma 1 as was done in the upper bound part, we conclude

$$\mathbb{P}(X_m(-T,0) - (R_m^{J_0^m} + \rho_m^{J_0^m} - \varepsilon_1)T < -\varepsilon_2 z - \alpha T(R^J + \rho^J + 1), \mathcal{A}_m)$$
$$\leq o(z^{-d})$$

for any $d > 0$. Then

$$\begin{aligned}
\mathbb{P}(W_0^m > z) &\geq \mathbb{P}(X_m(-x,0) > C_m x + z) \geq \mathbb{P}(X_m(-x,0) > C_m x + z, \mathcal{A}_m) \\
&\succeq \mathbb{P}((R_m^{J_0^m} + \rho_m^{J_0^m} - \varepsilon_1 - \delta_3)T > C_m x + (1+\varepsilon_2)z, \mathcal{A}_m) + o(z^{-d}) \\
&\succeq \mathbb{P}(A^{J_0^m}(x,\alpha))
\end{aligned}$$

For the last inequality, we used $R_m^{J_0^m} + \rho_m^{J_0^m} > C_m$ and chose $\alpha, \varepsilon_1, \varepsilon_2$ small enough and b large enough so that

$$(R_m^{J_0^m} + \rho_m^{J_0^m})(1-\alpha) > C_m + \frac{1-\varepsilon_2}{b}$$

Combining above with equation (14) gives

$$\mathbb{P}(W_0^m > z) \succeq \mathbb{P}(A^{J_0^m}(x,\alpha)) \sim O(z^{-\kappa_{J_0^m}})$$

□

Remark 5 *The uniqueness of J_0^m was assumed for convenience and is not necessary. Thus the results and proofs still hold when there are many optimal configuration of long sessions.*

Remark 6 *If there are input flows to the network with light tailed session lengths such that* $\lim_{x\to\infty} \log \mathbb{P}(\tau > x)/\log x = -\infty$*, then they can be ignored in calculating the buffer asymptotics. Such an input can be replaced with a Pareto tailed version with arbitrarily large rate* β *without decreasing the probability of large buffer occupancies. Therefore, in determining* J_0^m*, none of these inputs need to be considered.*

Remark 7 *If the session lengths are regularly varying (i.e.* $\lim_{x\to\infty} \mathbb{P}(\tau_n > tx)/\mathbb{P}(\tau_n > x) = t^{-(1+\beta_n)}$ *for* $\beta_n > 0$*), it can be shown that the buffer occupancy distributions are also regularly varying with the parameters as found in the main result.*

We can also find the joint distribution of buffer asymptotics. We only state the result since the proof follows mutatis mutandis as above.

Corollary 8 *Let* $\mathcal{S} \subset \{1, \ldots, M\}$ *and* $a_m > 0, m \in \mathcal{S}$*. Then as* $z \to \infty$*,*

$$\mathbb{P}(W_0^m > a_m z, m \in S) \sim O(z^{-\kappa_{\mathcal{S}}})$$

where $\kappa_{\mathcal{S}} = \max\{\kappa_{J_0^m} \mid m \in \mathcal{S}\}$

4 Examples

In order to illustrate the results, in particular how the rates $r_m^{n,J}$, $\rho_m^{n,J}$ are determined, we consider two examples of a simple network with two nodes. The first example is a feedforward network while the second example is a network where individual routes have no loops but the network is not feedforward.

Example 1: There are three classes of traffic, one of which uses resources from both nodes. The schema is illustrated in the figure below.

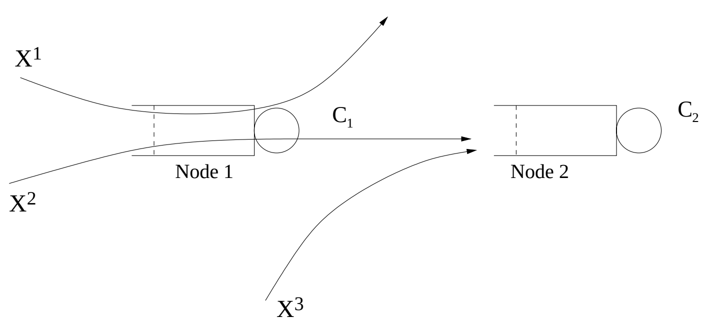

Fig. 3. Two node feedforward network

Now we calculate the large buffer asymptotics at the second node. First,

$$r_2^{1,J}=0,\ r_2^{2,J}=\frac{r_2C_1}{\max(J_1r_1+J_2r_2+\rho_1+\rho_2,C_1)},\ r_2^{3,J}=r_3$$
$$\rho_2^{1,J}=0,\ \rho_2^{2,J}=\frac{\rho_2C_1}{\max(J_1r_1+J_2r_2+\rho_1+\rho_2,C_1)},\ \rho_2^{3,J}=\rho_3$$

and thus

$$J_0^2=\arg\min_{J\in Z_+^3}\{\sum_{m=1}^{3}J_i\beta_i\ :\ \frac{(J_2r_2+\rho_2)C_1}{\max(C_1,J_1r_1+J_2r_2+\rho_1+\rho_2)}+J_3r_3+\rho_3>C_2\}$$

In special cases, J_0^2 can be obtained easily. For example, if classes 2 and 3 have the same transmission and decay rates ($r_2=r_3,\beta_2=\beta_3$), then $J_0^2=(0,0,\lceil\frac{C_2-\rho_2-\rho_3}{r_3}\rceil)$. This is because class 1 sessions do not contribute to the buffer occupancy at node 2 and we can take all the long sessions to be of class 3.

Example 2: In this example, we consider a non-feedforward network with two nodes and two types of traffic as illustrated in Fig. 4.

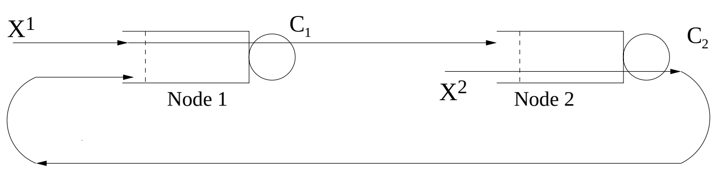

Fig. 4. Two node non-feedforward network

Then we obtain the following:

$$r_2^{1,J}=r_1/\max(J_1r_1+J_2r_1^{2,J}+\rho_1+\rho_1^{2,J},C_1),\ r_2^{2,J}=r_2,$$
$$r_1^{2,J}=r_2/\max(J_1r_2^{1,J}+J_2r_2+\rho_2+\rho_2^{1,J},C_2),\ r_1^{1,J}=r_1,$$
$$\rho_2^{1,J}=\rho_1/\max(J_1r_1+J_2r_1^{2,J}+\rho_1+\rho_1^{2,J},C_1),\ \rho_2^{2,J}=\rho_2,$$
$$\rho_1^{2,J}=\rho_2/\max(J_1r_2^{1,J}+J_2r_2+\rho_2+\rho_2^{1,J},C_2),\ \rho_1^{1,J}=\rho_1,$$

Hence

$$\kappa_{J_0^2}=\min\{\sum_{m=1}^{3}J_i\beta_i\ :J\in\Omega_1\cup\Omega_2\}\tag{15}$$

where

$$\Omega_1 = \{J \in \mathbb{Z}_+^3 : J_1 r_1 + \rho_1 + \frac{(J_2 r_2 + \rho_2) C_2}{J_1 r_1 + \rho_1 + J_2 r_2 + \rho_2} < C_1, J_1 r_1 + \rho_1 + J_2 r_2 + \rho_2 > C_2\}$$

$$\Omega_2 = \{J \in \mathbb{Z}_+^3 : J_1 r_1 + \rho_1 + \beta(J_2 r_2 + \rho_2) > C_1,\ \alpha(J_1 r_1 + \rho_1) + J_2 r_2 + \rho_2 > C_2\}$$

Here $\alpha = C_1/(J_1 r_1 + \rho_1 + \beta(J_2 r_2 + \rho_2))$ and $\beta = C_2/(\alpha(J_1 r_1 + \rho_1) + J_2 r_2 + \rho_2)$. Note that Ω_1 corresponds to the situation where large happens only at the second node and Ω_2 is where it happens at both nodes. In finding the optimal point in equation (15), we need to check whether a combination (J_1, J_2) is inside Ω_1 or Ω_2. For Ω_1, this is easy. For Ω_2, first α and β are algebraically or numerically solved and then substituted to the constraint in the definition of Ω_2. If classes 1 and 2 have the same transmission and decay rates ($r_1 = r_2, \beta_1 = \beta_2$), then it can be seen that it is enough to carry out the above calculations assuming only the long sessions of class 2. In addition, if $C_1 > C_2$, then $J_0^2 = (0, \lceil \frac{C_2 - \rho_1 - \rho_2}{r_2} \rceil)$ since now the most likely large buffer occupancy at the second node will happen without one at the first node.

5 Conclusion

In this paper, we have considered a loop-free network under the discrete-time heterogeneous $(M/G/\infty)^N$ model with long-tailed Pareto sessions. We have shown that the buffer occupancy is also Pareto and determined its parameters. The continuous-time case can be readily treated with slight changes in the technical details of the proofs. On the other hand, it seems difficult to obtain finer asymptotics, i.e. determining the prefactor constants because we can not parametrically model the traffic inside the network. We believe that the O-asymptotics will hold for more general session arrival distributions provided the session arrivals to the network are independent.

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A Proof of Lemma 1

We will show that $r_m^{n,J}$ and $\rho_m^{n,J}$ form a unique fixed point of a function for a given set of values of r_n, ρ_n. Let us now describe this function: Let $q = 2\sum_{n=1}^N l_n,\ s_1 = 1,\ s_n = 2\sum_{k=1}^{n-1} l_k,\ 1 < n \leq N$ and define

$$\Omega = \{v \in \mathbb{R}^q \ :\ v_{s_n+j} \in [0, C_{\pi_j^n}], n = 1, \ldots, N,\ j = 1, \ldots, l_n - 1\}$$

Here vector v corresponds to the transmission rates and average loads of the flows at all nodes. Now for $J \in \mathbb{Z}_+^N$, define the function $T^J : \Omega \to \Omega$ such that

$$T^J(v)_{s_n+j+1} = \frac{v_{s_n+j}}{\max(\sum_{k \in \mathcal{N}_{\pi_{\bar{j}}^n}} J_k v_{s_k+i(k,j,n)} + v_{s_k+l_k+i(k,j,n)}, C_{\pi_{\bar{j}}^n})}, \qquad \text{(A.1)}$$
$$j = 0, \ldots, l_n - 2, l_n, l_n + 1, \ldots, 2l_n - 2$$

where $i(k,j,n)$ is chosen such that $\pi_i^k = \pi_{\bar{j}}^n$ with $\bar{j} = j\ mod(l_n)$.

The function T^J expresses the input rates to a node in terms of the output rates of the upstream nodes. Since each input is either an external flow or the output of another node, we must have the relation $T^J(v) = v$. Now we need to show that this is indeed the case and for a fixed $2N$ dimensional vector w given by $w_n = v_{s_n}$ and $w_{n+N} = v_{s_n+l_n}$, i.e., for fixed values of the external input rates, this solution is unique. It is easy to check that $T^J(\Omega) \subset \Omega$. Let T_w^J be equal to T^J for a fixed w, i.e., $T_w^J = T^J|_{\Omega_w}$ with $\Omega_w = \{v \in \Omega\ :\ v_{s_n} = w_n,\ v_{s_n+l_n} = w_{n+N}\}$. Now we will show that T_w^J has a unique fixed point. From an extension of Banach fixed point theorem [20, p.187], it is enough to show that T_w^J is a condensing mapping which means that for a given metric d and for $u, v \in \Omega_w$, $d(T_w^J(u), T_w^J(v)) < d(u,v)$. We choose the metric d to be the one corresponding to the L_2 norm. To prove that T_w^J is condensing, it is sufficient to show that the transformation at each node between input and output rates is a condensing mapping. Indeed T_w^J can be written as a disjoint sum of such transformations by choosing the appropriate permutation of $\{v_i\}$. Hence we will consider a generic mapping of the form

$$F : D \to D,\ F(v)_j = \frac{v_j C}{\max(\sum_{i=1}^{p} J_i v_i, C)}$$

where $D \subset \mathbb{R}_+^p$ is a compact set for some $p > 0$ and $J \in \mathbb{Z}_+^p$. The compactness condition on the domain D is justified because a flow other than an external input is output of a node and hence it has a bounded rate. Define $\|v\| = \sum_{i=1}^{p} J_i v_i$. Take $u, v \in D$. If $\|u\| < C, \|v\| < C$, then $F(u) - F(v) = u - v$. If $\|u\| < C, \|v\| > C$, take $z = tu + (1-t)v,\ t \in (0,1)$ such that $\|z\| = C$. Then,

$$F(u) - F(v) = F(u) - F(z) + F(z) - F(v) = u - z + F(z) - F(v)$$

Now consider the case $\|u\| > C, \|v\| > C$. Then $\mathcal{J}_F$, the Jacobian of F, is given by

$$\mathcal{J}_F(v) = \frac{C}{\|v\|^2} A \text{ where } A_{ii} = \|v\| - J_i v_i,\ A_{ij} = -J_j v_i,\ i \neq j$$

Thus $A = \|v\| I_p - B$ where $B_{ij} = J_j v_i$ and I_p is the $p \times p$ identity matrix. Note that the matrix B has rank "1" and thus it has one eigenvalue at its trace equal to $\|v\|$ with the remaining eigenvalues being 0. By taking an invertible matrix G such that $G^{-1}BG$ is Jordan form [21] of B, it follows that $G^{-1}AG$ has one eigenvalue at 0 and $p-1$ eigenvalues at $\|v\|$. Therefore the largest eigenvalue of $\mathcal{J}_F(v)$ is $C/\|v\|$ which is less than 1. This implies that F is a condensing mapping.

As mentioned above, T_w^J can be written as the disjoint sum of F type transformations. Therefore T_w^J is also a condensing mapping and has a unique fixed point $v^0(w)$ provided that $v^0(w)$ satisfies $\sum_{n \in \mathcal{M}_k} J_n v^0(w)_{s_n + n_k} + v^0(w)_{s_n + l_n n_k} > C_k$ for at least one node k. Here n_k is chosen such that $\pi_{n_k}^n = k$. If this does not hold, then a fixed point must satisfy $v^0(w)_j = w_n$ for $s_n \leq j < s_n + l_n$ and $v^0(w)_j = w_{n+N}$ for $s_n + l_n \leq j < s_{n+1}$. Therefore if $\sum_{n \in \mathcal{M}_k} J_n v^0(w)_{s_n + n_k} + v^0(w)_{s_n + l_n n_k} < C_k$ holds for every node k, then $v^0(w)$ is a unique fixed point. Thus we have shown that for every w, there exists a unique $v^0(w)$ satisfying $T_w(v^0(w)) = v^0(w)$. In other words, for a given set of transmission rates r_n and average loads ρ_n of the external inputs, the corresponding internal ones are uniquely defined as in equations (3) and (4). Furthermore $v^0(w)$ is also a continuous function of w. Indeed assume that this is not true and there exists $n \to \infty$, $w_n \to w$ but $v^0(w_n) \nrightarrow v^0(w)$. But since $v^0(w_n)$ lies in a compact region, there exists $\bar{v}$ s.t. $v^0(w_{n_k}) \to \bar{v} \in \Omega_w$ and $\bar{v} \in \Omega_w$. Since T is continuous, $T(\bar{v}) = \bar{v}$. But this is in contradiction to the uniqueness of the fixed point in Ω_w and thus proves the continuity of $v^0(.)$. $\square$