Dynamic Vehicle Routing with Heterogeneous Demands

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***Abstract*— In this paper we study a variation of the Dynamic Traveling Repairperson Problem (DTRP) in which there are two classes of demands; high priority, and low priority. In the problem, demands arrive in the environment randomly over time and assume a random location and on-site service requirement. A service vehicle must travel to each demand location and provide the required on-site service. The quality of service provided to each class of demands is measured by the expected delay between a demand's arrival and its service completion. The goal is to design policies for the service vehicle which minimize a convex combination of the delays for each class. We provide a lower bound on the achievable delay for this problem, and propose a policy which performs within a known constant factor of the optimal in heavy load (i.e., when the fraction of time the service vehicle spends performing on-site service approaches one). The problem studied in this paper is analogous to the multi-class queuing problem in classical queuing theory.**

I. Introduction

A classical problem in queueing theory is that of priority queues, [1]. In the simplest setup, customers arrive at a single server sequentially over time. Each customer is a member of either the high-priority, or the low-priority class. High priority customers and low priority customers form separate queues. The goal is to provide the best possible quality of service to the high priority queue (Q_α) while maintaining stability of the low priority queue (Q_β). That is, to minimize the expected delay for high-priority customers while keeping the length of low-priority queue finite. When both the customer inter-arrival times and the customer service times are memory-less (i.e., distributed exponentially), the preemptive priority policy is known to be optimal [1]:

> When Q_α is nonempty, serve high priority customers; when Q_α is empty, serve low-priority customers; If a high priority customer arrives while serving Q_β, preempt service and immediately begin serving the high-priority customer.

A more general two-class queuing problem is to minimize a convex combination of the service delays for high- and low-priority customers

$$\min(cD_\alpha + (1-c)D_\beta) \quad \text{where } c \in (0,1).$$

In this case an optimal policy can be created by using a mixed policy that spends fraction c of the time serving Q_α as the high-priority queue, and fraction $(1-c)$ serving Q_β as the high-priority queue [2]. Lower bounds on the achievable delays have also been studied for queuing networks [3].

In this paper we consider a spatial two-class queuing problem. Customers (or demands for service) arrive in an environment according to a stochastic process, and upon arrival assume a location in the environment. A service vehicle must travel to each demand location and provide the required on-site service. The quality of service provided to each class of demands is measured by the expected delay between a demand's arrival, and its service completion.

When there is only one class of demands, the problem is known as the Dynamic Traveling Repairperson Problem (DTRP), first introduced by Bertsimas and van Ryzin [4], [5], [6]. The series of papers by Bertsimas and van Ryzin propose policies within a constant factor of the optimal in both heavy load (i.e., when the fraction of time the service vehicle spends performing on-site service approaches one), and in light load (i.e., when the fraction of time the service vehicle spends performing on-site service approaches zero). They also study the case of multiple service vehicles, vehicles with finite service capacity, and extend their results to arbitrary renewal arrival processes, and nonuniform demand location distributions. In [7], and [8], decentralized policies are developed for the DTRP. Spatial queuing problems have also been studied in the context of urban operations research [9], where approximations are used in order to cast the problems in the traditional queuing framework.

The main contribution of this paper is to introduce the multi-class DTRP. This problem has applications in areas such as UAV surveillance, where targets are given different priority levels based on their urgency or potential importance. We focus on the two-class problem for which we derive a lower bound on the achievable values of the convex combination of delays, and propose a simple policy. The policy is characterized by a parameter $p \in [0,1]$, whose optimal value is simply a function of the arrival rates of the demands of each class, and the convex combination coefficient c. We show that for all values of c, this policy is within a constant factor of the optimal in heavy load.

The paper is organized as follows. In Section II we give some asymptotic properties of the Traveling Salesperson Tour. In Section II-B we formalize the problem and in Section III we derive a lower bound, and in Section IV we introduce and analyze the Randomized Priority Policy. Finally, in Section IV-D we discuss an optimization procedure for p, and present simulation results. Due to space constraints, all proofs, with the exception of the proof of Theorem 3.1 have been omitted, but can be found in the technical report [10].

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II. Background and Problem Statement

In this section we give some results on the asymptotic properties of the traveling salesperson tour in the Euclidean plane and we formalize the two-class dynamic traveling repairperson problem.

A. The Euclidean Traveling Salesperson Problem

The Euclidean Traveling Salesperson Problem (TSP) is formulated as follows: given a set Q of n points in R^d, find the minimum-length tour (i.e., cycle that visits all nodes exactly once) of Q; the length of a tour is the sum of all Euclidean distances on the tour. Let $\mathrm{TSP}(Q)$ denote the minimum length of a tour through all the points in Q; by convention, $\mathrm{TSP}(\emptyset) = 0$. Assume that the locations of the n points are random variables independently and identically distributed in a compact set $\mathcal{E}$; in [11] it is shown that there exists a constant $\beta_{\mathrm{TSP},d}$ such that, almost surely,

$$\lim_{n\to+\infty} \frac{\mathrm{TSP}(D)}{n^{1-1/d}} = \beta_{\mathrm{TSP},d} \int_{\mathcal{E}} \bar{f}(q)^{1-1/d}\, dq \ \text{ a.s.}, \qquad (1)$$

where $\bar{f}$ is the density of the absolutely continuous part of the distribution of the points. The current estimate of the constant in the case $d = 2$ is $\beta_{\mathrm{TSP},2} \simeq 0.7120$, [12].

Notice that the bound (1) holds for all compact sets: the shape of the set only affects the convergence rate to the limit. According to [9], if $\mathcal{E}$ is a "fairly compact and fairly convex" set in the plane, then Eq. (1) provides an adequate estimate of the optimal TSP tour length for values of n as low as 15.

Remarkably, the asymptotic cost of the stochastic TSP for uniform point distributions is an upper bound on the asymptotic cost for general point distributions: i.e.,

$$\lim_{n\to+\infty} \frac{\mathrm{TSP}(Q)}{n^{1-1/d}} \le \beta_{\mathrm{TSP},d} |\mathcal{E}|^{1/d},$$

where $|\mathcal{E}|$ is the volume of $\mathcal{E}$; this follows directly from an application of Jensen's inequality for concave functions to the right hand side of (1).

$$\int_{\mathcal{E}} \bar{f}(q)^{1-\frac{1}{d}}\, dq \le |\mathcal{E}|^{1/d} \left(\int_{\mathcal{E}} \bar{f}(q)\, dq \right)^{1-\frac{1}{d}} \le |\mathcal{E}|^{1/d}.$$

B. Problem Statement

Consider a bounded environment $\mathcal{E}$ in the plane with area $|\mathcal{E}|$. In the environment is a vehicle with maximum speed v. Demands of type α arrive in $\mathcal{E}$ according to a Poisson process with rate λ_α. Similarly, demands of type β arrive in the environment according to a Poisson process with rate λ_β. Upon arrival, demands assume an independently and uniformly distributed location in $\mathcal{E}$. A demand of type α (respectively, β) is serviced when the vehicle spends an on-site service time that is generally distributed with finite mean $\bar{s}_\alpha$ (respectively, $\bar{s}_\beta$).

Consider the arrival of the ith demand of type α. The service delay for the ith demand, $D_\alpha(i)$, is the time elapsed between its arrival and its service completion. The wait time is then given by $W_\alpha(i) := D_\alpha(i) - s_\alpha(i)$, where $s_\alpha(i)$ is the on-site service time required by demand i. Given a stable policy P (i.e., a policy for which the α and β queue lengths remain finite), the steady-state service delay is defined as $D_\alpha(P) := \lim_{i\to+\infty} \mathbb{E}\left[D_\alpha(i)\right]$, and the steady-state wait is $W_\alpha(P) := D_\alpha(P) - \bar{s}_\alpha$. In a similar fashion, we define $D_\beta(P)$ and $W_\beta(P)$ for demands of type β. Then, given a stable policy P, the average delay per demand is given by

$$D(P) = \frac{\lambda_\alpha}{\lambda_\alpha + \lambda_\beta} D_\alpha(P) + \frac{\lambda_\beta}{\lambda_\alpha + \lambda_\beta} D_\beta(P).$$

The average delay per demand is the standard cost functional for queueing systems with multiples classes of demands. Notice that we can write $D(P) = cD_\alpha + (1-c)D_\beta$, with $c = \lambda_\alpha/(\lambda_\alpha + \lambda_\beta)$. Then, a possible way to model priority is to allow any convex combination of D_α and D_β. We are now ready to state our problem.

Problem Statement: Determine the vehicle routing policy P which minimizes

$$D(P) := cD_\alpha(P) + (1-c)D_\beta(P),$$

where $c \in (0, 1)$.

We restrict our attention to the case where

$$c \ge \frac{\lambda_\alpha}{\lambda_\alpha + \lambda_\beta} =: c^*; \qquad (2)$$

recalling the previous discussion, $c = c^*$ implies that we are not giving any priority, while $c > c^*$ implies that the α demands have higher priority. If for given values of λ_α and λ_β the desired value of c does not satisfy equation (2) then the labels α and β on the classes can simply be exchanged. Notice that a necessary condition for there to exist a policy which yields a finite $D(P)$ is

$$\varrho := \lambda_\alpha \bar{s}_\alpha + \lambda_\beta \bar{s}_\beta < 1.$$

The quantity ϱ is known as the "load factor" and captures the fraction of time the service vehicle must be busy in any stable policy.

III. Lower Bound in Heavy Load

In this section we present a lower bound on the two-class DTRP in the form of two results, the first holds only in heavy load, while the second (less tight) bound holds for all ϱ.

Theorem 3.1 (Heavy load lower bound): In heavy load ($\varrho \to 1^-$), for every routing policy, the delay D^* is bounded as

$$D^* \ge \frac{\beta_{\mathrm{TSP}}^2}{2} \cdot \frac{(2-c)\lambda_\alpha + (1-c)\lambda_\beta}{v^2(1-\varrho)^2} \cdot |\mathcal{E}|, \qquad (3)$$

where $\beta_{\mathrm{TSP}} := \beta_{\mathrm{TSP},2}$, and $c \ge c^*$.

Proof: Consider a tagged demand i of type α, and let us quantify its total service requirement. The demand requires on-site service time $s_\alpha(i)$. In addition, to service demand i, the vehicle must travel from the location of the demand served prior to i, to i's location. We denote this distance by $d_\alpha(i)$. Thus the total service requirement of demand i is $d_\alpha(i) + s_\alpha(i)$. The steady state expected travel requirement is $\bar{d}_\alpha := \lim_{i\to+\infty} \mathbb{E}\left[d_\alpha(i)\right]$ and in a similar manner we define $\bar{d}_\beta$. To maintain stability of the system we must require that

$$\lambda_\alpha \left(\frac{\bar{d}_\alpha}{v} + \bar{s}_\alpha \right) + \lambda_\beta \left(\frac{\bar{d}_\beta}{v} + \bar{s}_\beta \right) < 1. \qquad (4)$$

Recalling that $\varrho = \lambda_\alpha \bar{s}_\alpha + \lambda_\beta \bar{s}_\beta$, we rewrite equation (4) as

$$\lambda_\alpha \bar{d}_\alpha + \lambda_\beta \bar{d}_\beta < v(1-\varrho). \tag{5}$$

For a stable policy P, let N_α and N_β represent the number of demands of type α and β in the queue, respectively. From a key result in the DTRP literature (see [13], page 23), we have in heavy load ($\varrho \to 1^-$) the following

$$\bar{d}_\alpha, \bar{d}_\beta \geq \frac{\beta_{\mathrm{TSP}}}{\sqrt{2}} \frac{\sqrt{|\mathcal{E}|}}{\sqrt{N_\alpha + N_\beta}} =: \bar{d}. \tag{6}$$

Combining equations (5) and (6), squaring both sides, and rearranging we obtain

$$\frac{\beta_{\mathrm{TSP}}^2}{2} \frac{|\mathcal{E}|(\lambda_\alpha + \lambda_\beta)^2}{v^2(1-\varrho)^2} < N_\alpha + N_\beta.$$

From Little's law $N_\alpha = \lambda_\alpha W_\alpha$ and $N_\beta = \lambda_\beta W_\beta$, and thus

$$\frac{\lambda_\alpha W_\alpha + \lambda_\beta W_\beta}{\lambda_\alpha + \lambda_\beta} > \frac{\beta_{\mathrm{TSP}}^2}{2} \frac{|\mathcal{E}|(\lambda_\alpha + \lambda_\beta)}{v^2(1-\varrho)^2}. \tag{7}$$

Recall that $W_\alpha = D_\alpha - \bar{s}_\alpha$, and thus equation (7) gives us a constraint on the feasible values of the delays D_α and D_β. In fact, when $c = c^*$, equation (7) yields a lower bound on D^*. In general we require more constraints.

To determine another constraint, consider the case where we provide the best possible service to α-demands, while maintaining stability for β-demands. Since we are looking for a lower bound, we ignore the travel time needed for β-demands; at any instant the vehicle can serve a β-demand by simply stopping for an on-site service time $\bar{s}_\beta$. By reducing the workload of β-demands, we can reduce the delay for α-demands. In this scenario $\bar{d}_\beta = 0$, and the expected travel distance between successive α-demands is bounded by

$$\bar{d}_\alpha \geq \frac{\beta_{\mathrm{TSP}}}{\sqrt{2}} \frac{\sqrt{|\mathcal{E}|}}{\sqrt{N_\alpha}}.$$

Combining the above equation with equation (5), squaring both sides, and rearranging we obtain

$$W_\alpha \geq \frac{\beta_{\mathrm{TSP}}^2}{2} \frac{|\mathcal{E}|\lambda_\alpha}{v^2(1-\varrho)^2}. \tag{8}$$

Using the previous argument for β-demands we obtain

$$W_\beta \geq \frac{\beta_{\mathrm{TSP}}^2}{2} \frac{|\mathcal{E}|\lambda_\beta}{v^2(1-\varrho)^2}. \tag{9}$$

To simplify notation, let us define the linear operator

$$G(x) := \frac{\beta_{\mathrm{TSP}}^2}{2} \frac{|\mathcal{E}|}{v^2(1-\varrho)^2} x.$$

Thus, we can determine a lower bound by solving the linear program

$$\begin{aligned} &\textbf{minimize} \quad cW_\alpha + (1-c)W_\beta, \\ &\textbf{subject to} \quad \begin{bmatrix} \frac{\lambda_\alpha}{\lambda_\alpha+\lambda_\beta} & \frac{\lambda_\beta}{\lambda_\alpha+\lambda_\beta} \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} W_\alpha \\ W_\beta \end{bmatrix} \geq \begin{bmatrix} G(\lambda_\alpha+\lambda_\beta) \\ G(\lambda_\alpha) \\ G(\lambda_\beta) \end{bmatrix}. \end{aligned}$$

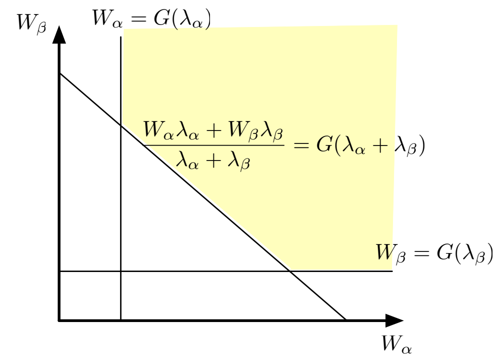


Fig. 1. The feasible region for the wait times W_α and W_β.

Assuming that $c \geq \lambda_\alpha/(\lambda_\alpha + \lambda_\beta)$, the linear program is optimized at the vertex corresponding to the first two constraints. Thus we obtain $W_\alpha \geq G(\lambda_\alpha)$, and

$$W_\beta \geq \frac{1}{\lambda_\beta} \left(G((\lambda_\alpha + \lambda_\beta)^2) - \lambda_\alpha G(\lambda_\alpha) \right).$$

Substituting the bounds on W_α and W_β into the cost function

$$\begin{aligned} cW_\alpha + (1-c)W_\beta &\geq cG(\lambda_\alpha) \\ &\quad + \frac{(1-c)}{\lambda_\beta}(G((\lambda_\alpha+\lambda_\beta)^2) - G(\lambda_\alpha^2)). \end{aligned}$$

Applying the definition of $G(\cdot)$ and simplifying yields

$$cW_\alpha + (1-c)W_\beta \geq \frac{\beta_{\mathrm{TSP}}^2}{2} \frac{((2-c)\lambda_\alpha + (1-c)\lambda_\beta)|\mathcal{E}|}{v^2(1-\varrho)^2}.$$

Finally, letting $W_\alpha = D_\alpha - \bar{s}_\alpha$ and $W_\beta = D_\beta - \bar{s}_\beta$ yields the desired result.

■

Remark 3.2 (Lower bound for all $\varrho \in [0,1)$): With slight modifications to the proof of Theorem 3.1 (see [10]), it is possible to obtain the following lower bound which is valid for all values of ϱ, although less tight

$$D^* \geq \gamma^2 \frac{((2-c)\lambda_\alpha + c\lambda_\beta)|\mathcal{E}|}{v^2(1-\varrho)^2} - \frac{c(1-2\varrho_\alpha)}{2\lambda_\alpha} + \frac{(1-c)\varrho_\beta}{\lambda_\beta}, \tag{10}$$

where $\varrho_\alpha = \lambda_\alpha \bar{s}_\alpha$ and $\varrho_\beta = \lambda_\beta \bar{s}_\beta$, which holds for all $\varrho \in [0,1)$ and $c \geq \lambda_\alpha/(\lambda_\alpha + \lambda_\beta)$. •

IV. RANDOMIZED PRIORITY POLICY

In this section we propose and analyze a policy that is within a constant factor of the previous lower bound. In Section IV we introduce the policy and in Section IV-B we establish two main results, Theorem 4.2 which characterizes the α and β queue lengths, and Theorem 4.3 which establishes the constant factor.

A. Randomized Priority Policy

In the following, we assume that information on outstanding demands of type α and of type β at time t is summarized, respectively, as a finite set of demand positions

$Q_\alpha(t)$ and $Q_\beta(t)$, with $N_\alpha(t) := \mathrm{card}(Q_\alpha(t))$ and $N_\beta(t) := \mathrm{card}(Q_\beta(t))$. Demands of type α (respectively of type β) are inserted in sets Q_α (Q_β) as soon as they are generated; removal from the set Q_α (respectively Q_β) requires that the service vehicle moves to the demand location, and stays there for the required on-site service time.

Randomized Priority Policy (RP)

if the set $Q_\alpha \cup Q_\beta$ is empty **then**
    Move toward the median of $\mathcal{E}$.
if the set $Q_\alpha \cup Q_\beta$ is non-empty **then**
    with probability p **do**
        Set $Q := Q_\alpha$; /* task TS_1 */
    and probability $1-p$ **do**
        Set $Q := Q_\alpha \cup Q_\beta$; /* task TS_2 */
    Compute the TSP tour through all demands in Q.
    Service all demands by following the TSP tour, starting at the demand closest to the vehicle's current position.
Repeat.
Optimize over p.

How close to the lower bound is the performance of the RP policy? How do we optimize over p? The following analysis will provide precise answers to these questions.

B. Analysis of the RP Policy in Heavy Load

We analyze the RP policy in the heavy load case, i.e., $\varrho \to 1^-$. We first introduce some notation. We will refer to the time instant t_i in which the vehicle computes a new TSP tour as the epoch i of the policy; we refer to the time interval between epoch i and epoch $i+1$ as the i-th iteration and we will refer to its length as T_i. For brevity we define

- $Q_\alpha(t_i) := Q_{\alpha,i}$: set of outstanding α-demands at beginning of iteration i;
- $N_\alpha(t_i) := N_{\alpha,i}$: number of outstanding α-demands at beginning of iteration i;
- $Q_\beta(t_i) := Q_{\beta,i}$: set of outstanding β-demands at beginning of iteration i;
- $N_\beta(t_i) := N_{\beta,i}$: number of outstanding β-demands at beginning of iteration i;

The following lemma, although straightforward, will be essential in deriving our main results (see [10] for its proof).

Lemma 4.1 (Queue size in heavy load): In heavy load (i.e., $\varrho \to 1^-$), after transients, the number of demands serviced in a single tour of the vehicle in the RP policy is very large with high probability (i.e., with probability that tends to 1 as ϱ approaches 1).

By definition of the policy, at iteration i the vehicle will choose to execute task TS_1 with probability p and task TS_2 with probability $1-p$. Then, by the total probability law

$$\begin{aligned}
\mathbb{E}\left[N_{\alpha,i+1}\right] &= p\mathbb{E}\left(N_{\alpha,i+1}|TS_1\right) + (1-p)\mathbb{E}\left(N_{\alpha,i+1}|TS_2\right)\\
\mathbb{E}\left[N_{\beta,i+1}\right] &= p\mathbb{E}\left(N_{\beta,i+1}|TS_1\right) + (1-p)\mathbb{E}\left(N_{\beta,i+1}|TS_2\right),
\end{aligned} \tag{11}$$

where the conditioning is with respect to the task being performed at iteration i. During iteration i of the policy, demands arrive according to a Poisson process. Call, respectively, $N^{\text{new}}_{\alpha,i}$ and $N^{\text{new}}_{\beta,i}$ the α- and β-demands newly arrived during iteration i; then, by the law of iterated expectation

$$\begin{aligned}
\mathbb{E}\left(N_{\alpha,i+1}|TS_1\right) &= \mathbb{E}\left(N^{\text{new}}_{\alpha,i}|TS_1\right) = \lambda_\alpha \mathbb{E}\left(T_i|TS_1\right),\\
\mathbb{E}\left(N_{\beta,i+1}|TS_1\right) &= \mathbb{E}\left(N^{\text{new}}_{\beta,i}|TS_1\right) + \mathbb{E}\left(N_{\beta,i}|TS_1\right)\\
&= \lambda_\beta \mathbb{E}\left(T_i|TS_1\right) + \mathbb{E}\left[N_{\beta,i}\right],\\
\mathbb{E}\left(N_{\alpha,i+1}|TS_2\right) &= \mathbb{E}\left(N^{\text{new}}_{\alpha,i}|TS_2\right) = \lambda_\alpha \mathbb{E}\left(T_i|TS_2\right),\\
\mathbb{E}\left(N_{\beta,i+1}|TS_2\right) &= \mathbb{E}\left(N^{\text{new}}_{\beta,i}|TS_2\right) = \lambda_\beta \mathbb{E}\left(T_i|TS_2\right).
\end{aligned} \tag{12}$$

In the second equality, notice that the number of β-demands outstanding at the beginning of iteration i is independent of the task that will be chosen, therefore $\mathbb{E}\left(N_{\beta,i}|TS_1\right) = \mathbb{E}\left[N_{\beta,i}\right]$. Therefore, we are left with computing the conditional expected values of T_i.

The length of T_i is given by the time needed by the vehicle to travel along the TSP tour plus the time spent to service demands. Assuming i large enough, Lemma (4.1) holds, and we can apply Eq. (1) to estimate from $N_{\alpha,i}$ and $N_{\beta,i}$ the length of the TSP tour at iteration i. Then, conditioning on task 1 (when only α-demands are serviced)

$$\begin{aligned}
\mathbb{E}\left(T_i|TS_1\right) = &\frac{\beta_{\text{TSP}}\sqrt{|\mathcal{E}|}}{v}\mathbb{E}\left(\sqrt{N_{\alpha,i}}|TS_1\right) +\\
&+ \mathbb{E}\left(\textstyle\sum_{k=1}^{N_{\alpha,i}} s_{\alpha,k}|TS_1\right)\\
\leq &\frac{\beta_{\text{TSP}}\sqrt{|\mathcal{E}|}}{v}\sqrt{\mathbb{E}\left[N_{\alpha,i}\right]} + \mathbb{E}\left[N_{\alpha,i}\right]\bar{s}_\alpha,
\end{aligned} \tag{13}$$

where we have

- applied Eq. (1);
- applied Jensen's inequality for concave functions, in the form $\mathbb{E}\left[\sqrt{X}\right] \leq \sqrt{\mathbb{E}\left[X\right]}$;
- removed the conditioning on TS_1, since random variable $N_{\alpha,i}$ is independent from *future* events, and thus from the choice of the task at iteration i;
- used the fact that the on-site service times are independent from the number of outstanding demands.

Similarly,

$$\begin{aligned}
\mathbb{E}\left(T_i|TS_2\right) \leq &\frac{\beta_{\text{TSP}}\sqrt{|\mathcal{E}|}}{v}\sqrt{\mathbb{E}\left[N_{\alpha,i}+N_{\beta,i}\right]} +\\
&+ \mathbb{E}\left[N_{\alpha,i}\right]\bar{s}_\alpha + \mathbb{E}\left[N_{\beta,i}\right]\bar{s}_\beta.
\end{aligned} \tag{14}$$

Collecting all results (for short $\mathbb{E}\left[X\right]$ is denoted by $\bar{X}$, where X is any random variable):

$$\begin{aligned}
&\bar{N}_{\alpha,i+1} \leq p\lambda_\alpha\Big(\frac{\beta_{\text{TSP}}\sqrt{|\mathcal{E}|}}{v}\sqrt{\bar{N}_{\alpha,i}} + \bar{N}_{\alpha,i}\bar{s}_\alpha\Big) +\\
&(1-p)\lambda_\alpha\Big(\frac{\beta_{\text{TSP}}\sqrt{|\mathcal{E}|}}{v}\sqrt{\bar{N}_{\alpha,i}+\bar{N}_{\beta,i}} + \bar{N}_{\alpha,i}\bar{s}_\alpha + \bar{N}_{\beta,i}\bar{s}_\beta\Big),
\end{aligned} \tag{15}$$

and

$$\bar{N}_{\beta,i+1} \leq p\left[\bar{N}_{\beta,i} + \lambda_\beta\left(\frac{\beta_{\mathrm{TSP}}\sqrt{|\mathcal{E}|}}{v}\sqrt{\bar{N}_{\alpha,i}} + \bar{N}_{\alpha,i}\bar{s}_\alpha\right)\right] +$$
$$(1-p)\lambda_\beta\left(\frac{\beta_{\mathrm{TSP}}\sqrt{|\mathcal{E}|}}{v}\sqrt{\bar{N}_{\alpha,i} + \bar{N}_{\beta,i}} + \bar{N}_{\alpha,i}\bar{s}_\alpha + \bar{N}_{\beta,i}\bar{s}_\beta\right). \tag{16}$$

The two above inequalities describes a system of recursive relations for an upper bound on $\bar{N}_{\alpha,i}$ and $\bar{N}_{\beta,i}$. The following theorem (whose proof is given in [10]) bounds the values to which they converge.

Theorem 4.2 (Queue length): For every set of initial conditions $(\bar{N}_{\alpha,0}, \bar{N}_{\beta,0}) \in \mathbb{R}^2_{>0}$, the trajectories $i \mapsto (\bar{N}_{\alpha,i}, \bar{N}_{\beta,i})$, resulting from equations (15) and (16), satisfy

$$\limsup_{i\to+\infty} \bar{N}_{\alpha,i} \leq \frac{\lambda_\alpha^2\beta_{\mathrm{TSP}}^2|\mathcal{E}|(p+\sqrt{(1-p)^2+\mu(1-p)})^2}{v^2(1-\varrho)^2},$$

and

$$\limsup_{i\to+\infty} \bar{N}_{\alpha,i} \leq \frac{\lambda_\alpha\lambda_\beta\beta_{\mathrm{TSP}}^2|\mathcal{E}|(p+\sqrt{(1-p)^2+\mu(1-p)})^2}{v^2(1-\varrho)^2(1-p)},$$

where $\mu = \lambda_\beta/\lambda_\alpha$.

With the previous theorem we can prove (see [10]) that the upper and lower bound differ by a known constant factor.

Theorem 4.3 (Constant factor): The Randomized Priority policy performs within a factor

$$\frac{D(RP)}{D^*} \leq 2\frac{\frac{1-pc}{1-p}\left(p+\sqrt{(1-p)^2+(1-p)\mu}\right)^2}{2-c+(1-c)\mu}, \tag{17}$$

of the optimal policy as $\varrho \to 1^-$. By optimizing over p, this factor is bounded by a constant, independent of c and μ.

C. Optimizing the RP Policy

Ideally we would obtain a closed form expression for value of p, as a function of c and $\mu := \lambda_\beta/\lambda_\alpha$, which minimizes the right-hand side of equation (17). Unfortunately this does not seem to be a simple task. However, given values for μ and c, finding the optimal value of p is a simple one-dimensional constrained optimization that can be readily solved. In Fig. 2 the left figure shows optimal values of p as a function of c, for several values of μ. The right figure shows the constant factor at the optimal value of p. These constant factor curves are plotted for all values of c. At $c = c^*$ there is a "kink" corresponding to the swapping of α and β labels as the high-priority demand changes.

One can see that when μ is small there is a threshold below which the optimal value of p is $p = 0$. We let c_{crit} denote the minimum c value at which the optimal value of p is positive. By differentiating the right-hand side of equation (17) with respect to p, and setting $p = 0$, we can determine c_{crit} to be

$$c_{\mathrm{crit}} = 1 + \frac{2}{\sqrt{1+\mu}} - \frac{2+\mu}{1+\mu}. \tag{18}$$

A plot of c_{crit} as a function on μ is shown in Fig. 3. From this figure one can see that for small μ values, the optimal p is $p = 0$, unless c is very close to one. Conversely, when μ is large, $p = 0$ is never the optimal value.

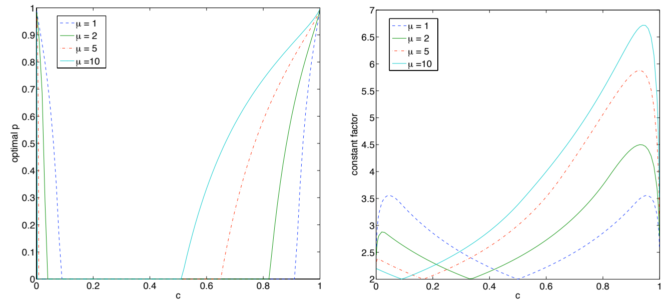


Fig. 2. Left Figure: The optimal value of p as a function of c for several μ values. Right Figure: The constant factor at optimal p for several values of μ. Notice that the curves are plotted for all $c \in (0, 1)$. The "kink" in the constant factor curve occurs at $c = c^*$.

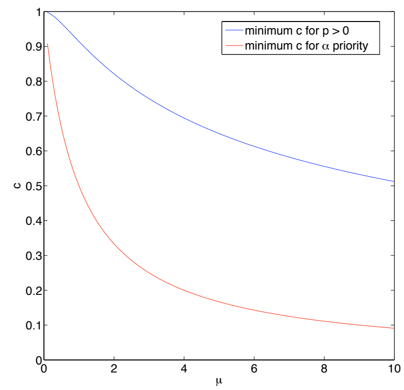
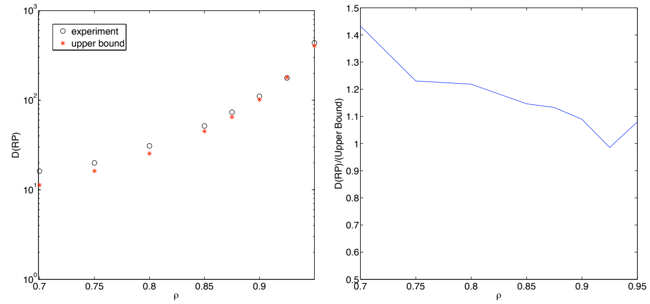


Fig. 3. Minimum value of c at which the optimal p is greater than zero, and the line $c = c^* := 1/(\mu + 1)$.

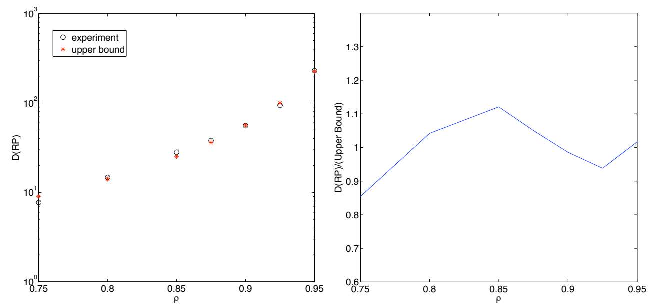
D. Simulations of the RP Policy

Simulations of the Randomized Priority policy were performed using `linkern` [1] as a solver to generate approximations to the optimal TSP tour. Fig. 4 shows a comparison between experimental results and the theoretical upper bound. Each experimental data point represents the average of the steady state delay of ten runs, where each run consists of 300 repetitions of the RP policy. To ensure convergence to steady state and avoid effects due to the transient response, only the last 50 iterations in each run were used to calculate the delay. Changes in the load factor were made by altering the on-site service times, $\bar{s}_\alpha$ and $\bar{s}_\beta$. Fig. 4(a) shows a comparison for equal α and β arrival rates, $c = 0.75$, and an optimal p of zero. Fig. 4(b) shows simulations when the arrival rate of β tasks is five times that of α tasks, $c = 0.8$, and an optimal p value of 0.585. One can see that the upper bound provides a good approximation to the actual performance even for load factors as low as $\varrho = 0.7$. The right hand figures show that as ϱ approaches one, the ratio between the experimental results and the theoretical upper bound decreases and there thus there appears to be evidence that the theoretical upper bound is tight. Note that in some runs the experimental delay is larger than the theoretical upper bound. This unexpected fact is due to one or a combination of the following reasons:

[1] `linkern` is written in ANSI C and is freely available for academic research use at `http://www.tsp.gatech.edu//concorde.html`.



(a) Simulations for $\lambda_\alpha = \lambda_\beta$, $c = 0.75$, and $p_{\text{opt}} = 0$.



(b) Simulations for $5\lambda_\alpha = \lambda_\beta$, $c = 0.8$, and $p_{\text{opt}} = .585$.

Fig. 4. Left Figures: Experimental delay values and delay values predicted by theoretical upper bound. Right Figures: The ratio between the experimental delay values and the theoretical upper bound delay values.

we are using an approximate solution for the optimal TSP, we have not reached the limit as $\varrho \to 1^-$, the transients may not have completely been eliminated in each run.

V. Discussion and Conclusions

A. Using the results

The results of this paper could be applied in the following scenario. A system designer is given the parameters $\lambda_\alpha, \bar{s}_\alpha, \lambda_\beta, \bar{s}_\beta$. For each $c \in (0, 1)$ there is a corresponding optimal value of p, and thus upper bounds on the α and β delays (when $c < c^*$ the labels are switched). Thus, given a tolerance on the α delay, c could be selected such that the tolerance is satisfied and the β delay is kept within a constant factor of the minimum.

B. Light load policy

In light load, when $\varrho \to 0^+$, existing DTRP policies can be used to achieve optimal performance. Indeed the following policy, first introduced in [4] is known to be optimal.

> Locate the vehicle at the median of $\mathcal{E}$. When a demand arrives (α or β), service them first-come-first-served, returning to the median location after each service is completed.

The performance of this policy is independent of c, since in light load, the server has enough "free time" to optimally service both queues.

C. Conclusions

In this paper we have introduced the multi-class Dynamic Traveling Repairperson problem. We derived a lower bound on the achievable performance and proposed the Randomized Priority Policy which performs within a constant factor of the optimal in heavy load. This paper provides an important first step into into the broad class of problems in the dynamic servicing of heterogeneous demands. For future work we would like to extend our results to multiple service vehicles, nonuniform spatial densities, and m classes of demands where $m > 2$.

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