# Distributed Dominating Sets on Grids

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*Abstract*— **This paper presents a distributed algorithm for finding near optimal dominating sets on grids. The basis for this algorithm is an existing centralized algorithm that constructs dominating sets on grids. The size of the dominating set provided by this centralized algorithm is upper-bounded by $\left\lceil \frac{(m+2)(n+2)}{5} \right\rceil$ for $m \times n$ grids and its difference from the optimal domination number of the grid is upper-bounded by five. Both the centralized and distributed algorithms are generalized for the $k$-distance dominating set problem, where all grid vertices are within distance $k$ of the vertices in the dominating set.**

## I. INTRODUCTION

Significant attention has been devoted in recent years to the study of large-scale sensor and robotic networks due to their promise in a variety of fields [1]. One of the key objectives in such networks is to ensure *coverage* of a given area, where every point in the space is within the sensing radius of one or more of the agents (i.e., sensors or robots).

In certain scenarios, the environment may impose restrictions on the feasible locations and motion of the agents [2]. In such cases, it is natural to model the environment as a *graph*, where each node represents a feasible location for an agent, and edges between nodes indicate available paths for the agents to follow. The coverage capabilities of any given agent are then related to the shortest-path distance metric on the graph: an agent located on a node can cover all nodes within a certain distance of that node. The goal of selecting certain nodes in a graph so that all other nodes are within a specified distance of the selected nodes is known as the *dominating set problem* [3]. Versions of this problem appear in settings such as multi-agent security and pursuit [4], routing in communication networks [5], and sensor placement in power networks [6]. Finding the *domination number* (i.e., the size of a smallest dominating set) of arbitrary graphs is NP-hard [3] and in fact one cannot obtain better approximation ratios than $c \log n$ for any $c < \frac{1}{4}$ for general graphs [7].

Grid graphs are a special class of graphs that have attracted attention due to their ability to model and discretize rectangular environments [8], [9]. Due to the special structure of grids, their domination number can in fact be determined optimally, although this number was unknown until recently. For $m \times n$ grids, an upper bound of $\left\lfloor \frac{(m+2)(n+2)}{5} \right\rfloor - 4$ was shown in [10] for $8 \leq m \leq n$ using a constructive method. Various attempts have been made in recent years to find a tight lower bound on the size of the optimal dominating set. In [11], the authors used brute-force computational

techniques to find optimal dominating sets in grids of size up to $n = m = 29$. The paper [12] showed finally that the lower bound on the domination number is equal to the upper bound for $16 \leq m \leq n$, thus characterizing the domination number in grids.

In this paper, we make two contributions to the study of dominating sets on grids and their application to multi-agent coverage. First, we provide a distributed algorithm that locates a set of agents on the vertices of an $m \times n$ grid such that they construct a dominating set for the grid, where the required number of agents is within a constant error from the optimal. The agents require only limited memory, sensing and communication abilities, and thus the solution is applicable to multi-robot coverage applications where the environment can be discretized as a grid. Our distributed algorithm is based on a simple constructive method to obtain *near-optimal* dominating sets (i.e., that require no more than 5 vertices over the optimal number) in grids by Chang [10]. This approach is based on a systematic tiling pattern that we call a *diagonalization*. Second, we generalize Chang's construction to the $k$-distance dominating set problem, where a given vertex can cover all other vertices within a distance $k$ from it. We show that our distributed algorithm can also be generalized to work in the $k$-distance domination scenario.

## II. BACKGROUND

A graph $G = (V, E)$ is defined as a set of vertices $V$ connected by a set of edges $E \subseteq V \times V$. We assume the graph is *undirected*, i.e., $(v, u) \in E \Leftrightarrow (u, v) \in E, \forall v, u \in V$. A vertex $u \in V$ is defined as a *neighbour* of vertex $v \in V$, if $(u, v) \in E$. The set of all neighbours of vertex $v$ is denoted by $N(v)$. For a set of vertices $U \subseteq V$, we define $N(U)$ as $\bigcup_{u \in U} N(u)$. For a set of vertices $U \subseteq V$, we say the vertices in $N(U)$ are *dominated* by the vertices in $U$. For graph $G$, a set of vertices $S \subseteq V$ is a *dominating set* if each vertex $v \in V$ is either in $S$ or is dominated by $S$.

A dominating set with minimum cardinality is called an *optimal dominating set* of a graph $G$; its cardinality is called the *domination number* of $G$ and is denoted by $\gamma(G)$. Note that although the domination number of a graph, $\gamma(G)$, is unique, there may be different optimal dominating sets.

Here, we study the dominating set problem on *grid graphs*. An $m \times n$ grid graph $G = (V, E)$ has vertex set $V = \{v_{i,j} | 1 \leq i \leq m, 1 \leq j \leq n\}$ and edge set $E = \{(v_{i,j}, v_{i,j'}) | |j - j'| = 1\} \bigcup \{(v_{i,j}, v_{i',j}) | |i - i'| = 1\}$ [13]. For ease of exposition, we will fix an orientation and labelling of the vertices, so that vertex $v_{1,1}$ is the lower-left vertex and vertex $v_{m,n}$ is the upper-right vertex of the grid. We denote the domination number of an $m \times n$ grid $G$ by $\gamma_{m,n} = \gamma(G)$.
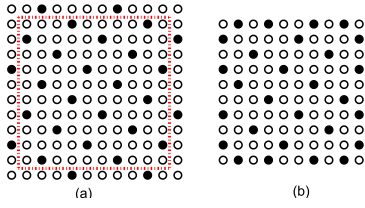
Fig. 1: In (a), a $12 \times 12$ grid $G'$ is demonstrated and its $10 \times 10$ sub-grid $G$ is highlighted by a red dashed square. $G'$ is diagonalized by a set $U'$ of 28 vertices. In (b), vertices in $U' \backslash V$ are projected onto their neighbours in $G$.

**Theorem II.1** (Gonçalves *et al*, [12]). *For an $m \times n$ grid with $16 \leq m \leq n$, $\gamma_{m,n} = \left\lfloor \frac{(m+2)(n+2)}{5} \right\rfloor - 4.$* ∎

We also require the following definitions.

**Definition II.2.** (Grid Boundary) For an $m \times n$ grid $G = (V, E)$, we define the *boundary* of $G$, denoted by $B(G)$, as the set of vertices with less than 4 neighbours. ∎

**Definition II.3.** (Sub-Grids and Super-Grids) An $m \times n$ grid $G = (V, E)$ is called a *sub-grid* of an $m' \times n'$ grid $G' = (V', E')$ if $G$ is induced by vertices $v'_{i,j} \in V'$, where $2 \leq i \leq m' - 1$ and $2 \leq j \leq n' - 1$. If $G$ is a sub-grid of $G'$, $G'$ is called the *super-grid* of $G$ (see Figure 1(a)). ∎

### III. OVERVIEW OF CENTRALIZED GRID DOMINATION ALGORITHM

In this section, we provide an overview of a construction presented in [10] which we will use in the subsequent sections. To begin we require a few definitions.

**Definition III.1.** (Diagonal Pattern) A set of vertices $U \subset V$ constitutes a *diagonal pattern* on grid $G = (V, E)$ if there exists a fixed $r \in \{0, 1, 2, 3, 4\}$ such that for any vertex $v_{x,y} \in U$ we have $y - 2x \equiv r \pmod 5$. ∎

**Definition III.2.** (Diagonalization) A set of vertices $U \subset V$ *diagonalizes* grid $G = (V, E)$ if it constitutes a diagonal pattern and there exists no vertex $v \in V \backslash U$ that can be added to $U$ so that $U$ remains a diagonal pattern. ∎

An example of a diagonalization is shown in Figure 1(a).[1] The algorithm derived from Chang's construction consists of the following two main steps:

(i) Diagonalization: A set of vertices $U$ that diagonalizes the grid is provided.

(ii) Projection: New vertices are added to $U$ to dominate the vertices that are not in $N(U)$.

We now discuss these two steps in more detail. Chang showed in [10] that if a grid $G = (V, E)$ is diagonalized by a set of vertices $U \subset V$, then any vertex $v \in (V \backslash U)$ that is not located on the grid's boundary shares an edge with exactly one vertex in $U$. Moreover, [10] proved that if a set of vertices $U \subset V$ diagonalizes an $m \times n$ grid $G = (V, E)$, then $U$ contains at most $\left\lceil \frac{mn}{5} \right\rceil$ vertices. To construct a dominating set for $G$ it only remains to add some vertices to $U$ so that the resulting set dominates the vertices on the boundary as

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[1]One can also define a diagonal pattern as a set of vertices whose $(x, y)$ coordinates satisfy $x - 2y \equiv r \pmod 5$, for some fixed $r$. This corresponds to swapping the $x$ and $y$ axes. For the proofs we only analyze the case mentioned in Definition III.1; the other case can be treated similarly.

well. The vertices located on $B(G)$ with no neighbour in $U$ are called *orphans* and are defined formally as follows.

**Definition III.3.** (Orphans) Let $U \subset V$ be a set of vertices that diagonalizes grid $G = (V, E)$. A vertex $v \in V$ that has no neighbour in $U$ is called an *orphan* (see Figure 1(a)). ∎

To dominate orphans, Chang used $G' = (V', E')$, the super-grid of $G$. Since the vertices on the boundary of $G$ lie inside grid $G'$, a set of vertices $U' \subset V'$ that diagonalizes $G'$ dominates all vertices of $G$. Moreover, it can be seen that $U = U' \cap V$ is a diagonalization for grid $G$.

Recall that diagonalization results in every vertex being dominated by at most one vertex in the diagonal pattern. Therefore, if a set of vertices $U' \subset V'$ diagonalizes $G'$, then there are vertices in $B(G)$ that are dominated by vertices in $U' \backslash V$. Hence, the *orphan* of a vertex $v \in U' \backslash V$ is a vertex $u \in B(G)$ such that $u \in N(v)$, and is denoted by $u = \text{orphan}(v)$.

Since by diagonalizing $G'$ the orphans in $G$ are dominated by the dominating vertices on the boundary of $G'$, a procedure called *projection* is introduced that projects the dominating vertices in $B(G')$ inside sub-grid $G$. Hence, projection results in having all vertices in $G$ being dominated. This procedure is defined formally as follows.

**Definition III.4.** (Projection) Consider a grid $G = (V, E)$ and its super-grid $G' = (V', E')$. For a set $U' \subseteq V'$, its *projection* is defined as the set $U'' = \left( N(U' \backslash V) \cup U' \right) \cap V$. Similarly, we say a vertex $v \in U' \backslash V$ is *projected* if it is mapped to its neighbour in $V$. ∎

Figure 1(b) shows an example of a projection. For grid $G = (V, E)$, its super-grid $G' = (V', E')$ and set $U' \subset V'$ that diagonalizes $G'$, the size of the obtained dominating set of $G$ using projection is at most $|U'|$. The reason is that vertices in $U'$ that are located at the corners of $G'$ have no neighbours in $V$ and hence, after projection they are not mapped into $V$. Thus, $|U'|$ is an upper-bound on the number of dominating vertices obtained by diagonalization and projection. As the size of the super-grid of an $m \times n$ grid $G$ is $(m+2) \times (n+2)$, one has $|U'| \leq \left\lceil \frac{(m+2)(n+2)}{5} \right\rceil$, leading to the following result.

**Theorem III.5** (Chang, [10]). *For any $m \times n$ grid $G = (V, E)$ with $m, n \in \mathbb{N}$, a dominating set $S \subset V$ can be constructed in polynomial-time, such that $|S| \leq \left\lceil \frac{(m+2)(n+2)}{5} \right\rceil$.* ∎

By virtue of Theorem II.1, the set $S$ provided by diagonalization and projection satisfies $|S| \leq \gamma_{m,n} + 5$, when $16 \leq m \leq n$. An example of constructing dominating sets for grids using diagonalization and projection is shown in Figure 1. In the following lemma, we show that although in diagonal patterns no vertex is dominated by more than one dominating vertex, a simple greedy algorithm does not necessarily result in diagonalizing the grid or using at most $\left\lceil \frac{(m+2)(n+2)}{5} \right\rceil$ dominating vertices. Note that a greedy algorithm adds a vertex to the dominating set that is a neighbour of the most undominated vertices at each step.

**Lemma III.6.** *The size of the dominating set obtained by a greedy algorithm on an $m \times n$ grid $G$ might be as large as $\left\lceil \frac{m}{3} \right\rceil \left\lceil \frac{n}{3} \right\rceil + 2 \left\lfloor \frac{m}{3} \right\rfloor \left\lfloor \frac{n}{3} \right\rfloor.$*

*Proof.* The proof can be found in [14]. □

## IV. DISTRIBUTED GRID DOMINATION

### A. Model and Notation

Here we assume that the environment is an $m \times n$ grid $G = (V, E)$ with $m, n \in \mathbb{N}$. The goal is to dominate the grid environment in a distributed fashion using several robots (or agents) without any knowledge of environment size. Initially, there exist $k$ agents in the environment, where $k$ can be smaller or greater than the number of agents needed to dominate the grid. The following assumptions are made for the grid and agents.

*Grid Assumptions:* Agents can be located only on the vertices of the grid and are able to move between the grid vertices only on the edges of the grid. At each moment, a vertex can contain more than one agent.

*Agent Assumptions:* The agents, denoted by $a_1, \ldots, a_k$, are initially located at arbitrary vertices on the grid. The agents have three modes: (a) *sleep*, (b) *active*, and (c) *settled*. The mode of an agent $a$ and the vertex it is located at are denoted by $\text{mode}(a)$ and $v(a)$, respectively. Only agents in the active and settled modes are able to communicate. At the beginning of the procedure, all the agents are in sleep mode. During each *epoch* (a time interval with a specified length), one agent goes to active mode. The activation sequence of agents is arbitrary. The active agent can communicate with the settled agents to perform the distributed dominating set algorithm. Once an agent activates and performs its part in the algorithm, it goes to settled mode. Ultimately, all settled agents go back to sleep mode and will not activate again.

Here, each agent is equipped with suitable angle-of-arrival (bearing) and range sensors. Using these sensors, agent $a$ computes the coordinates of other agents in its own coordinate frame $\Sigma_a$ with its origin at $v(a)$ and an arbitrary orientation, fixed relative to agent $a$. Each agent also has a compass to determine its heading direction. Additionally, agents are equipped with short-ranged proximity sensors to sense the environment boundary. Agents are able to sense the boundary only if they are on a vertex $v$ whose neighbour is a boundary vertex of the grid, i.e., $N(v) \cap B(G) \neq \emptyset$. The compass helps agents to distinguish which of the four boundary edges they are approaching.

### B. Overview of Algorithm

The main idea in this algorithm is to implement the diagonal pattern defined in Section III on grid $G = (V, E)$, using communications among active and settled agents. A special unit called a *module* is defined for the active and settled agents. A module is a cross-like shape consisting of the agent at its center with the associated dominated vertices in the arms of the cross (see Figure 2(a)). For each module $m$, the vertex that contains the agent, i.e., the center vertex, is referred to as the *module center*, denoted by $c(m)$. Modules $m_1$ and $m_2$ with module centers $c(m_1) = v_{i,j}$ and $c(m_2) = v_{i',j'}$ can connect to each other if $v_{i',j'} \in \{v_{i+1,j+2}, v_{i+2,j-1}, v_{i-1,j-2}, v_{i-2,j+1}\}$ (see Figure 2(f)). This condition is called the *module connection condition*. The set of centers of the connected modules is called a *cluster*.

*Valid Slots:* Let $G' = (V', E')$ be the super-grid of $G$. A vertex $v_{a,b} \in V'$ is called a *slot* if there exists a module $m$ in the cluster with center $v_{i,j}$ such that $v_{a,b} \in \{v_{i+1,j+2}, v_{i+2,j-1}, v_{i-1,j-2}, v_{i-2,j+1}\}$ and $v_{a,b}$ is not already a center for a module in the cluster. For a settled agent $a$ located at $v(a)$, denote the set of all its slots by $\text{slots}(a)$. Recall that the *orphan* of a vertex $v \in V' \backslash V$, i.e., $\text{orphan}(v)$, is a vertex $u \in B(G)$ such that $u \in N(v)$. The set of all *valid slots* for settled agent $a$, denoted by $\text{vslots}(a)$, is defined as $(\text{slots}(a) \cap V) \cup \text{orphan}(\text{slots}(a) \backslash V)$. Newly activated agents can settle only on the valid slots of the settled agents.

*Updating Valid Slots:* When an active agent settles, it creates the list of its valid slots as follows. If a settled agent $a$ cannot sense the boundary (i.e., it has no neighbour on the boundary), $\text{slots}(a) \backslash V = \emptyset$ and hence $\text{vslots}(a) = \text{slots}(a)$. Conversely, a settled agent can also determine which of its slots lie outside the grid boundary (Figure 3(a)). Each newly settled agent marks the vertices on the grid boundary that are neighbours of $\text{slots}(a) \backslash V$ as *orphans* and so $\text{vslots}(a) = (\text{slots}(a) \cap V) \cup \text{orphan}(\text{slots}(a) \backslash V)$ (Figure 3(b)). By the definition of valid slots, no valid slot exists in an orphan's neighbourhood. Therefore, each orphan needs one agent to be located on itself or one of its neighbours to be dominated. For simplicity we always put an agent on the orphan itself.

When an agent activates, it transmits a signal to find the settled agents on the grid and waits for some specified time for a response from them. Since there is no settled agent in the environment when the first agent activates, it receives no signal and concludes it is the first one activated. Thus, the agent stays at its initial location and goes to the settled mode. Subsequently, each active agent translates to the closest settled agent.[2]

### C. Distributed Grid Domination Algorithm

During the distributed grid domination algorithm, active agents can either contribute to grid diagonalization by locating on non-orphan valid slots or can settle on orphans. In each epoch, the set of the non-orphan vertices containing the previously settled agents is called the *cluster* and is denoted by $C$, while the set of occupied orphans is denoted by $P$. At the beginning of the algorithm $C = P = \emptyset$. It should be mentioned that $C$ and $P$ are not saved by any agent, and are used only to aid in the presentation of the algorithm. Moreover, we denote the set of all settled agents at each moment by $A_s$, where at the beginning of the algorithm $A_s = \emptyset$. If agent $a$ is already settled and is now in sleep mode, $\text{done}(a) = 1$; otherwise $\text{done}(a) = 0$.

### D. Distributed Algorithm Analysis

We now prove that the set of vertices determined by Algorithm 1, i.e., $C \cup P$, creates a dominating set for the grid. Recall that at each epoch, $C$ is the set of non-orphan vertices containing the previously settled agents and $P$ is the set of occupied orphans.

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[2]Note that for completeness of the algorithm, it is not necessary for the active agents to go to the closest settled agents. An active agent can go toward any arbitrary settled agent to occupy its valid slot.

## Algorithm 1: DISTRIBUTED GRID DOMINATION

**Input**: An $m \times n$ Grid and a set of agents $A$

1 **while** $\exists$ *agent* $a \in A$ *with* $\text{mode}(a) = sleep$ *and* $\text{done}(a) = 0$ **do**
2      $\text{mode}(a) := active$, $a$ sends out signal to $A_s$ (Figure 2(b)).
3      **if** $A_s \neq \emptyset$ **then**
4          At least one agent in $A_s$ sends a signal out to $a$.
5      **if** *a receives no signal* **then**
6          $\text{mode}(a) := settled$ (Figure 2(a)).
7          Set $A_s := \{a\}$, $C := \{v(a)\}$, and skip to Step 20.
8      **if** $\text{vslots}(A_s) \neq \emptyset$ **then**
9          Agent $a$ computes the closest settled agent $s \in A_s$ and notifies $A_s$.
10          Agent $s$ sends the coordinates of $\text{vslots}(s)$ to $a$.
11          Agent $a$ moves toward the closest $v \in \text{vslots}(s)$.
12          **if** $v(a) = v$ **then**
13              $\text{mode}(a) := settled$ (Figure 2(d)).
14              $A_s := A_s \cup \{a\}$.
15          **if** $v(a)$ *and* $v(s)$ *satisfy connection condition* **then**
16              $C := C \cup \{v(a)\}$.
17          **else**
18              $P := P \cup \{v(a)\}$ (Figure 3(c)).
19              $\text{mode}(a) := sleep$.
20          **for** $i = 1 \to |A_s|$ **do**
21              **if** $v(A_s(i)) \in C$ *and* $\text{mode}(A_s(i)) \neq sleep$ **then**
22                  Update $\text{vslots}(A_s(i))$ (Figures 2(e) and 3(c)).
23                  **if** $\text{vslots}(A_s(i)) = \emptyset$ **then**
24                      $\text{mode}(A_s(i)) := sleep$ (Figure 2(f)).
25                      $\text{done}(A_s(i)) := 1$.
26      **else**
27          Break.
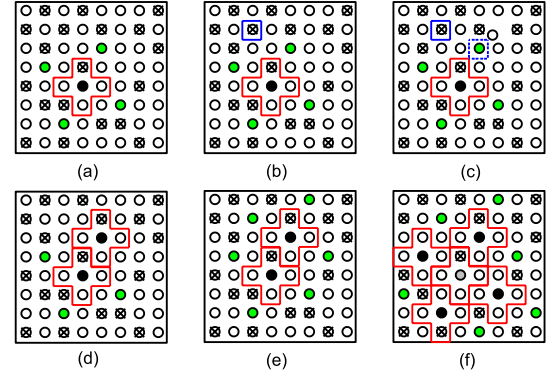28 The remaining non-activated agents leave the grid.



Fig. 2: Non-activated agents are marked by black crosses and the already settled agents are shown by black circles. Agents in $C$ have red crosses as their modules. Figure (a) shows the first active agent, as in Step 6. The green circles show the valid slots. In (b), an active agent is highlighted by a blue square. Step 11 is depicted in (c), where a dashed blue square shows the closest valid slot to the active agent. In (d), the active agent moves to the valid slot and joins $C$, as in Step 13. In (e), the list of valid slots is updated as in Step 22. In (f), the grey circle shows an agent that goes from settled to sleep mode.
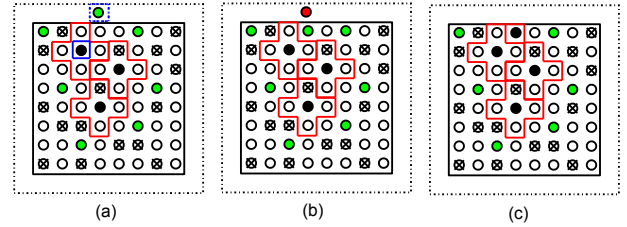


Fig. 3: Non-activated agents are marked by black crosses and the already settled agents are shown by black circles, with red crosses as their modules. The green circles demonstrate the valid slots. In (a), a settled agent, highlighted by a solid blue square, realizes one of its slots, shown by a dashed blue square, is outside the grid boundary. In (b), the settled agent replaces the slot outside the grid boundary with its orphan and names the resulting set as its valid slots. In (c), an active agent locates at the orphan and it is shown that it has no valid slot.

**Lemma IV.1.** *During the operation of Algorithm 1, the module connection condition forces the vertices in $C$ to create a diagonal pattern.*

*Proof.* This will be proved by induction on the size of $C$ during the operation of the algorithm. According to the module connection condition, the module of agent $a$ located at $v(a) = v_{i',j'} \notin C$ can connect to the module of $v_{i,j} \in C$ if $v_{i',j'} \in \{v_{i+1,j+2}, v_{i+2,j-1}, v_{i-1,j-2}, v_{i-2,j+1}\}$. The base of induction is $|C| = 0$, when the first agent is about to be added to $C$. In this case, the first agent settles at its current location $v(a) = v_{i,j}$ and establishes the value $r \equiv j - 2i \pmod 5$.

For $|C| > 1$, $C$ has a diagonal pattern and an active agent $a$ at $v(a) = v_{i',j'}$ aims to join it by connecting to a module centered at $v_{i,j}$. Since $v_{i,j}$ is already in $C$, $j - 2i \equiv r \pmod 5$. It can be seen that for a vertex $v_{i',j'}$ that satisfies the module connection condition with respect to $v_{i,j}$ we have $j' - 2i' \equiv r \pmod 5$. Thus, the resulting set has a diagonal pattern. $\square$

**Theorem IV.2.** *The number of agents used to dominate an $m \times n$ grid $G = (V, E)$ by Algorithm 1 is upper-bounded by $\left\lceil \frac{(m+2)(n+2)}{5} \right\rceil$. For grids with $16 \leq m \leq n$, the number of agents used is upper-bounded by $\gamma_{m,n} + 5$.* ∎

*Proof.* Here we only provide a sketch of the proof, a more detailed proof can be found in [14]. Set $C$ denotes the set of the non-orphan vertices in $G'$, the super-grid of $G$, occupied by the previously settled agents when Algorithm 1 finishes and $P$ is the set of occupied orphans. It can be seen that $C$ diagonalizes $G$ and $P$ contains the projected vertices and hence by Theorems II.1 and III.5 this bound holds. $\square$

Note that while the agents do not form a dominating set for $G$, an active agent finds a valid slot in at most $n + m$ steps. A step is a specified time duration within which an agent performs its basic operation, such as traversing an edge or transmitting signals. Since the number of agents needed to dominate an $m \times n$ grid is less than $mn$, Algorithm 1 takes at most $mn(m + n)$ steps to construct a dominating set for $G$.

Figure 4 shows snapshots from a simulation of Algorithm 1. Figure 4(a) shows the initial positions of 41 agents on a $10 \times 15$ grid graph. The first agent that activates is located on vertex $(5, 9)$ and hence stays on that vertex. Figure 4(b) shows the location of agents when Algorithm 1 is completed. It can be seen that every vertex is dominated.
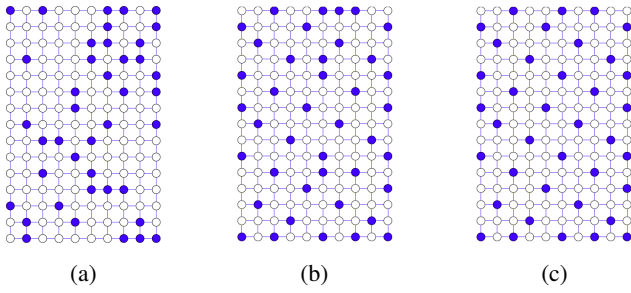
Fig. 4: A $10 \times 15$ grid with agents shown in blue. In (a), the initial configuration of the agents is shown and (b) shows the agents' configuration when Algorithm 1 is finished. In (c), all non-settled and non-asleep agents leave the grid.

## V. $k$-DISTANCE DOMINATION ON GRIDS

In this section we generalize the construction introduced in [10] for grid domination, discussed in Section III, to the $k$-distance dominating set problem, where a vertex dominates all the vertices within distance $k$ from it. Before defining the problem formally, let $d(u,v)$ denote the shortest path distance between vertices $v, u \in V$ in $G = (V, E)$. Moreover, vertex $u \in V$ is defined as a $k$-neighbour of vertex $v \in V$ if $0 < d(u,v) \le k$. The set of all $k$-neighbours of $v$ is denoted by $N^k(v)$. Moreover, for a set of vertices $W \subset V$ and a vertex $v \in V \backslash W$, we have $u = \text{friend}^k(v, W)$ if (a) $u \in W$, (b) $u \in N^k(v)$, and (c) $d(v,u) \le d(v,w), \forall w \in W$.

**Definition V.1.** ($k$-Distance Dominating Set Problem) Given a graph $G = (V, E)$, the $k$-distance dominating set problem is to find a set of vertices $S \subseteq V$ such that for every vertex $v \in V \backslash S$ there exists a vertex $u \in S$ where $u \in N^k(v)$. The cardinality of a smallest $k$-distance dominating set for $G$ is called the $k$-distance domination number of $G$ and is denoted by $\gamma^k(G)$ [15]. ∎

We say that vertex $u \in S$ $k$-distance dominates $v \in V \backslash S$ if $d(u,v) \le k$. The 1-distance dominating set problem is a special case of the $k$-distance dominating set problem, and thus, $k$-distance domination is also NP-hard on general graphs. However, to the best of our knowledge the $k$-distance domination number of grids is not known and the complexity of the problem is open. In Section V-A, we generalize the approaches in Section III to provide a $k$-distance dominating set for an $m \times n$ grid graph $G$.

### A. Centralized $k$-Distance Domination on Grids

Before discussing the $k$-distance domination algorithms on grids we introduce the following definitions. Also note that due to space constraints some of the proofs are not demonstrated here and they can be found in [14].

**Definition V.2.** ($k$-Sub-Grids and $k$-Super-Grids) An $m \times n$ grid $G = (V, E)$ is called a $k$-sub-grid of an $m' \times n'$ grid $G' = (V', E')$ if $G$ is induced by vertices $v'_{i,j} \in V'$, where $k+1 \le i \le m' - k$ and $k+1 \le j \le n' - k$. If $G$ is a $k$-sub-grid of $G'$, $G'$ is called the $k$-super-grid of $G$. ∎

**Lemma V.3.** For an $m \times n$ grid $G$, $|N^k(v)| \le 2k^2 + 2k + 1$.

*Proof.* Since $G$ is a grid, the $k$-neighbours of $v$ form a diamond around it with a diameter of $2k + 1$ (see the red regions in Figure 5(a)). Thus $|N^k(v)|$ is upper-bounded by the area of this region, which is $\left\lceil \frac{(2k+1)^2}{2} \right\rceil = 2k^2 + 2k + 1$. □

In what follows we define $N^k_{\max} = 2k^2 + 2k + 1$.

**Definition V.4.** ($k$-Diagonal Pattern) A set of vertices $U \subset V$ constitutes a $k$-diagonal pattern on grid $G = (V, E)$ if there exists a fixed $0 \le r < N^k_{\max}, r \in \mathbb{Z}_+$ such that for any vertex $v_{x,y} \in U$ we have $ky - (k+1)x \equiv r \pmod{N^k_{\max}}$ (see Figure 5(a)). ∎

The $k$-distance diagonalization of $G$ is a natural generalization of Definition III.2. Moreover, for a grid $G = (V, E)$ and its $k$-super-grid $G' = (V', E')$, the $k$-projection is defined as a special mapping from the vertices in $V' \backslash V$ to their $k$-neighbours in $V$. It is defined formally as follows.

**Definition V.5.** ($k$-Projection) Consider a grid $G = (V, E)$ and its $k$-super-grid $G' = (V', E')$. The $k$-projection for a set $U' \subseteq V'$ is defined as the set $U'' = \{ \bigcup_{v \in U' \backslash V} \text{friend}^k(v, V) \} \cup \{U' \cap V\}$, (see Figure 5(b)). ∎

**Lemma V.6.** Let $U$ be a set of vertices that $k$-diagonalizes a grid $G = (V, E)$. For any two vertices $v_{x,y}, v_{x',y'} \in U$ we have $d(v_{x,y}, v_{x',y'}) \ge 2k + 1$.

**Lemma V.7.** Consider a grid $G = (V, E)$ and its $k$-super-grid $G' = (V', E')$. If $U' \subset V'$ $k$-diagonalizes $G'$, then each vertex in $V$ is $k$-dominated by exactly one vertex from $U'$.

*Proof.* For each vertex $v_{x,y} \in V$ let $r_{v_{x,y}} \equiv ky - (k+1)x \pmod{N^k_{\max}}$. Consider any vertex $v \in V$ and its $k$-neighbourhood $N^k(v)$. The distance between any two vertices in $J = \{v\} \cup N^k(v)$ is at most $2k$. Also, there are exactly $N^k_{\max}$ vertices in this set. Thus, for any two distinct vertices $u, w \in J$ we have $r_u \neq r_w$ by Lemma V.6. Hence each vertex $u \in N^k(v)$ has a distinct value of $r_u$. Consequently, for the value of $r$ that corresponds to the diagonalization $U'$, there is exactly one vertex in the $k$-neighbourhood of $v$ such that $r_v = r$ and thus $v$ is $k$-dominated by exactly one vertex from $U'$. □

**Lemma V.8.** If a set of vertices $U \subset V$ $k$-diagonalizes an $m \times n$ grid $G = (V, E)$, then $|U| \le \left\lceil \frac{mn}{N^k_{\max}} + \frac{N^k_{\max}}{4} \right\rceil$.

**Theorem V.9.** For an $m \times n$ grid $G = (V, E)$, a $k$-distance dominating set $S \subset V$ can be constructed using $k$-diagonalization and $k$-projection in polynomial-time such that $|S| \le \left\lceil \frac{(m+2k)(n+2k)}{N^k_{\max}} + \frac{N^k_{\max}}{4} \right\rceil$. ∎

*Proof.* The proof follows from Lemmas V.3, V.7 and V.8, by replacing the diagonalization and projection operations with the $k$-diagonalization and $k$-projection operations in the proof of Theorem III.5 [10]. □

**Lemma V.10.** If $S \subset V$ is a $k$-distance dominating set for an $m \times n$ grid $G = (V, E)$, $|S| \ge \left\lceil \frac{mn}{N^k_{\max}} \right\rceil$.

*Proof.* According to Lemma V.3, a vertex $v \in V$ $k$-distance dominates at most $N^k_{\max}$ vertices. Hence, at least $\left\lceil \frac{mn}{N^k_{\max}} \right\rceil$ dominating vertices are needed to $k$-dominate an $m \times n$ grid.
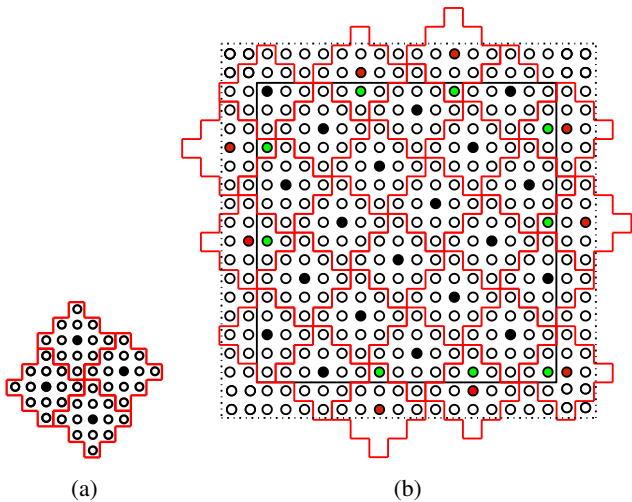
Fig. 5: A 2-diagonal pattern with similar structure to a 1-diagonal pattern is shown in Figure (a). In (b), a $16 \times 16$ grid $G$ and its 2-super-grid $G'$ are shown by solid and dashed squares, respectively. Both grids are 2-diagonalized. The black circles are the vertices that 2-diagonalize $G$. The union of red and black circles 2-diagonalizes $G'$. The green circles are the 2-projections of the red circles onto $G$. Before projection these vertices are called orphans.

Note that we use $|S| \geq \left\lceil \frac{mn}{N^k_{\max}} \right\rceil$ instead of $|S| \geq \left\lfloor \frac{mn}{N^k_{\max}} \right\rfloor$ since dominating vertices in the $k$-neighbourhood of vertices on the grid boundary do not have all their $k$-neighbours in $V$. $\quad\square$

**Corollary V.11.** *Let $S$ be a $k$-distance dominating set for an $m \times n$ grid $G = (V, E)$ obtained by $k$-diagonalization and $k$-projection and let $L$ denote the lower-bound for $S$ from Lemma V.10. For any constant $k \in \mathbb{Z}_+$, the approximation ratio $\frac{|S|}{L}$ satisfies $\lim_{n,m \to \infty} \frac{|S|}{L} = 1$.*

*Remark* V.12 (*$k$-th Power of Grids*). For a graph $G$, its *$k$-th power*, denoted by $G^k$, is a graph with the same vertex set in which two distinct vertices share an edge if and only if their distance in $G$ is at most $k$ [13]. It might seem that a reasonable approach for $k$-distance domination on a grid $G$ is to take its $k$-th power graph $G^k$ and perform 1-distance domination on it. However, $G^k$ is no longer a grid and thus direct analysis of the power graph is potentially difficult. $\blacksquare$

### B. Distributed k-Distance Domination on Grids

Using the algorithm explained in Section V-A, a distributed $k$-distance domination algorithm can be designed for grids. In practice, the $k$-distance dominating set problem corresponds to settings where agents are equipped with longer range sensory equipment and can sense vertices up to distance $k$ from them. Therefore, the goal is to arrange the agents on the grid vertices in a distributed way such that for each vertex there exists at least one agent with distance at most $k$ from it.

This algorithm is similar to Algorithm 1 in Section IV-C, except for two modifications. The first modification is that module $m_2$ and module $m_1$ with module centers $c(m_1) = v_{i,j}$ and $c(m_2) = v_{i',j'}$ can now connect to each other if $v_{i',j'} \in \{v_{i+k,j+k+1}, v_{i+k+1,j-k}, v_{i-k,j-k-1}, v_{i-k-1,j+k}\}$ (see Figure 5(a)). These constitute the *slots*. The second

modification is the definition of *orphans*. If $U'$ is a set of vertices that $k$-diagonalizes the $k$-super-grid of $G$, vertex $v \in V$ is an orphan if it satisfies the two following conditions: (a) $v$ has no $k$-neighbour in $U' \cap V$, and (b) there exists a vertex $u \in U' \setminus V$ so that $v$ is the closest vertex in $V$ in its $k$-neighbourhood. Hence, *valid slots* are defined for each settled agent as the union of its slots located inside the grid and the orphans of its slots located outside the grid (see Figure 5(b)).

## VI. SUMMARY AND OPEN PROBLEMS

In this paper we studied the dominating set and $k$-distance dominating set problems on $m \times n$ grids. We discussed a construction from [10] to obtain dominating sets for grids with near optimal size and generalized it to work in the $k$-distance domination scenario. We used these methods in distributed algorithms and showed that the resulting dominating sets have the same upper-bounds as the centralized algorithm.

There are many open problems in this area. The $k$-distance domination number of grids is still unknown. It is also of interest to find centralized and distributed algorithms for dominating sub-graphs of grids. Generalizing these algorithms to the cases where the underlying graphs are cubic or hyper-cubic grids is another direction for research.

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