Differential form of Maxwells

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} - \vec{M}$$

This is the Faraday Maxwell Eqn it is similar to lenzs law stating that a changing magnetic flux induces a curling E field

$$\nabla \times \vec{B} = \frac{\partial \vec{D}}{\partial t} + \vec{J}$$ Ampere maxwell law

$\nabla \cdot \vec{D} = \rho$ → gauss law E field

$\nabla \cdot \vec{B} = 0$ → gauss law B field

Maxwells + Boundary conditions
↳ effect of dielectric & magnetic materials

$\vec{\mathcal{E}}$ = E field

$\mathcal{B}$ magnetic field

Φ magnetic flux

Integral form of gauss / maxwell

$$\oint_S \vec{D} \cdot ds = \int_{\text{Volume of surface}} \rho \, dv = Q_{\text{enc total}}$$

$$\oint_S \vec{B} \cdot ds = 0$$

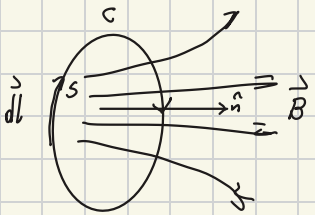
Integral forms of Ampere / Faraday

$$\oint_C \vec{E} \cdot dl = \frac{-\partial}{\partial t} \int_S B \cdot ds - \int_S \vec{M} \cdot ds$$

$$\oint_C \vec{H} \, dl = \frac{\partial}{\partial t} \int_S \vec{D} \cdot ds + I \qquad I = \int_S \vec{J} \cdot ds$$

$$\oint \vec{D} \cdot ds = \int_v \rho \, dv = Q \qquad \vec{B} = \mu_0 \vec{H} \qquad \vec{D} = \varepsilon_0 \vec{E}$$

$$\oint B \cdot dS =$$



$$\vec{\mathcal{E}}(x,y,z,t) = \hat{x} A(x,y,z) \cos(\omega t + \phi)$$

A → real amplitude

ω → rad f

Φ → phase ref

$$\vec{E}(x,y,z) = \hat{x} A(x,y,z) e^{i\phi}$$

$$\vec{\mathcal{E}}(x,y,z,t) = Re\{E(x,y,z) e^{i\omega t}\}$$

$$\nabla \times \vec{E} = -\frac{\partial B}{\partial t} - \vec{M}$$

$$\nabla \times \vec{B} = \frac{\partial D}{\partial t} + \vec{J} \qquad \vec{B} = \mu_0 \vec{H} \qquad \vec{D} = \mu_0 \vec{E}$$

$\nabla \cdot \vec{D} = \rho$

$\nabla \cdot \vec{B} = 0$

Maxwells Equations in Phasor form

$\nabla \times \vec{E} = -j\omega\vec{B} - \vec{M}$

$\nabla \times \vec{H} = j\omega D + \vec{J}$ J, M volume

$\nabla \cdot \vec{D} = \rho$ current densities actual

$\nabla \cdot \vec{B} = 0$ currents written out of form current

sheet line current infinitesimal dipole current

Fields in Media / Boundary Conditions

↳ Assumed E field B field in free space with no material boundary which isn't the case

Electromagnetic fields in other material mediums field vectors are related to each other

Transmision lines stuff...

$$\frac{\partial V}{\partial z} = -(R + j\omega L) I$$

$$\frac{\partial I}{\partial z} = -(G + j\omega C) V$$

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)}$$

$$Z_0 = \sqrt{(R + j\omega L)/(G + j\omega C)}$$

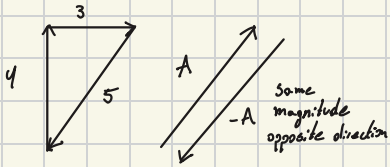
$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0}$$

$$\nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

$$\nabla \cdot V = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

Start with griffiths instead

Chapter 1. Vector Anaylsis



- Vectors have a magnitude + direction
- Scalars only have magnitude

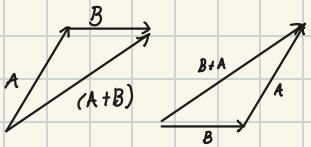
$\vec{A}, \vec{B}$ magnitude $|A|, |B|$

i) Addition of two vectors is tip to tail

↳ commutative $A+B = B+A$

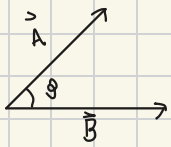
↳ Addition is also associative $(A+B)+C = A+(B+C)$

Subtraction is addition by opposite



ii) multiplication by a scalar multiplies magnitude direction unchange

↳ distributive; $a(A+B) = aA + aB$



iii) Dot product of 2 vectors

$A \cdot B = AB\cos\theta$

- θ is the angle they form when placed tail to tail

↳ Dot product is scalar it is commutative

$A \cdot B = B \cdot A$

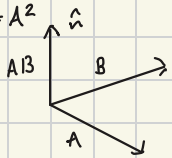
↳ and Distributive

$A \cdot (B+C) = A \cdot B + A \cdot C$

Geometrically $A \cdot B$ product of A times proj of B along A

if vectors are // $A \cdot B = AB$ $\vec{A} \cdot \vec{A} = A^2$ $\hat{n}$

vectors are ⊥ $A \cdot B = 0$



iv) Cross product of 2 vectors

$A \times B \equiv AB\sin\theta\,\hat{n}$

$\hat{n}$ → unit vector points perpindicular to plane formed by A and B

2 directions possible for any plane "in" and "out"

rhl: finger point in direction of first vector curl around toward second thumb is direction

Cross product is distributive

$A \times (B+C) = (A \times B) + (A \times C)$

not commutative

$B \times A = -(A \times B)$

$|A \times B|$ → Area of parallelogram

if vectors // cross product 0

$A \times A = 0$

Component form of Vectors

$\hat{x}, \hat{y}, \hat{z}$ unit vectors parallel to x, y, z axis

arbitrary A vector can be made from these terms

$$A = A_x \hat{x} + A_y \hat{y} + A_z \hat{z}$$

A_x, A_y, A_z are the components projections of A along coordinate axis

i) Add vectors, add like components

$A + B$

$(A_x + B_x)\hat{x} + (A_y + B_y)\hat{y} \cdots$

ii) multiply by scalar

$aA = aA_x \hat{x} \cdots$

$\hat{x} \cdot \hat{x} = \hat{y} \cdot \hat{y} \cdots = 0 \quad x \cdot y = 0 = x \cdot z = y \cdot z$

$A \cdot B = A_x B_x + A_y B_y + A_z B_z$

iii) to calculate dot product multiply like and Add

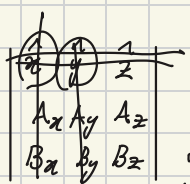
$\hat{x} \times \hat{x} = \hat{y} \times \hat{y} = \hat{z} \times \hat{z} = 0$

$\hat{x} \times \hat{y} = -\hat{y} \times \hat{x} = \hat{z}$

$\hat{y} \times z = -z \times \hat{y} = \hat{x}$

$\hat{z} \times \hat{x} = -\hat{x} \times \hat{z} = \hat{y}$

$A \times B =$



cross product can be written as a determinant

cofactor

$1,1^{+2}$	$1,2^{-3}$	$1,3^{+4}$
+	−	+
−	+	−
+	−	+

$$A \times B = (A_y B_z - A_z B_y)\hat{x} + (A_z B_x - A_x B_z)\hat{y} + (A_x B_y - A_y B_x)\hat{z}$$

$A \cdot B = 1 \cdot 0 + 0 \cdot 1 + 1 \cdot 1 = 1$

$A \cdot B = AB\cos\theta$

$\sqrt{2}\sqrt{2}\cos\theta = 1$

$\cos\theta = \frac{1}{2}$

$\theta = 60°$

Triple products

· product of 2 vectors is a vector so it can be dotted or crossed with a 3rd vector

i) Scalar triple product

$A \cdot (B \times C)$

$A \cdot (B \times C) = B \cdot (C \times A) = C \cdot (A \times B)$

$$A\cdot(B\times C) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

$A\cdot(B\times C) = (A\times B)\cdot C$

you can't $(A\cdot B)\times C$ (scalor → vector)

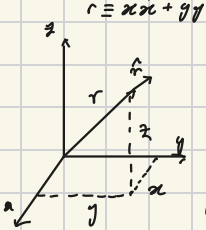
ii) Vector triple product

$A\times(B\times C) = B(A\cdot C) - C(A\cdot B)$

Position, Displacement, and seperation Vectors

- location of a point can be described by cartesian coords
  - ↳ that point from origin is the position vector

$r \equiv x\hat{x} + y\hat{y} + z\hat{z}$



dist from origin

$r = \sqrt{x^2+y^2+z^2}$

$\hat{r} = \frac{\vec{r}}{r} = \frac{x\hat{x}+y\hat{y}+z\hat{z}}{\sqrt{x^2+y^2+z^2}}$

↳ unit vector pointing radially out

infitesimal displacement vector

$dl = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$

$\mathcal{L} \equiv r - r'$ → seperation vector source point to field point

$\mathcal{L} = |r-r'|$

$\hat{\mathcal{L}} = \frac{\mathcal{L}}{\mathcal{L}} = \frac{r-r'}{|r-r'|}$

$\vec{\mathcal{L}} = (x-x')\hat{x} + (y-y')\hat{y} + (z-z')\hat{z}$

$\hat{\mathcal{L}} = \frac{(x-x')\hat{x} + (y-y')\hat{y} + (z-z')\hat{z}}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}}$

Vector transformations

$A_y = A\cos\vartheta \quad A_z = A\sin\vartheta$

$$\begin{bmatrix} \bar{A}_y \\ \bar{A}_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} A_y \\ A_z \end{bmatrix} \qquad \bar{A}_i = \sum_{j=1}^{3} R_{ij} A_j \qquad \bar{T}_{ij} = \sum_{k=1}^{3}\sum_{l=1}^{3} R_{ik} R_{jl} T_{kl}$$

Differential Calculus

Ordinary derivatives

$\frac{df}{dx}$ → tells us how rapidly the function $f(x)$ varies when we change argument x by tiny dx

$df = \left(\frac{df}{dx}\right) dx$ increment x by a infitesimal amount dx f changes by df derivitive proportional factor

$\frac{df}{dx}$ → slope of graph f verses x

Gradient

- function of 3 variables temp of room $T(x,y,z)$

each point x,y,z gives temp T at that spot depends on 3 vars

derivative tells how fast function varies by moving tiny distance

here it depends on what direction you move in? increase horizontal not much

↳ how fast does T vary infinite solutions

$$dT = \frac{\partial T}{\partial x} dx + \frac{\partial T}{\partial y} dy + \frac{\partial T}{\partial z} dz$$

↳ How T changes when you alter 3 variables by infitesimal amounts dx, dy, dz

Dot product like

$$dT = \left(\frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}\right) \cdot dx\hat{x} + dy\hat{y} + dz\hat{z}$$

$$= (\nabla T) \cdot (dl)$$

∇T is the gradient vector

$$dT = \nabla T \cdot dl = |\nabla T||dl| \cos\vartheta$$

The gradient ∇T points in direction of max increase of function T

The magnitude $|\nabla T|$ gives the slope along maximal direction

Standing on a hill → steepest ascent = direction of gradient

Slope of direction → magnitude of gradient

$\vartheta = 90$ slope is 0

$\nabla T = 0$ $dT = 0$ stationary point max / min / saddle point

Del Operator

$$\nabla T = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right) T$$

$$\nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

del is not a vector differentiates a function

∇ is a vector operator that acts upon T

Vector can multiply 3 ways

1. By scalar
2. By a vector dot product $A \cdot B$
3. By a vector cross product $A \times B$

Same with operator ∇

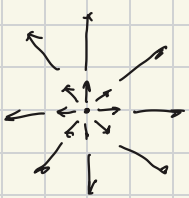
1. On scalar function T: ∇T gradient
2. On vector function dot $\nabla \cdot V$ (divergence)
3. Cross product on Vector function $\nabla \times V$ (curl)

The divergence

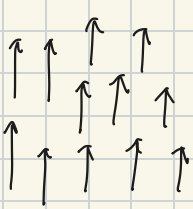
$$\nabla \cdot V = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right) \cdot (v_x \hat{x} + v_y \hat{y} + v_z \hat{z})$$

$$= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$ } divergence is a scalar

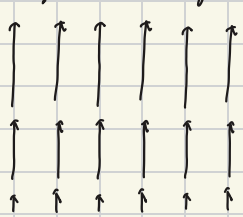
→ divergence is a measure of how much the vector V spreads out (diverges) from a point



large positive divergence (Arrows in negative)



Zero divergence



positive divergence

Imagine a pond sprinkling dust into it if the material spreads out like a faucet positive divergence if it colects together negative divergence / drain / sink

worked example of divergence

$v_a = r = x\hat{x} + y\hat{y} + z\hat{z}$

$V_b = \hat{z} \quad V_c = z\hat{z}$

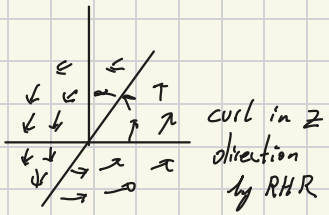
$\nabla \cdot V_a = \frac{\partial}{\partial x} x + \frac{\partial}{\partial y} y + \frac{\partial}{\partial z} z = 1+1+1 = 3$

$\nabla \cdot V_b = \frac{\partial}{\partial x} 0 + \frac{\partial}{\partial y} 0 + \frac{\partial}{\partial z} 1 = 0+0+0 = 0$

$\nabla \cdot V_c = \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(z) = 0+0+1 = 1$

Curl

$$\nabla \times \vec{V} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ V_x & V_y & V_z \end{vmatrix}$$



curl in z direction by RHR

$$= \hat{x}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) + \hat{y}\left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right) + \hat{z}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)$$

→ measure of how much vector $\vec{V}$ swirls around a point

whirlpool region with lots of curl

worked example

$V_a = -y\hat{x} + x\hat{y} \quad V_b = x\hat{y}$

$$\nabla \times V_a = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y & x & 0 \end{vmatrix} = 2\hat{z}$$

(constant)

$$\hat{x}\left(\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z} x\right) + \hat{y}\left(\frac{\partial}{\partial z}(-y) - \frac{\partial}{\partial x}(0)\right) + \hat{z}\left(\frac{\partial}{\partial x} x - \left(\frac{\partial}{\partial y}(-y)\right)\right)$$

$0\hat{x} + 0\hat{y} + 2\hat{z} = 2\hat{z}$ $\qquad$ $1 - (-1) = 2$

Product rules

- Sum rule

$$\frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{dg}{dx}$$

- multiplying by constant

$$\frac{d}{dx}(kf) = k\frac{df}{dx}$$

- product rule

$$\frac{d}{dx}(fg) = f\frac{dg}{dx} + g\frac{df}{dx}$$

- quotient rule

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g\left(\frac{df}{dx}\right) - f\left(\frac{dg}{dx}\right)}{g^2}$$

Same holds true for vector derivatives

- $\nabla(f+g) = \nabla f + \nabla g$
- $\nabla\cdot(A+B) = (\nabla\cdot A) + (\nabla\cdot B)$
- $\nabla\times(A+B) = (\nabla\times A) + (\nabla\times B)$
- $\nabla(kf) = k\nabla f$

$\nabla\cdot(kA) = k(\nabla\cdot A)$ $\nabla\times(kA) = k(\nabla\times A)$

6 product rules exist for vector function derivatives

i) $\nabla(fg) = f\nabla g + g\nabla f$ } gradients

ii) $\nabla(A\cdot B) = A\times(\nabla\times B) + B\times(\nabla\times A) + (A\cdot\nabla)B + (B\cdot\nabla)A$ } gradients

iii) $\nabla\cdot(fA) = f(\nabla\cdot A) + A\cdot(\nabla f)$ } divergences

iv) $\nabla\cdot(A\times B) = B\cdot(\nabla\times A) - A\cdot(\nabla\times B)$ } divergences

v) $\nabla\cdot(fA) = f(\nabla\times A) - A\times(\nabla f)$

vi) $\nabla\times(A\times B) = (B\cdot\nabla)A - (A\cdot\nabla)B + A(\nabla\cdot B) - B(\nabla\cdot A)$

Second derivatives

- gradient/divergence/curl first derivatives made with ∇
- You can apply ∇ twice to make a second derivative

↳ gradient ∇T is a vector we can take divergence + curl of it

1) Divergence of gradient $\nabla\cdot(\nabla T)$
2) Curl of gradient $\nabla\times(\nabla T)$

divergence is a scalar $\nabla\cdot V$ only gradient

3) gradient of divergence $\nabla(\nabla\cdot V)$

Curl is a vector $\nabla\times\vec{V}$ can take div and curl

4) Divergence of curl $\nabla\cdot(\nabla\times V)$
5) Curl of Curl $\nabla\times(\nabla\times V)$

1) $\nabla \cdot (\nabla T) = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right) \cdot \left(\frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}\right)$

$= \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}$

$\nabla^2 T \rightarrow$ Laplacian of T (scalar)

$\nabla^2 v \equiv (\nabla^2 v_x)\hat{x} + (\nabla^2 v_y)\hat{y} + (\nabla^2 v_z)\hat{z}$

2. The curl of a gradient is 0

$\nabla \times (\nabla T) = 0$

3. $\nabla(\nabla \cdot v) \rightarrow$ gradient of the divergence rarely any applications

4. The divergence of a curl is 0

$\nabla \cdot (\nabla \times v) = 0$

5. $\nabla \times (\nabla \times v) = \nabla(\nabla \cdot v) - \nabla^2 v$

Laplacian very important

Integral Calculus

↳ Several kinds of integrals exist most important are line/path, surface/flux, and volume integrals

a) Line Integral

$\int_a^b v \cdot dl$ → v is vector function

dl is infinitesimal displacement vector

↳ Integral carried along a prediscribed path P from a to b

If the path forms a loop $b = a$

$\oint v \cdot dl$

At each point along the path we take the dot product of v with the displacement dl to next point on path

Work done by a force is a line integral

$W = \int F \cdot dl$ value of line integral depends critically on path taken a to b

Special class of vector functions for which the line integral is independent of the path

(based on endpoint) force with this property is conservative

line integral example

$v = y^2\hat{x} + 2x(y+1)\hat{y}$ $a = (1,1,0)$ $b = (2,2,0)$

$dl = dx\hat{x} + dy\hat{y} + dz\hat{z}$ (1) 2 parts $dy = dz = 0$

:) $dl = dx\hat{x}$ $y = 1$

$v \cdot dl = y^2\,dx$

$= dx$

$\int v \cdot dl = \int_1^2 dx = 1$

} evaluating horizontal segment

::) evaluating vertical segments

$dl = dy\hat{y}$ $x = 2$

$v \cdot dl = 2x(y+1)\,dy$ $x=2$

$= 4(y+1)\,dy$

$4\int_1^2 (y+1)\,dy = 4\left[\frac{y^2}{2} + y\right]_1^2$

$\frac{4}{2} + 2 - \left(\frac{1}{2} + 1\right)$

$\frac{8}{2} - \frac{3}{2} = \frac{5}{2}$

$\frac{5}{2} \times 4 = 10$

via path 1 then

$\int_a^b v \cdot dl = 1 + 10 = 11$

Path 2

$x = y$ $dx = dy$ $dz = 0$

$dl = dx\hat{x} + dx\hat{y}$

$v \cdot dl = x^2\,dx + 2x(x+1)\,dx$

$= 3x^2 + 2x\,dx$

$\int_a^b v \cdot dl = \int_1^2 3x^2 + 2x\,dx$

$= \left[x^3 + x^2\right]_1^2$

$= 10$

for a loop

$\oint v \cdot dl = 11 - 10 = 1$

Surface integrals

$\int_S v \cdot da$ $v \rightarrow$ some vector function

integral is over some specified surface S

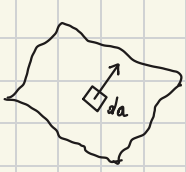
$da \rightarrow$ infinitesimally small patch of area direction $\perp$ to the surface

2 directions for $\perp$ so sign has no meaning

Closed surface $\oint v \cdot da$

traditionally outward is positive but for open surfaces it does not matter

if v describes the flow of a fluid then $\int v \cdot da$ represents flux mass per unit time passing through surface



Surface integral worked example

$V = 2xz\,\hat{x} + (x+2)\hat{y} + y(z^2-3)\hat{z}$

over the 5 sides of the cubical box

:) (out of page)

$x=2 \quad da = dy\,dz\,\hat{x}$

$V \cdot da = 2xz\,dy\,dz = 4z\,dy\,dz$

$4\int_0^2 dy \int_0^2 z\,dz$

$= 4(2) \times \frac{2^2}{2}$

$= 8 \times 2 = 16$

::) $\underline{x=0} \quad da = -dy\,dz\,\hat{x}$

$V \cdot da = -2xz\,dy\,dz = 0$

$\int V \cdot da = 0$

:::) $y=2 \quad da = dx\,dy\,\hat{y}$

$V \cdot da = (x+2)\,dx\,dz$

$\int V \cdot da = \int_0^2 (x+2)\,dx \int_0^2 dz = -12$

::::) $y=0 \quad da = -dx\,dz\,\hat{y}$

$V \cdot da = -(x+2)\,dx\,dz$

$\int V \cdot da = \int_0^2 (x+2)\,dx \int_0^2 dz = 12$

v) $z=2 \quad da = dx\,dy\,\hat{z}$

$V \cdot da = y(z^2-3)\,dx\,dy$

$= y\,dx\,dy$

$\int V \cdot da = \int_0^2 dx \int_0^2 y\,dy = 4$

$\int_{\text{surface}} V \cdot da = 16 + 0 + 12 - 12 + 4 = 20$

Volume integrals

$\int_V T\,d\tau$ T: scalar function

$d\tau$: infinitesimal volume element

Cartesian Coords $d\tau = dx\,dy\,dz$

T could be the density of a substance which can vary point to point

↳ Volume integral would give total mass

$$\int v\,d\tau = \int (v_x \hat{x} + v_y \hat{y} + v_z \hat{z})\,d\tau$$

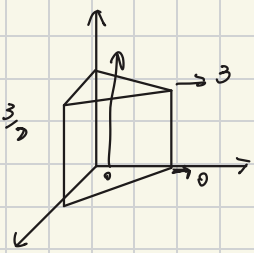
$$= \hat{x}\int v_x\,d\tau + \hat{y}\int v_y\,d\tau + \hat{z}\int v_z\,d\tau$$

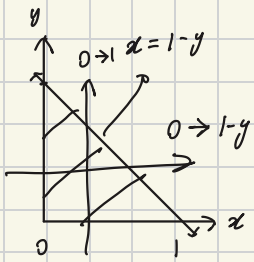
unit vectors are constants

$T = xyz^2$

$$\int T\,d\tau = \int_0^3 z^2 \left\{ \int_0^1 y \left[\int_0^{1-y} x\,dx \right] dy \right\} dz$$

$$\frac{1}{2}\int_0^3 z^2\,dz \int_0^1 (1-y)^2 y\,dy = \frac{1}{2}\cdot 9\left(\frac{1}{12}\right) = \frac{3}{8}$$





FTC. $\int_a^b \left(\frac{df}{dx}\right) dx = f(b) - f(a)$

$$\int_a^b F(x)\,dx = f(b) - f(a)$$

Fundamental theorem for gradients

↳ Suppose a scalar function of 3 variables $T(x,y,z)$, start from a moving a distance dl,

$dT = (\nabla T)\cdot dl$, → function changes by an amount

$$\int_a^b (\nabla T)\cdot dl = T(b) - T(a)$$

$\int_a^b (\nabla T)\cdot dl$ independent of path taken

$\oint (\nabla T)\cdot dl = 0$ start + end same

$T(b) - T(a) = 0$

Fundemental theorem for divergences

$$\int_V (\nabla \cdot v)\, d\tau = \oint_S v \cdot da$$ Gauss theorem / Greens theorem
divergence theorem

$$\int (\text{faucets within volume}) = \oint (\text{flow through the surface})$$

The integral of a derivative (divergence) over a region (volume V)
is equal to the value of the function at boundary (surface S which bounds volume V)
Geometric Interpetation: If v is the flow of a incompressible fluid the flux of v
is total amount of fluid passing through surface

divergence → "spreading out" high div ≡ faucet a bunch of faucets in a region filled with fluid
then a equal amount is forced out the boundary

a) count all faucets to record output
b) go around boundary measuring flow

Ex. Div theo..

$v = y^2 \hat{x} + (2xy + z^2)\hat{y} + 2yz\,\hat{z}$

$\nabla \cdot v = 2(x+y)$

$$\int_V 2(x+y)\, d\tau = 2\int_0^1 \int_0^1 \int_0^1 (x+y)\, dx\, dy\, dz$$

$$\left[\frac{x^2}{2} + yx\right]_0^1$$

$$\int_0^1 \left(\frac{1}{2} + y\right)$$

$$\left[\frac{1}{2}y + \frac{y^2}{2}\right]_0^1$$

$$= 1$$

$$\int_V \nabla \cdot v\, d\tau = 2$$

Surface integral requires you to evaluate all 6 faces

$$\oint v \cdot da = \frac{1}{3} - \frac{1}{3} + \frac{4}{3} - \frac{1}{3} + 1 + 0 = 2$$

Theorem valid
$\int_V \nabla \cdot v = \oint v \cdot da$

Fundemental theorem for curls

$$\int_S (\nabla \times v) \cdot da = \oint_P v \cdot dl$$ AKA Stokes theorem
smaller curls make a large curl around perimeter loop

The integral of a derivative (curl) over a region S is equal to the value of the function at the boundary (perimeter) closed line integral

Geometric view: Curl measures twist of vectors region of high curl ≡ whirlpool

The integral of curl over a surface (flux of curl through surface) is total amount of swirl this total can also be found by going around the edge finding flow at the boundary $\oint v \cdot dl$ → circulation of v

does not matter if CW or CCW just be consistent with sign convention

what direction is da for open surface: its outwards for closed surfaces
with stokes theorem just use rhr thumb direction of da

$\int (\nabla \times v) \cdot da$ depends only on the boundary line not the surface used

$\oint (\nabla \times \vec{v}) \cdot da = 0$ for any closed surface since the boundary line shrinks and vanishes

Example of stokes theorem

$v = (2xz + 3y^2)\hat{y} + (yz^2)\hat{z}$

$\nabla \times \vec{v} = (yz^2 - 2x)\hat{x} + 2z\hat{z}$

$da = dy\,dz\,\hat{x}$ (consistency)

$$\int (\nabla \times v) \cdot da = \int_0^1 \int_0^1 yz^2\,dy\,dz = \frac{1}{3}$$

$$\left[\frac{yz^2}{2}y\right]_0^1$$

$$\int_0^1 \frac{y}{2} z^2\,dz$$

$$\left[\frac{y}{3}z^3\right]_0^1 = \frac{y}{3}$$

i) line integral can be split into 4 segments

$x=0$ $z=0$ $v \cdot dl = 3y^2 dy$ $\int v \cdot dl = \int_0^1 3y^2\,dy = 1$

ii) $x=0$ $y=1$ $v \cdot dl = yz^2\,dz$ $\int v \cdot dl = \int_0^1 yz^2\,dz = \frac{1}{3}$

....

$$\oint v \cdot dl = 1 + \frac{1}{3} - 1 + 0 = \frac{1}{3}$$

Integration by parts

$$\frac{d}{dx}(fg) = f\left(\frac{dg}{dx}\right) + g\left(\frac{df}{dx}\right)$$

$$\int_a^b f\left(\frac{dg}{dx}\right)dx = -\int_a^b g\left(\frac{df}{dx}\right)dx + fg\Big|_b^a$$

$$\int_0^\infty x e^{-x}\,dx$$

$= f(x) = x$

$g(x) = -e^{-x}$

$$\int_0^\infty e^{-x}\,dx - xe^{-x}\Big|_0^\infty$$

$= -e^{-x}\Big|_0^\infty$

$= 1$

IBP for vector calculus

$\nabla \cdot (fA) = f(\nabla \cdot A) + A \cdot (\nabla f)$

$$\int \nabla \cdot (fA)\,d\tau = \int f(\nabla \cdot A)\,d\tau + \int A \cdot (\nabla f)\,d\tau = \oint fA \cdot da$$

$$\int_V f(\nabla \cdot A)\,d\tau = \oint_S fA \cdot da - \int_V A \cdot (\nabla f)\,d\tau$$

Spherical Coordinates

Can label points by Cartesian coordinates (x, y, z)

may be better to use spherical coordinates (r, θ, ϕ)

→ r: radius

θ: polar angle (around z axis)

ϕ: azimuthal angle

$x = r\sin\theta\cos\phi \quad y = r\sin\theta\sin\phi \quad z = r\cos\theta$

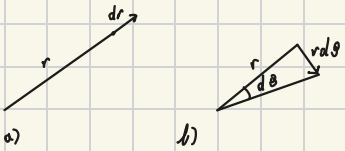
$A = A_r \hat{r} + A_\theta \hat{\theta} + A_\phi \hat{\phi}$

$\hat{r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$

$\hat{\theta} = \cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z}$

$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$

$\hat{r}, \hat{\theta}, \hat{\phi}$ not constant



$dl_\theta = r\,d\theta$

$dl_\phi = r\sin\theta\,d\phi$

$dl = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}$

$d\tau = dl_r\,dl_\theta\,dl_\phi = r^2\sin\theta\,dr\,d\theta\,d\phi$

2 types of surface elements da

$da_1 = dl_\theta\,dl_\phi\,\hat{r} = r^2\sin\theta\,d\theta\,d\phi\,\hat{r}$



$da_2 = dl_r\,dl_\phi\,\hat{\theta} = r\,dr\,d\phi\,\hat{\theta}$



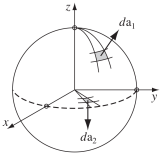


FIGURE 1.39

$0 < r < \infty$

$0 < \phi < 2\pi$

$0 < \theta < \pi$

Vector derivatives in spherical coords

Gradient

$$\nabla T = \frac{\partial T}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{\phi}$$

Divergence

$$\nabla \cdot V = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 V_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta V_\theta) + \frac{1}{r\sin\theta}\frac{\partial V_\phi}{\partial \phi}$$

Curl

$$\nabla \times V = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta V_\phi) - \frac{\partial V_\theta}{\partial \phi}\right]\hat{r} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial V_r}{\partial \phi} - \frac{\partial}{\partial r}(rV_\phi)\right]\hat{\theta} + \frac{1}{r}\left[\frac{\partial}{\partial r}(rV_\theta) - \frac{\partial V_r}{\partial \theta}\right]\hat{\phi}$$

Laplacian

$$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial^2 T}{\partial \phi^2}\right)$$

Cylindrical Coordinates

$x = s\cos\phi \quad y = s\sin\phi \quad z = z$

$\hat{s} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}$

$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$

$\hat{z} = \hat{z}$

$dl_s = ds \quad dl_\phi = s\,d\phi \quad dl_z = dz$

$d\vec{l} = ds\,\hat{s} + s\,d\phi\,\hat{\phi} + dz\,\hat{z}$

$d\tau = s\,ds\,d\phi\,dz$

$0 < s < \infty \quad 0 < \phi < 2\pi \quad -\infty < z < \infty$

gradient: $\nabla T = \frac{\partial T}{\partial s}\hat{s} + \frac{1}{s}\frac{\partial T}{\partial \phi}\hat{\phi} + \frac{\partial T}{\partial z}\hat{z}$

divergence: $\nabla \cdot V = \frac{1}{s}\frac{\partial}{\partial s}(sV_s) + \frac{1}{s}\frac{\partial V_\phi}{\partial \phi} + \frac{\partial V_z}{\partial z}$

Curl: $\nabla \times V = \left(\frac{1}{s}\frac{\partial V_z}{\partial \phi} - \frac{\partial V_\phi}{\partial z}\right)\hat{s} + \left(\frac{\partial V_s}{\partial z} - \frac{\partial V_z}{\partial s}\right)\hat{\phi} + \frac{1}{s}\left[\frac{\partial}{\partial s}(sV_\phi) - \frac{\partial V_s}{\partial \phi}\right]\hat{z}$

Laplacian:

$$\nabla^2 T = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial T}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2}$$

Differential form of maxwells

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} - \vec{M}$$

$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J}$$

$$\nabla \cdot \vec{D} = \rho$$

$$\nabla \cdot \vec{B} = 0$$

$\vec{E}$ = E field

$\vec{H}$ = Magnetic field

$\vec{D}$ = electric flux density

$\vec{B}$ = magnetic flux density

$\vec{M}$ = magnetic current dens

$\vec{J}$ = electric current densit

ρ = electric Chogen den

$\vec{B} = \mu_0 H \quad \vec{D} = \varepsilon_0 \vec{E}$

Integral form of maxwells

$$\oint_S D \cdot ds = \int_V \rho\,dv = Q_{enc}$$

$$\oint_S \vec{B} \cdot ds = 0$$

The Dirac Delta function

Consider vector function

$$v = \frac{1}{r^2}\hat{r}$$

At all points v is directed radially outward when you calculate div you get 0

$$\nabla \cdot v = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{1}{r^2}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}(1) = 0$$

$$\oint v \cdot da = \int \left(\frac{1}{R^2}\hat{r}\right) \cdot (R^2 \sin\theta \, d\theta \, d\phi \, \hat{r})$$

$$= \left(\int_0^{\pi} \sin\theta \, d\theta\right)\left(\int_0^{2\pi} d\phi\right) = 4\pi$$

$\int \nabla \cdot v \, d\tau$ is zero

but div theorem should yield $\int (\nabla \cdot v) \, d\tau = 4\pi$

entire contribution of a singular point vanishes elsewhere but integral over volume containing point is 4π

One dimensional Dirac Delta Function

$$\delta(x) = \begin{cases} 0, & \text{if } x \neq 0 \\ \infty, & \text{if } x = 0 \end{cases}$$

not a function

distribution / generalized

and $\int_{-\infty}^{\infty} \delta(x) = 1$

$f(x)\delta(x)$ zero everywhere except $x = 0$

$$f(x)\delta(x) = f(0)\delta(x)$$

$$\int_{-\infty}^{\infty} f(x)\delta(x)\,dx = f(0)\int_{-\infty}^{\infty} \delta(x)\,dx = f(0)$$

under integral delta picks out the value of $f(x)$ at $x = 0$

$$\delta(x-a) = \begin{cases} 0, & \text{if } x \neq a \\ \infty, & \text{if } x = a \end{cases} \text{ with } \int_{-\infty}^{\infty} \delta(x-a)\,dx = 1$$

$$f(x)\delta(x-a) = f(a)\delta(x-a)$$

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)\,dx = f(a)$$

Example

$$\int_0^3 x^3 \delta(x-2)\,dx \quad x^3 @ x=2 \;\; 2^3 = 8 \quad \text{if } \int_0^1 \text{ ans } 0 \text{ spike outside domain}$$

$$\int_{-\infty}^{\infty} f(x) D_1(x)\,dx = \int_{-\infty}^{\infty} f(x) D_2(x)\,dx$$