SMT Solver Z3

Testing, Quality Assurance, and Maintenance
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UNIVERSITY OF WATERLOO

Satisfiability Modulo Theory (SMT)

Satisfiability is the problem of determining wither a formula F has a model

- if F is ***propositional***, a model is a truth assignment to Boolean variables
- if F is ***first-order formula***, a model assigns values to variables and interpretation to all the function and predicate symbols

SAT Solvers

- check satisfiability of propositional formulas

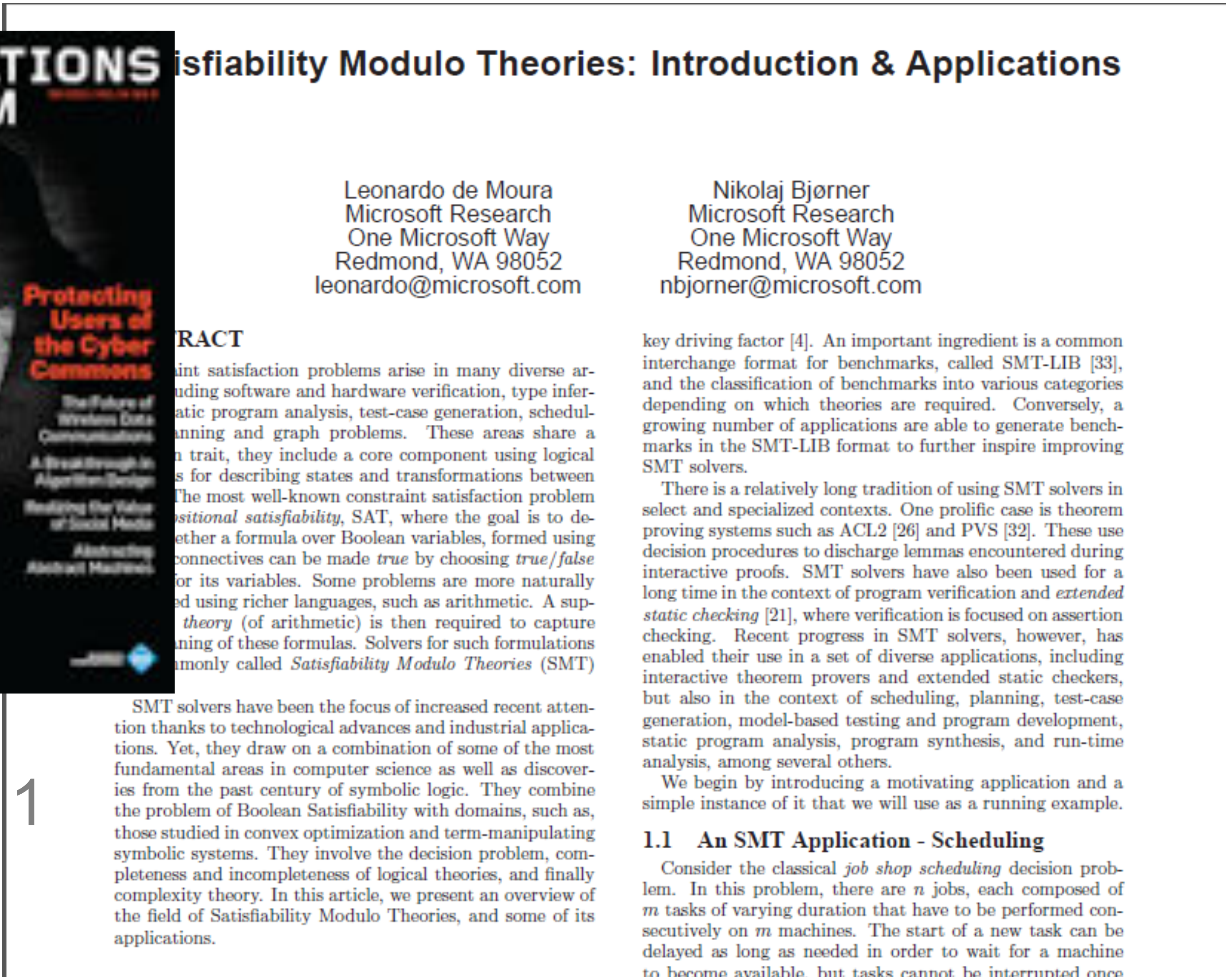
SMT Solvers

- check satisfiability of formulas in a ***decidable*** first-order theory (e.g., linear arithmetic, uninterpreted functions, array theory, bit-vectors)

(Optional) Background Reading: SMT

COMMUNICATIONS OF THE ACM
Protecting Users of the Cyber Commons
The Future of Wireless Data Communications
A Breakthrough in Algorithm Design
Realizing the Value of Social Media
Abstracting Abstract Machines
The Cyberwar Threat
Secure as a sieve
Titans can't

September 2011



isfiability Modulo Theories: Introduction & Applications

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RACT

aint satisfaction problems arise in many diverse ar-
uding software and hardware verification, type infer-
atic program analysis, test-case generation, schedul-
anning and graph problems. These areas share a
n trait, they include a core component using logical
s for describing states and transformations between
The most well-known constraint satisfaction problem
ositional satisfiability*, SAT, where the goal is to de-
ether a formula over Boolean variables, formed using
connectives can be made *true* by choosing *true*/*false*
for its variables. Some problems are more naturally
ed using richer languages, such as arithmetic. A sup-
theory (of arithmetic) is then required to capture
ning of these formulas. Solvers for such formulations
amonly called *Satisfiability Modulo Theories* (SMT)

SMT solvers have been the focus of increased recent attention thanks to technological advances and industrial applications. Yet, they draw on a combination of some of the most fundamental areas in computer science as well as discoveries from the past century of symbolic logic. They combine the problem of Boolean Satisfiability with domains, such as, those studied in convex optimization and term-manipulating symbolic systems. They involve the decision problem, completeness and incompleteness of logical theories, and finally complexity theory. In this article, we present an overview of the field of Satisfiability Modulo Theories, and some of its applications.

key driving factor [4]. An important ingredient is a common interchange format for benchmarks, called SMT-LIB [33], and the classification of benchmarks into various categories depending on which theories are required. Conversely, a growing number of applications are able to generate benchmarks in the SMT-LIB format to further inspire improving SMT solvers.

There is a relatively long tradition of using SMT solvers in select and specialized contexts. One prolific case is theorem proving systems such as ACL2 [26] and PVS [32]. These use decision procedures to discharge lemmas encountered during interactive proofs. SMT solvers have also been used for a long time in the context of program verification and *extended static checking* [21], where verification is focused on assertion checking. Recent progress in SMT solvers, however, has enabled their use in a set of diverse applications, including interactive theorem provers and extended static checkers, but also in the context of scheduling, planning, test-case generation, model-based testing and program development, static program analysis, program synthesis, and run-time analysis, among several others.

We begin by introducing a motivating application and a simple instance of it that we will use as a running example.

1.1 An SMT Application - Scheduling

Consider the classical *job shop scheduling* decision problem. In this problem, there are n jobs, each composed of m tasks of varying duration that have to be performed consecutively on m machines. The start of a new task can be delayed as long as needed in order to wait for a machine to become available, but tasks cannot be interrupted once

Example

$$b + 2 = c \wedge f(\mathtt{read}(\mathtt{write}(a, b, 3), c - 2)) \neq f(c - b + 1)$$

Example

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Arithmetic

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Example

$$b + 2 = c \wedge f(\mathtt{read}(\mathtt{write}(a, b, 3), c - 2)) \neq f(c - b + 1)$$

Array theory

Example

$$b + 2 = c \land f(\texttt{read}(\texttt{write}(a, b, 3), c - 2)) \neq f(c - b + 1)$$

Uninterpreted function

Example

$$b + 2 = c \wedge f(\texttt{read}(\texttt{write}(a, b, 3), c - 2)) \neq f(c - b + 1)$$

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$$b + 2 = c \wedge f(3) \neq f(3)$$

then, the formula is unsatisfiable

Example 2

$$x \geq 0 \wedge f(x) \geq 0 \wedge y \geq 0 \wedge f(y) \geq 0 \wedge x \neq y$$

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This formula is satisfiable

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$$x \geq 0 \land f(x) \geq 0 \land y \geq 0 \land f(y) \geq 0 \land x \neq y$$

This formula is satisfiable:

Example model:

$$x \rightarrow 1$$

$$y \rightarrow 2$$

$$f(1) \rightarrow 0$$

$$f(2) \rightarrow 1$$

$$f(\ldots) \rightarrow 0$$

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SMT-LIB: http://smt-lib.org

International initiative for facilitating research and development in SMT

Provides rigorous definition of syntax and semantics for theories

SMT-LIB syntax

- based on s-expressions (LISP-like)
- common syntax for interpreted functions of different theories
  - e.g. (and (= x y) (<= (* 2 x) z))
- commands to interact with the solver
  - (declare-fun …) declares a constant/function symbol
  - (assert p) conjoins formula p to the curent context
  - (check-sat) checks satisfiability of the current context
  - (get-model) prints current model (if the context is satisfiable)

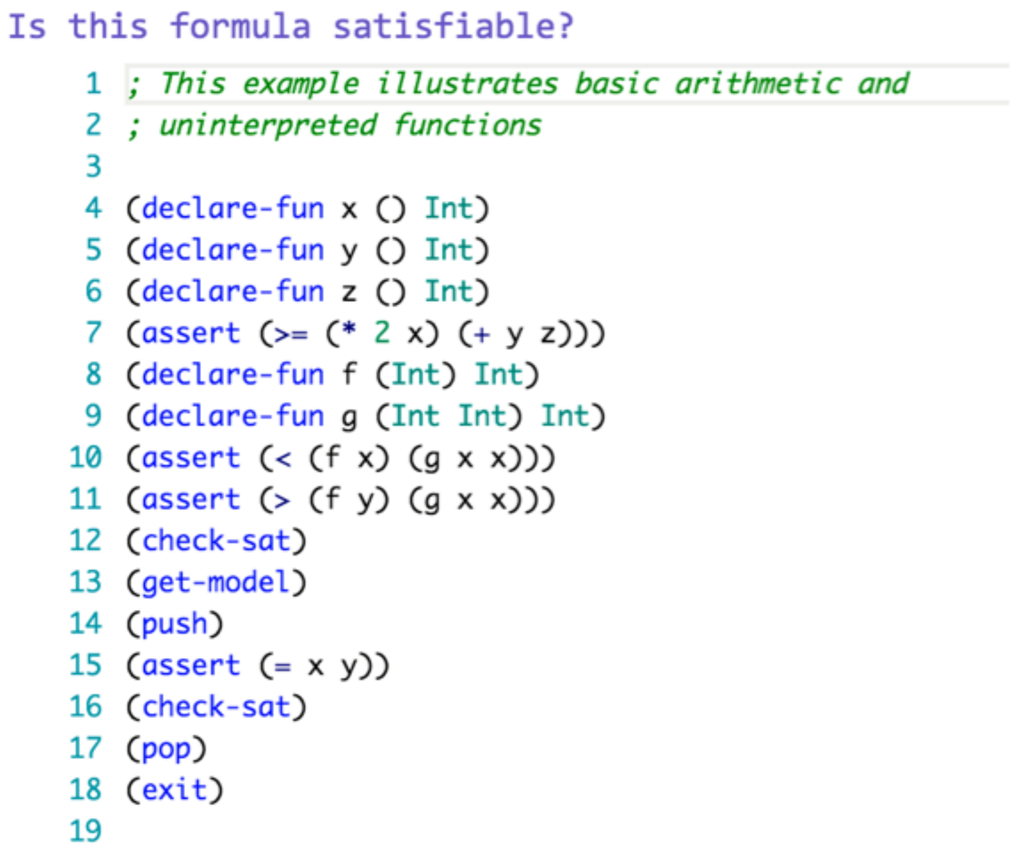
- see examples at http://rise4fun.com/z3

SMT-LIB Syntax

```
(declare-fun x () Int)
(declare-fun y () Int)
(declare-fun z () Int)
(assert (>= (* 2 x) (+ y z)))
(declare-fun f (Int) Int)
(declare-fun g (Int Int) Int)
(assert (< (f x) (g x x)))
(assert (> (f y) (g x x)))
(check-sat)
(get-model)
```

z3 Microsoft Research

SMT Example



Is this formula satisfiable?

```
; This example illustrates basic arithmetic and
; uninterpreted functions

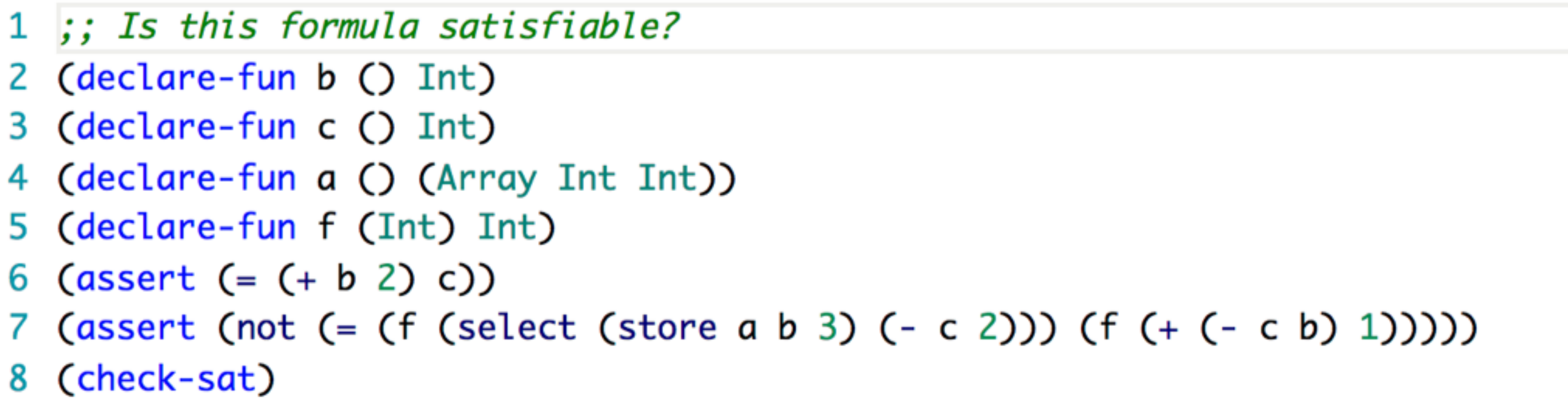
(declare-fun x () Int)
(declare-fun y () Int)
(declare-fun z () Int)
(assert (>= (* 2 x) (+ y z)))
(declare-fun f (Int) Int)
(declare-fun g (Int Int) Int)
(assert (< (f x) (g x x)))
(assert (> (f y) (g x x)))
(check-sat)
(get-model)
(push)
(assert (= x y))
(check-sat)
(pop)
(exit)
```

http://rise4fun.com/z3

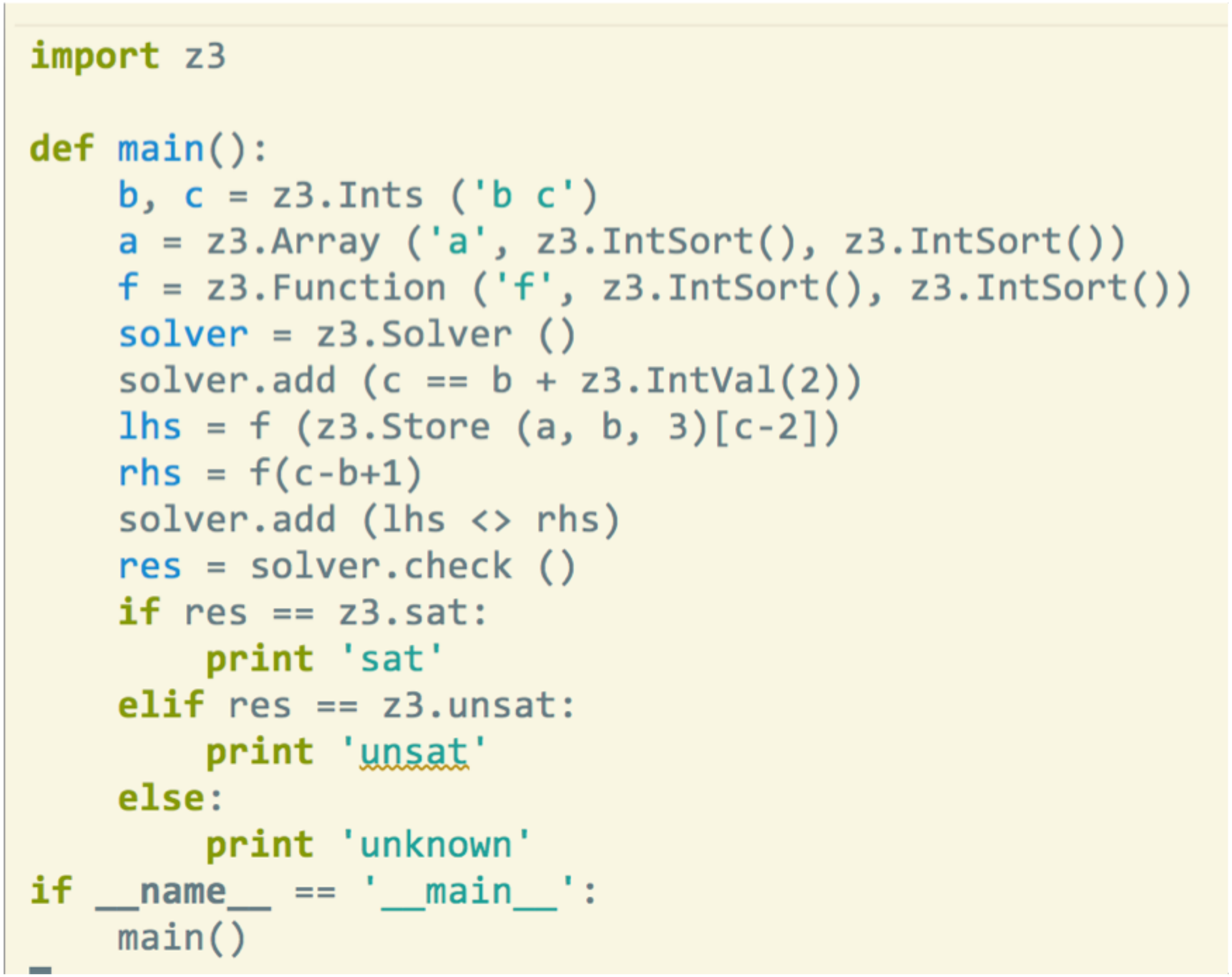
Example

$$b + 2 = c \wedge f(\mathtt{read}(\mathtt{write}(a, b, 3), c - 2)) \neq f(c - b + 1)$$

Is this formula satisfiable?

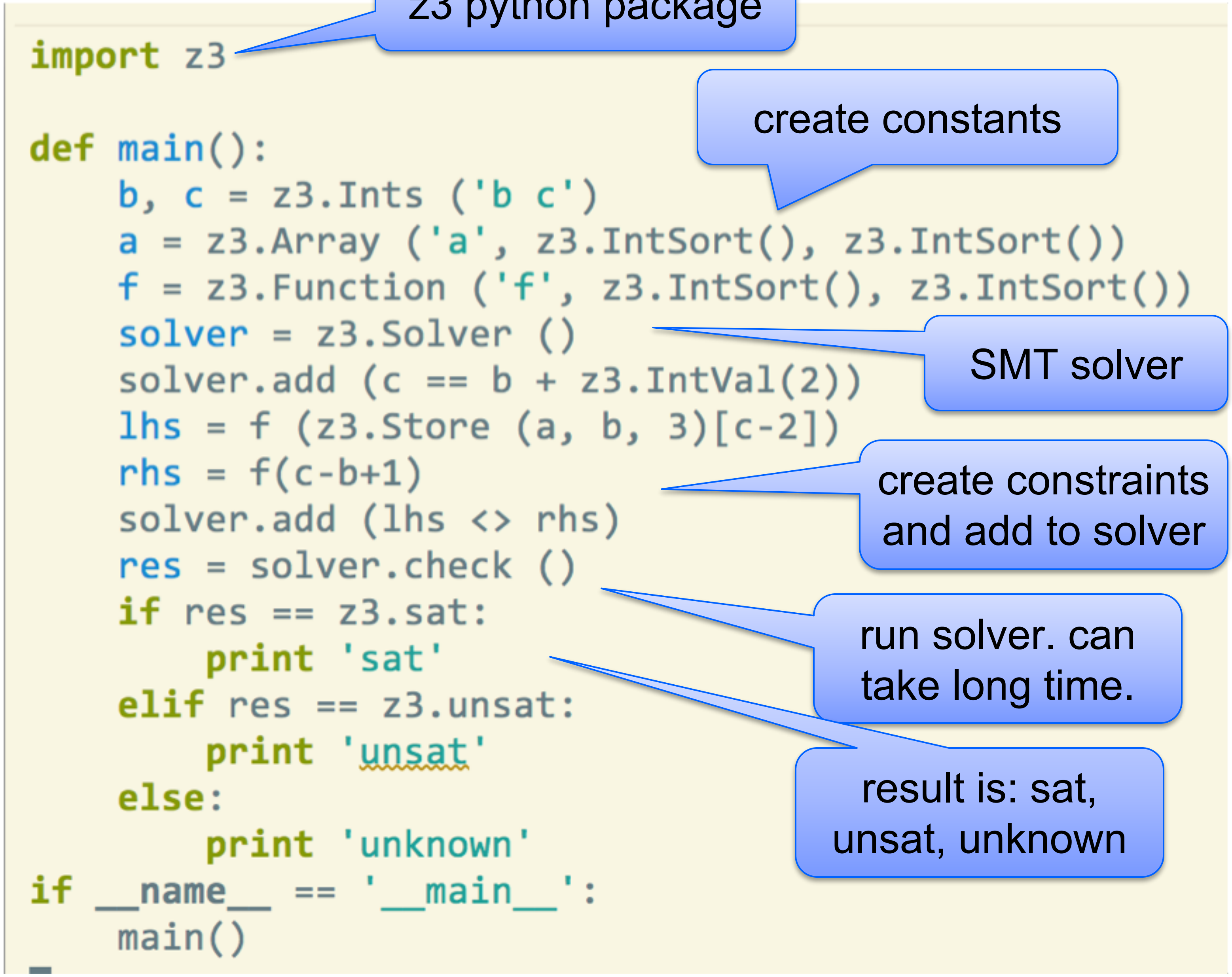


```
;; Is this formula satisfiable?
(declare-fun b () Int)
(declare-fun c () Int)
(declare-fun a () (Array Int Int))
(declare-fun f (Int) Int)
(assert (= (+ b 2) c))
(assert (not (= (f (select (store a b 3) (- c 2))) (f (+ (- c b) 1)))))
(check-sat)
```



```
import z3

def main():
    b, c = z3.Ints ('b c')
    a = z3.Array ('a', z3.IntSort(), z3.IntSort())
    f = z3.Function ('f', z3.IntSort(), z3.IntSort())
    solver = z3.Solver ()
    solver.add (c == b + z3.IntVal(2))
    lhs = f (z3.Store (a, b, 3)[c-2])
    rhs = f(c-b+1)
    solver.add (lhs <> rhs)
    res = solver.check ()
    if res == z3.sat:
        print 'sat'
    elif res == z3.unsat:
        print 'unsat'
    else:
        print 'unknown'
if __name__ == '__main__':
    main()
```



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    if res == z3.sat:
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    else:
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if __name__ == '__main__':
    main()
```

z3 python package

create constants

SMT solver

create constraints and add to solver

run solver. can take long time.

result is: sat, unsat, unknown

Useful Z3Py Functions

All these functions are under python package `z3`

Create constants and values

- `Int(name)` – an integer constant with a given name
- `FreshInt(name)` – unique constant starting with name
- `IntVal(v), BoolVal(v)` – integer and boolean values

Arithmetic functions and predicates

- `+,-,/,<,<=,>,>=,==,` etc.
- Distinct(a, b, …) – the arugments are distinct (expands to many disequalities)

Propositional operators

- `And, Or, Not`

Methods of the `z3.Solver` class

- `add(fml)` – add formula fml to the solver
- `check()` – returns z3.sat, z3.unsat, or z3.unknown (on failure to solve)
- `model()` – model if the result is sat

Methods of `z3.Model` class

- `eval(fml)` – returns the value of fml in the model

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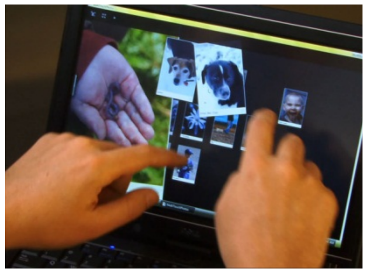
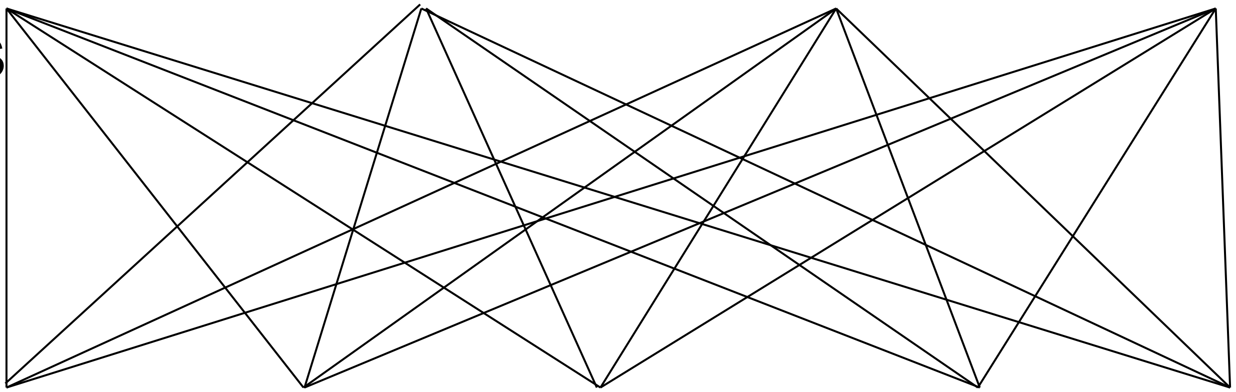
Job Shop Scheduling



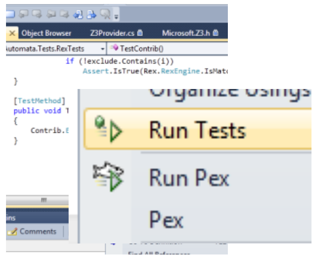
Machines

Tasks

Jobs



P = NP?



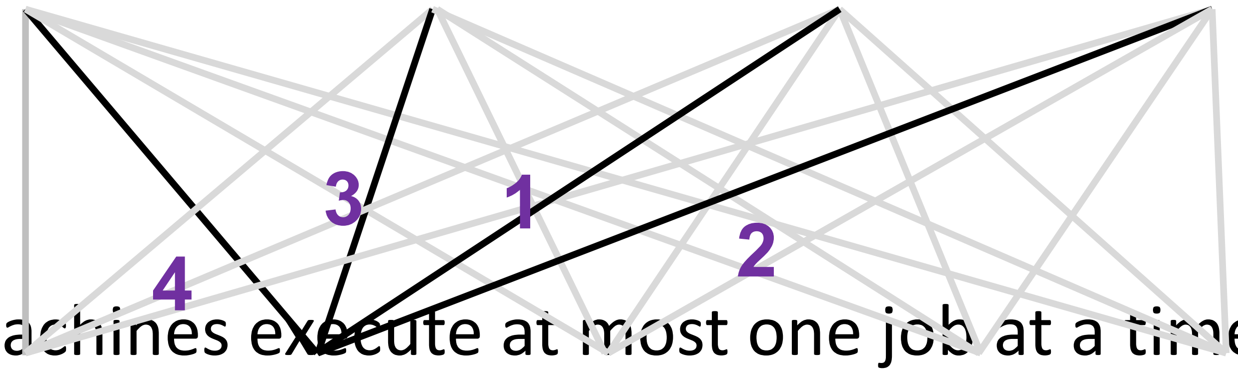
```
if (!exclude.Contains(i))
    Assert.IsTrue(Rex.RexEngine.IsMat
}
[TestMethod]
public void T
{
    Contrib.
}
```

$$\zeta(s) = 0 \Rightarrow s = \frac{1}{2} + ir$$

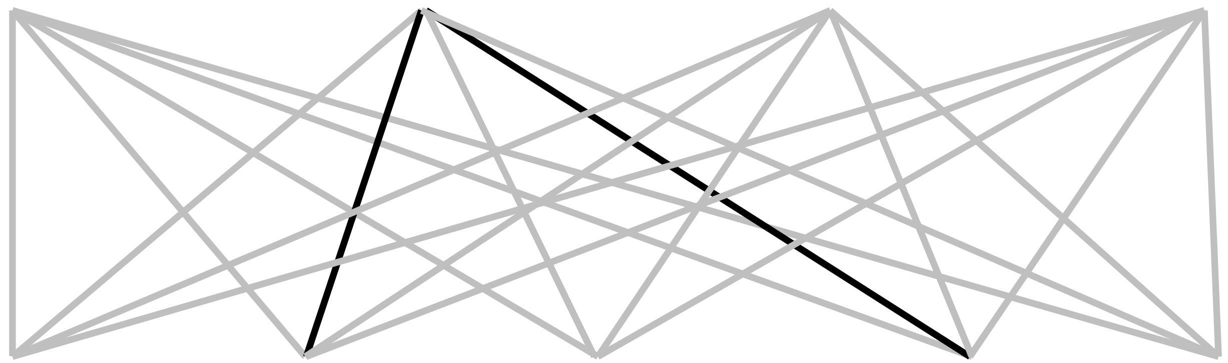
Job Shop Scheduling

Constraints:

Precedence: between two tasks of the same job



Resource: Machines execute at most one job at a time

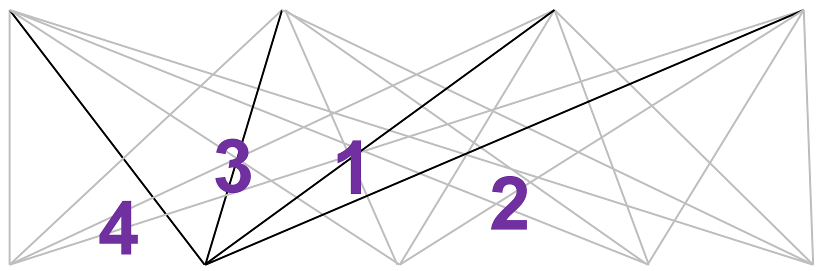


$$[start_{2,2}..end_{2,2}] \cap [start_{4,2}..end_{4,2}] = \emptyset$$

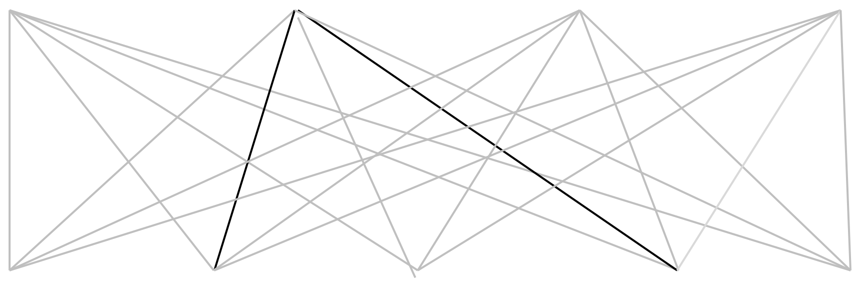
Job Shop Scheduling

Constraints:

Precedence:



Resource:



$$[start_{2,2}..end_{2,2}] \cap [start_{4,2}..end_{4,2}] = \emptyset$$

Encoding:

$t_{2,3}$ - start time of job 2 on mach 3

$d_{2,3}$ - duration of job 2 on mach 3

$$t_{2,3} + d_{2,3} \leq t_{2,4}$$

Not convex

$$t_{2,2} + d_{2,2} \leq t_{4,2}$$
$$\vee$$
$$t_{4,2} + \mathrm{d}_{4,2} \leq t_{2,2}$$

Job Shop Scheduling

$d_{i,j}$	Machine 1	Machine 2
Job 1	2	1
Job 2	3	1
Job 3	2	3

$max = 8$

Solution

$t_{1,1} = 5,\ t_{1,2} = 7,\ t_{2,1} = 2,$
$t_{2,2} = 6,\ t_{3,1} = 0,\ t_{3,2} = 3$

Encoding

$$
\begin{aligned}
&(t_{1,1} \geq 0) \wedge (t_{1,2} \geq t_{1,1} + 2) \wedge (t_{1,2} + 1 \leq 8) \wedge \\
&(t_{2,1} \geq 0) \wedge (t_{2,2} \geq t_{2,1} + 3) \wedge (t_{2,2} + 1 \leq 8) \wedge \\
&(t_{3,1} \geq 0) \wedge (t_{3,2} \geq t_{3,1} + 2) \wedge (t_{3,2} + 3 \leq 8) \wedge \\
&((t_{1,1} \geq t_{2,1} + 3) \vee (t_{2,1} \geq t_{1,1} + 2)) \wedge \\
&((t_{1,1} \geq t_{3,1} + 2) \vee (t_{3,1} \geq t_{1,1} + 2)) \wedge \\
&((t_{2,1} \geq t_{3,1} + 2) \vee (t_{3,1} \geq t_{2,1} + 3)) \wedge \\
&((t_{1,2} \geq t_{2,2} + 1) \vee (t_{2,2} \geq t_{1,2} + 1)) \wedge \\
&((t_{1,2} \geq t_{3,2} + 3) \vee (t_{3,2} \geq t_{1,2} + 1)) \wedge \\
&((t_{2,2} \geq t_{3,2} + 3) \vee (t_{3,2} \geq t_{2,2} + 1))
\end{aligned}
$$

Bit Tricks

Let `x, y` be a 32 bit machine integers (a bit-vector)

Show that `x!=0 && !(x & (x-1))` is true iff `x` is a power of 2

Show that `x` and `y` have different signs iff `x^y < 0`

Dog, Cat, Mouse

Spend exactly 100 dollars and buy exactly 100 animals.

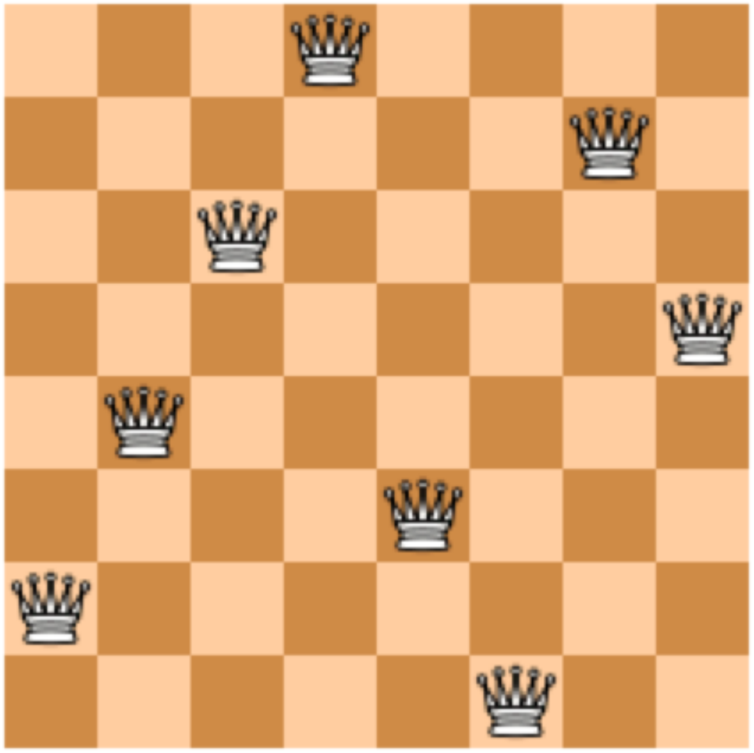
- Dogs cost 15 dollars,
- cats cost 1 dollar,
- and mice cost 25 cents each.

You have to buy at least one of each.

How many of each should you buy?

Eight Queens Problem

Place 8 queens on an 8x8 chess board so that no two queen attacks one another



Incremental Interface

Z3 provides two interfaces for incremental solving that allow for adding and removing constraints

- push/pop, and assumptions

Constraints can be added at any time. This is not called incremental ☺

Push/Pop Interface

- Store current solver state by a call to push
  - `s.push ()` in Python, and `(push)` in SMT-LIB
- Restore previous state by a call to pop
  - `s.pop ()` in Python and `(pop)` in SMT-LIB

Incremental Interface: Assumptions

Requires two steps, but much more flexible than push/pop

1. tag constraints by fresh Boolean constants
   - e.g., use `(assert (=> p phi))` instead of `(assert phi)`
2. during check-sat, enable constraints by forcing tags to be true
   - e.g., use (check-sat p)

For example,

```
(assert (=> a0 c0))
(assert (=> a1 c1))
(assert (=> a2 c2))
(check-sat a0)          ; check whether c0 is sat
(check-sat a0 a2)       ; check whether c0 and c2 are sat
(check-set a1 a2)       ; check whether c1 and c3 are sat
```

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Assumptions in Python Interface

Methods of `z3.Solver` class

- `check(self, *assumptions)` – check with assumptions
- `unsat_core(self)` – if the last call to `check` was `unsat`, returns the subset of assumptions that were actually used to show `unsat`