Induction, k-Induction, and Symbolic Model Checking

Automated Program Verification (APV)
Fall 2019

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Symbolic model checking

Model is represented symbolically using Boolean formulas

Model checking is performed on the symbolic representation **directly**

BDD-based

- Use specialized data structure, Binary Decision Diagrams, to represent and manipulate sets of states

SAT-based (most of this class)

- Represent sets of executions using Boolean formulas in Conjunctive Normal Form (CNF)
- Use efficient SAT(isfiability)-solvers for reasoning

SAT-based Model Checking

Bounded Model Checking

- Is there a counterexample of k-steps

Unbounded Model Checking

- Induction and *k*-Induction (k-IND)
- Interpolation Based Model Checking (IMC)
- Property Directed Reachability (IC3/PDR)

Mathematical Induction

To proof that a property P(n) holds for all natural numbers n

1. Show that P(0) is true

2. Show that P(k+1) is true for some natural number k, using an Inductive Hypothesis that P(k) is true

Example: Mathematical Induction

Show by induction that P(n) is true

$$0 + \cdots + n = \frac{n(n+1)}{2}$$

Base Case: P(0) is $0 = \frac{0(0+1)}{2}$

IH: Assume P(k), show P(k+1)

$$\begin{aligned} & 0 + \cdots + k + (k+1) \\ = \quad & \frac{k(k+1)}{2} + (k+1) \\ = \quad & \frac{k(k+1)+2(k+1)}{2} \\ = \quad & \frac{(k+1)((k+1)+1)}{2} \end{aligned}$$

Symbolic Safety and Reachability

A transition system P = (V, Init, Tr, Bad)

P is UNSAFE if and only if there exists a number N s.t.

$$Init(X_0) \wedge \left(\bigwedge_{i=0}^{N-1} Tr(X_i, X_{i+1}) \right) \wedge Bad(X_N) \not\Rightarrow \bot$$

P is SAFE if and only if there exists a safe inductive invariant Inv s.t.

$$Init \Rightarrow Inv$$

$$Inv(X) \wedge Tr(X, X') \Rightarrow Inv(X')$$

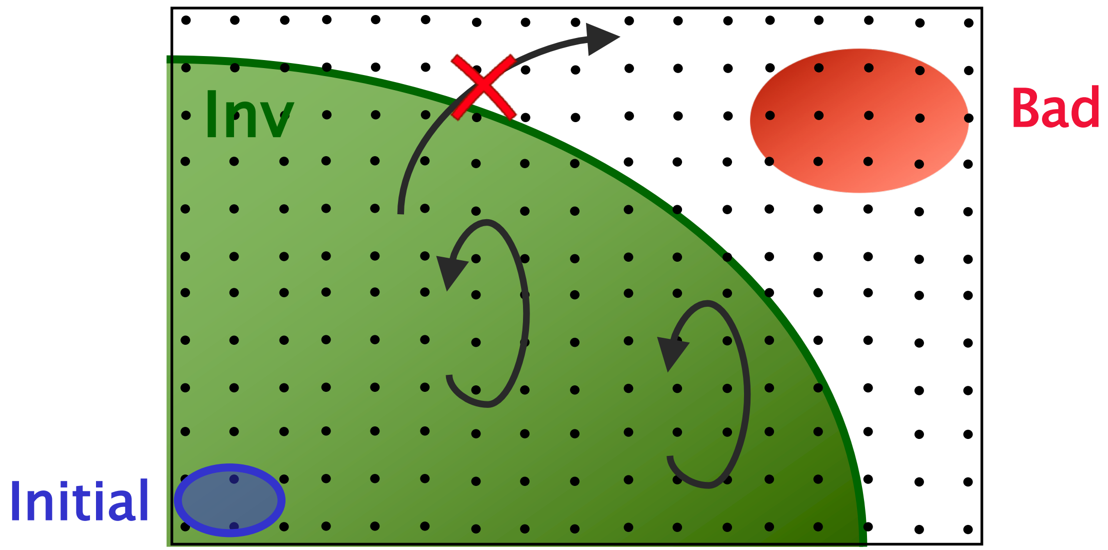
Inductive

$$Inv \Rightarrow \neg Bad$$

Safe

Inductive Invariants

System State Space

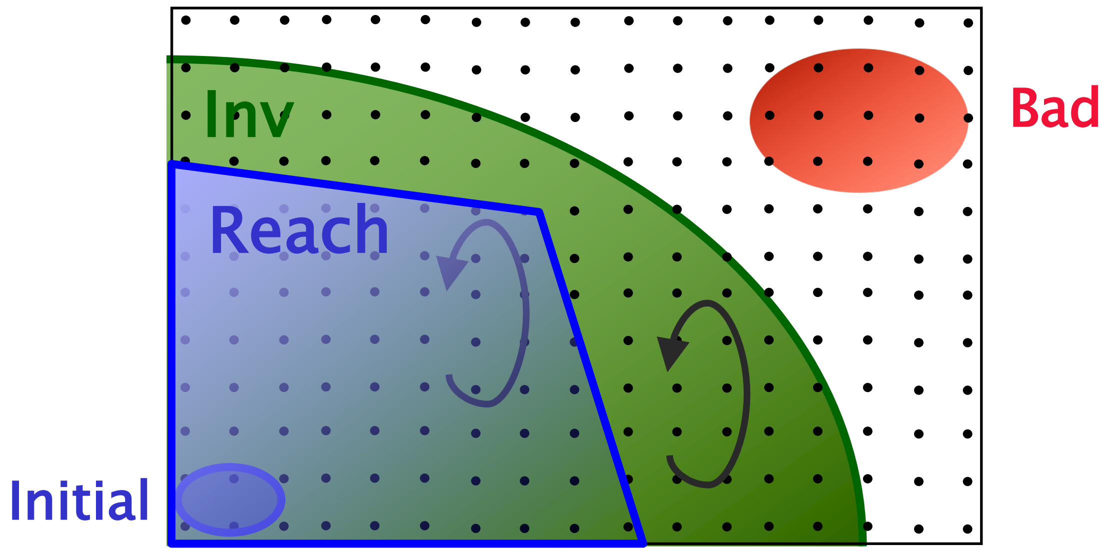


System S is safe iff there exists an inductive invariant Inv

- **Initiation** Initial ⊆ Inv
- **Safety** Inv ∩ Bad = ∅
- **Consecution** TR(Inv) ⊆ Inv

i.e., if s ∈ Inv and s⤳t then t ∈ Inv

Inductive Invariants



System S is safe iff there exists an inductive invariant Inv

- **Initiation** Initial ⊆ Inv
- **Safety** Inv ∩ Bad = ∅
- **Consecution** TR(Inv) ⊆ Inv

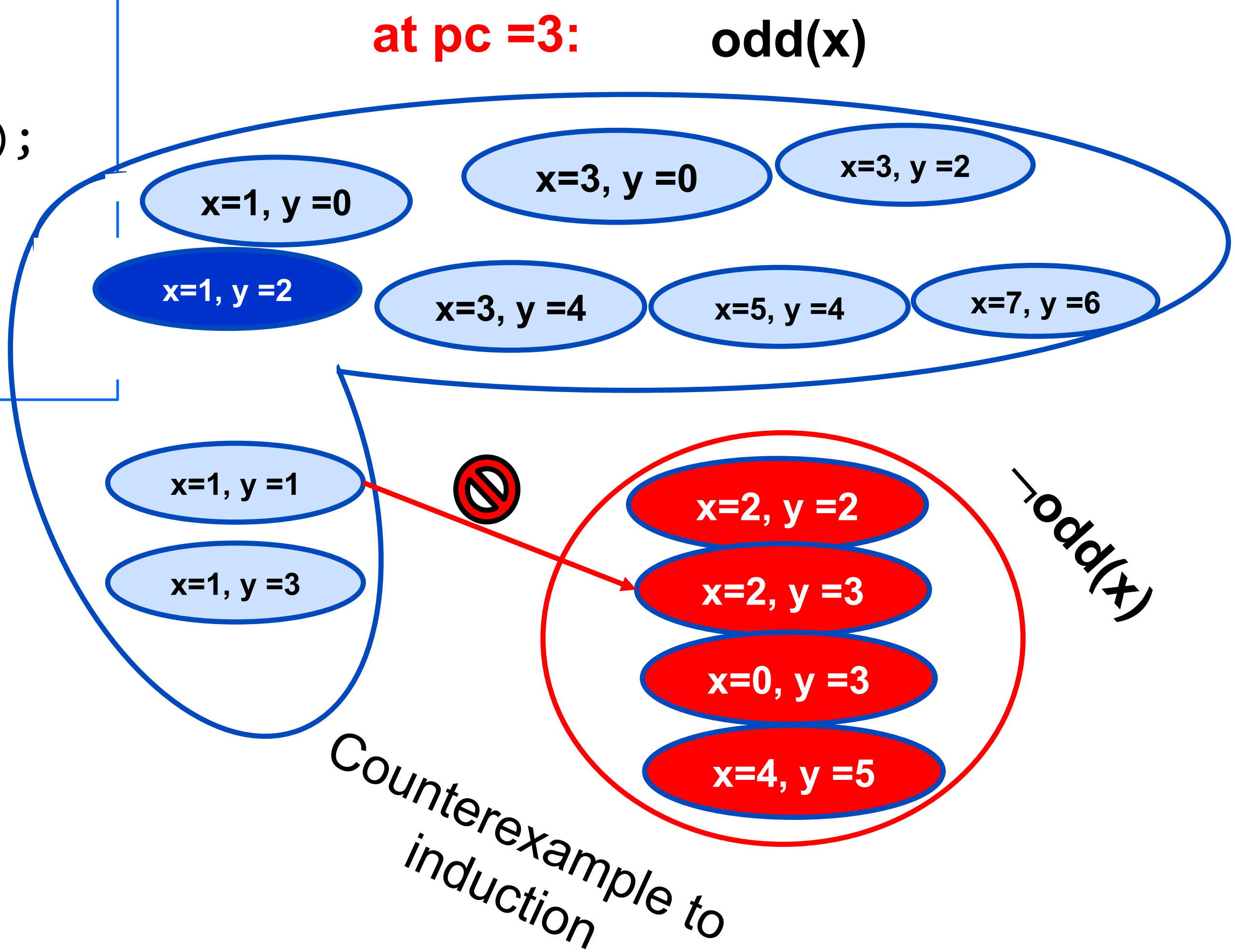
i.e., if s ∈ Inv and s⤳t then t ∈ Inv

System S is safe if Reach ∩ Bad = ∅

Induction: Simple Example

Is **pc=3 ⇒ odd(x)** an inductive invariant?

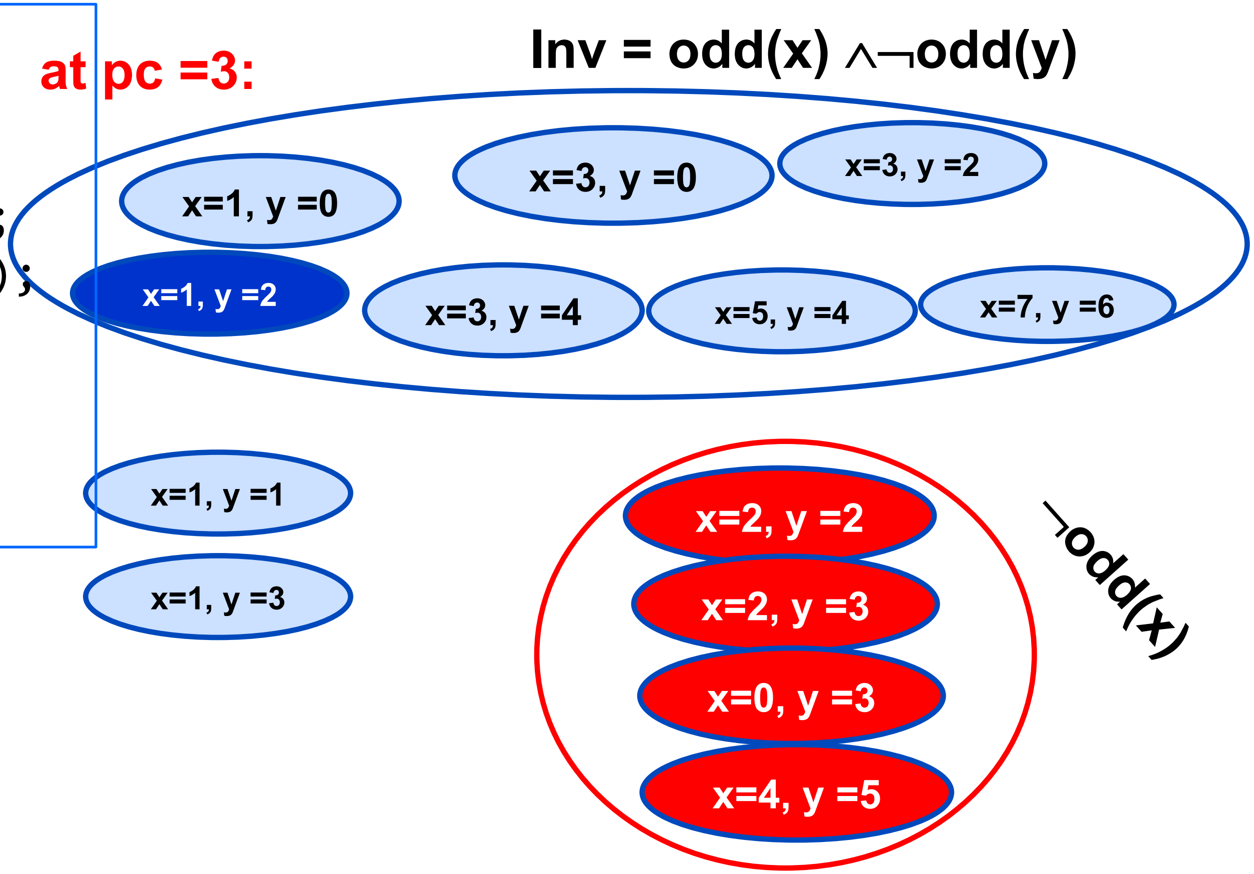
```
1: x := 1;
2: y := 2;
while * do {
   3: assert odd(x);
   4:  x:= x + y;
   5:  y := y + 2
}
6:
```



Inductive Invariants: Simple Example

Is **pc=3 ⇒ (odd(x) ∧ ¬odd(y))** an inductive invariant?

```
1: x := 1;
2: y := 2;
while * do {
   3: assert odd(x);
      assert !odd(y);
   4:  x:= x + y;
   5:  y := y + 2
}
6:
```



Checking Invariance is reducible to SAT!

Inputs

- A transition system P = (V, Init, Tr, Bad)
- A formula I(V) over variables V

Decide whether I is a safe inductive invariant

- Use SAT to check that $Init \wedge \neg I$ is UNSAT
- Use SAT to check that $I(V) \wedge Tr(V, V') \wedge \neg I(V')$ is UNSAT
- Use SAT to check that $I \wedge Bad$ is UNSAT

If all checks are UNSAT, I(V) is a safe inductive invariant

If a check fails, interpretation depends on the failing check:

- Check 1: missing initial states
- Check 2: not closed under a step of transition relation
- Check 3: not safe (true invariant, but not good enough for property)

Complete SAT-based Model Checker

(Don't try this at home)

Inputs

- A transition system P = (V, Init, Tr, Bad)

For every propositional formula Cand(V) over variables V

- If Cand(V) is a safe inductive invariant, return True

If got here, return False

Is this algorithm sound?

Is this algorithm complete?

Is this algorithm efficient?

Maximal Inductive Subset

Let *L* be a set of formulas, *P=(V, Init, Tr, Bad)* a program

A subset *X* of *L* is a *maximal inductive subset* iff it is the largest subset of *X* such that

$$Init(u) \Rightarrow \bigwedge_{\ell \in X} \ell(u)$$

$$\bigwedge_{\ell \in X} \ell(u) \wedge Tr(u, v) \Rightarrow \bigwedge_{\ell \in X} \ell(v)$$

A Maximal Inductive Subset is unique

- inductive invariants are closed under conjunction

Cormac Flanagan, K. Rustan M. Leino¬ Houdini, an Annotation Assistant for ESC/Java. FME 2001¬ 500-517

Minimal Unsatisfiable Subset

Let φ be a formula and $A = \{a_1, \ldots, a_n\}$ be atomic propositions occurring negatively in φ

Assume $\varphi \wedge a_1 \wedge \cdots \wedge a_n$ is UNSAT

A *minimal unsatisfiable subset (MUS)* of φ is the smallest subset $X \subseteq A$ such that $\varphi \wedge X$ is UNSAT

There are efficient algorithms for computing MUS (a.k.a. UNSAT core) for propositional formulas

Solving MIS via MUS

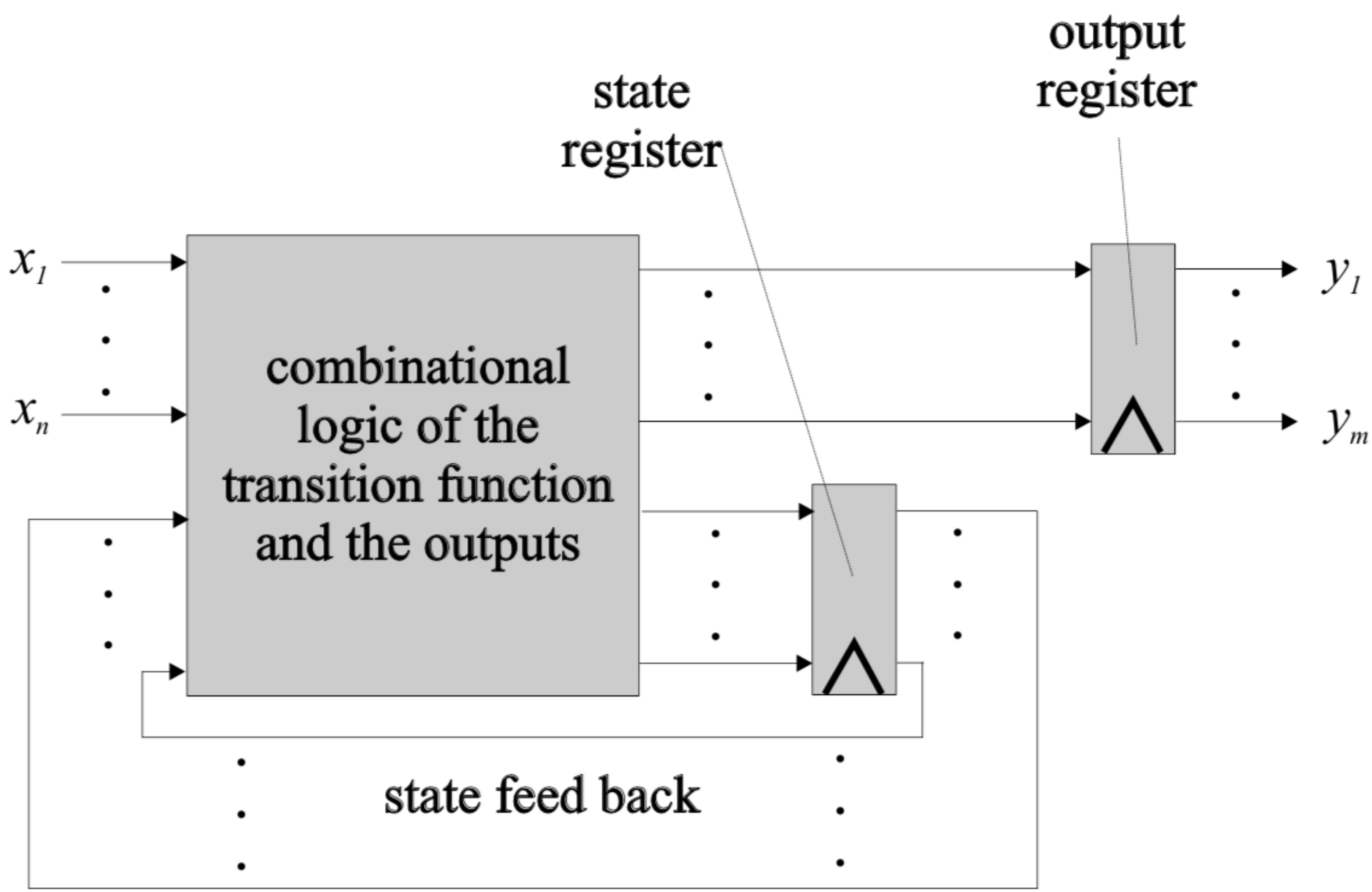
Reduce MIS to multiple calls to MUS

fresh propositional variables

Input : $\mathcal{L}, Tr$ — a set of lemmas and the transition relation (in BV)
Output: $\mathcal{L}' \subseteq \mathcal{L}$ the MIS of $\mathcal{L}$ relative to Tr

$\varphi \leftarrow (\bigwedge_{L_i \in \mathcal{L}} (pre_i \Rightarrow L_i(u))) \land Tr(u,v) \land (\bigvee_{L_i \in \mathcal{L}} (post_i \land \neg L_i(v)))$
`Sat_Add(B2P(`φ`))` — called once
$\mathcal{L}' \leftarrow \mathcal{L}$
forever do
    `Sat_Checkpoint()` — incremental SAT
    `Sat_Add(`pre_i`)` for all $L_i \in \mathcal{L}'$
    $C =$ `MUS`$(\{\neg post_i \mid L_i \in \mathcal{L}'\})$ — SAT MUS
    if $|C| = |\mathcal{L}'|$ **then return** $\mathcal{L}'$
    $\mathcal{L}' \leftarrow \{L_i \mid (\neg post_i) \in C\}$
    `Sat_Rollback()` — incremental SAT
end

A Synchronous Mealy Machine



Molitor and Mohnke. Equivalence Checking of Digital Circuits. 2004

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Terminology for Sequential Synthesis

The ***set of reachable states*** is the set of all possible valuations of the registers after arbitrary long execution from the initial state

Combinational synthesis – changing the combinational logic of the circuit without knowledge of reachable states

Sequential synthesis – modifies the circuit so that its behavior is preserved in the reachable states, but arbitrary changes are allowed on the unreachable states

Sequentially equivalent nodes – nodes having the same or opposite polarity in all reachable states

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https://people.eecs.berkeley.edu/~alanmi/publications/2007/tech07_pss.pdf

AIG: And-Inverter-Graph

A data structure for representing and manipulating arbitrary propositional formulas

A graph with 3 kinds of nodes

- input: one output, correspond to variables
- output: one input, correspond to functions, outputs
- AND: two (or more) inputs, one outputs, correspond to AND

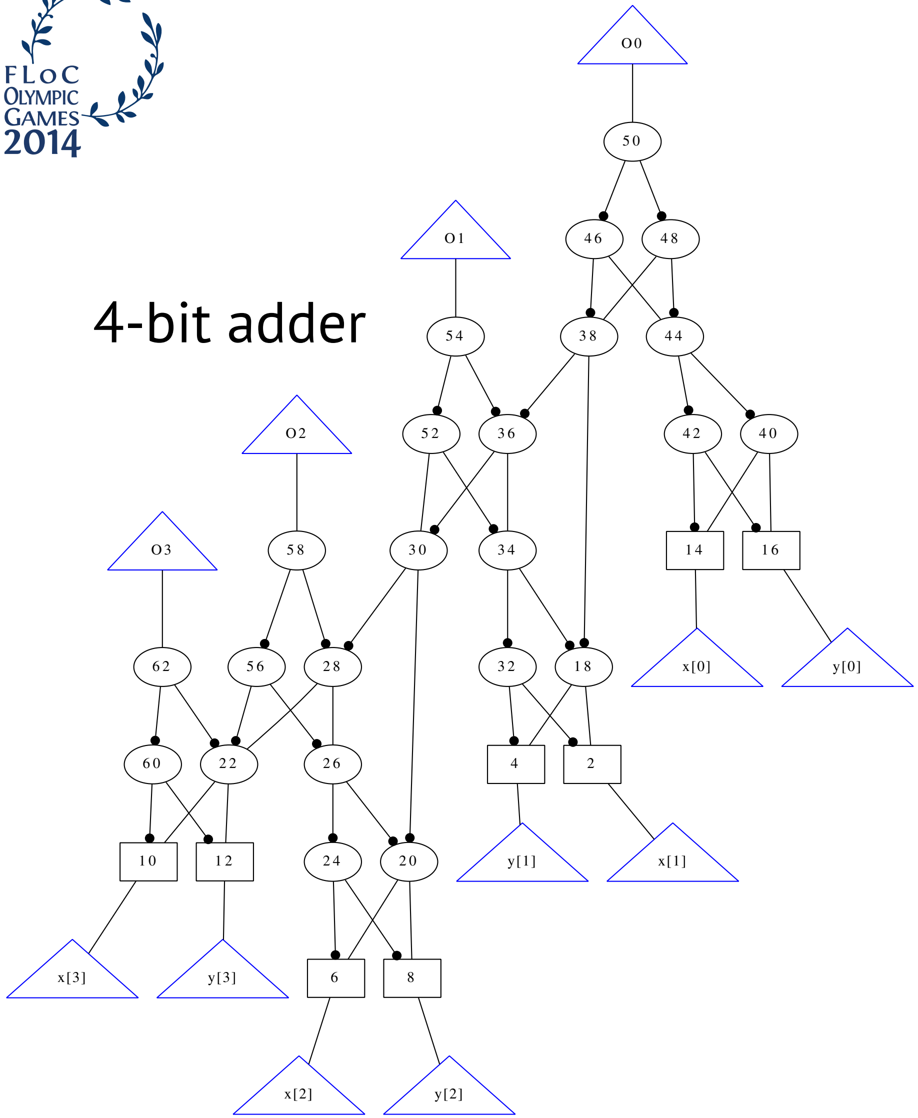
An input/output of any node can be negated

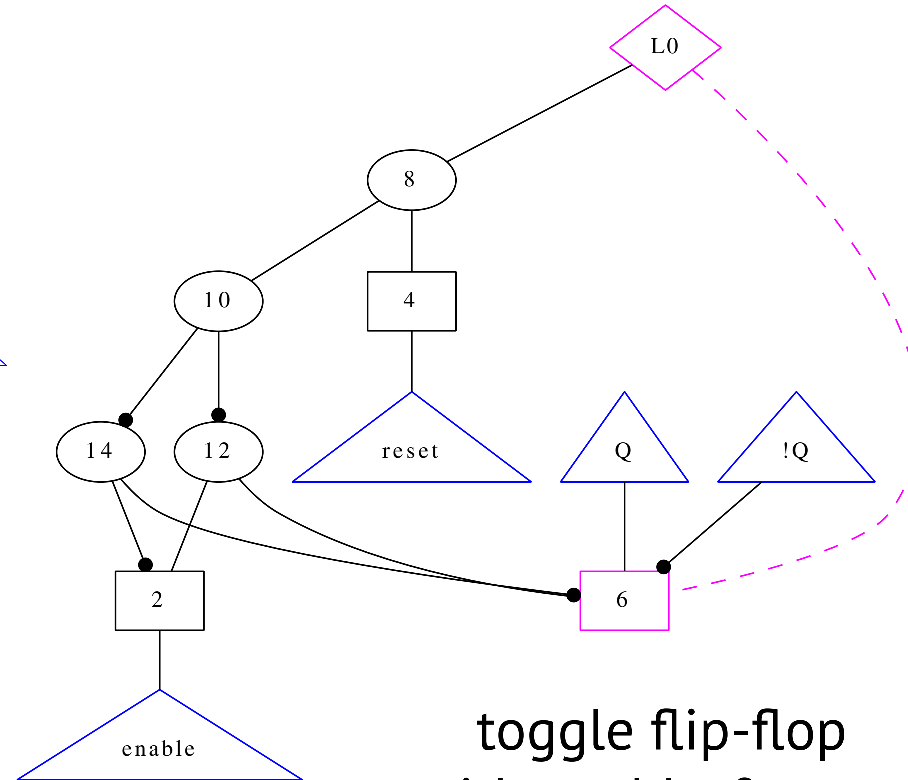
Hash-Cons

- AND nodes are kept in a hash table keyed on their children
- only one node is created for any syntactic function

AIGER

4-bit adder





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with enable & reset

http://fmv.jku.at/aiger/

Latch Correspondence Problem

Definition 10.1 (latch permutation problem) *Given two sequential circuits $F^{(1)}, F^{(2)} \in \mathcal{F}_{n,m,k}$, the latch permutation equivalence problem which is also referred as latch correspondence problem is the decision problem as to whether a correspondence π between the latches of $F^{(1)}$ and $F^{(2)}$ exists, such that the two synchronous sequential circuits $F^{(1)}$ and $F^{(2)}$ have their combinational parts functionally equivalent using this correspondence. More formally, the problem is to find a permutation $\pi \in \mathcal{P}er(\mathbb{N}_k)$ such that for all $j \in \mathbb{N}_m$*

$$\lambda_j^{(1)}(x_1, \ldots, x_n, u_{\pi(1)}^{(1)}, \ldots, u_{\pi(k)}^{(1)}) = \lambda_j^{(2)}(x_1, \ldots, x_n, u_1^{(2)}, \ldots, u_k^{(2)})$$

and for all $j \in \mathbb{N}_k$

$$\delta_{\pi(j)}^{(1)}(x_1, \ldots, x_n, u_{\pi(1)}^{(1)}, \ldots, u_{\pi(k)}^{(1)}) = \delta_j^{(2)}(x_1, \ldots, x_n, u_1^{(2)}, \ldots, u_k^{(2)})$$

hold. (For the notations, we refer to Chapter 8 Section 1.)

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Molitor and Mohnke. Equivalence Checking of Digital Circuits. 2004

Solving Latch Correspondence by MIS

Simulate the circuit with random inputs

Identify candidate equivalence classes

- latches H and K are candidates if in every simulation either
  - $H = K$ or $H = \neg K$

Refine candidate equivalences using BMC

- for every candidate H=K, use BMC to find a (short) counterexample

For all remaining candidates, compute Maximal Inductive Subset

- each call to SAT removes at least one candidate
- converges in linear time in the number of candidates

K-induction

Sheeran, Singh, Stålmarck Checking Safety Properties Using Induction and a SAT-Solver. FMCAD 2000

Induction

$$\frac{P(s_0) \qquad \forall i \,.\, P(s_i) \Rightarrow P(s_{i+1})}{\forall i \,.\, P(s_i)}$$

k-step Induction

$$\frac{P(s_{0..k-1}) \qquad \forall i \,.\, P(s_{i..i+k-1}) \Rightarrow P(s_{i+k})}{\forall i \,.\, P(s_i)}$$

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http://www.ccs.neu.edu/home/wahl/Publications/k-induction.pdf

2-Induction: Simple Example

Is pc=3 -> odd(x) 2-inductive invariant?

✓

Program

```
1: x := 1;
2: y := 2;
while * do {
   3: assert odd(x);
   4:  x := x + y;
   5:  y := y + 2
}
6:
```

2-Base

```
x := 1;
y := 2;
assert odd(x)
x := x + y;
y := y + 2;
assert odd(x)
```

2-IND

```
assume odd(x)
x := x + y;
y := y + 2;
assume odd(x)
x := x + y;
y := y + 2;
assert odd(x)
```

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Induction and Strong Induction

Induction Principle

$Init(v_0) \rightarrow Inv(v_0)$

$Inv(v_0) \wedge Tr \rightarrow Inv(v_1)$

$\neg Bad(v)$

$Tr[Inv]^1$
$\rightarrow Inv(v_1)$

Strong Induction Principle

$Inv(v_0) \wedge Tr \wedge Inv(v_1) \wedge Tr \wedge \cdots \wedge Inv(v_{k-1}) \wedge Tr \rightarrow Inv(v_k)$

$Init($

$Tr[Inv]^k \rightarrow Inv(v_k)$

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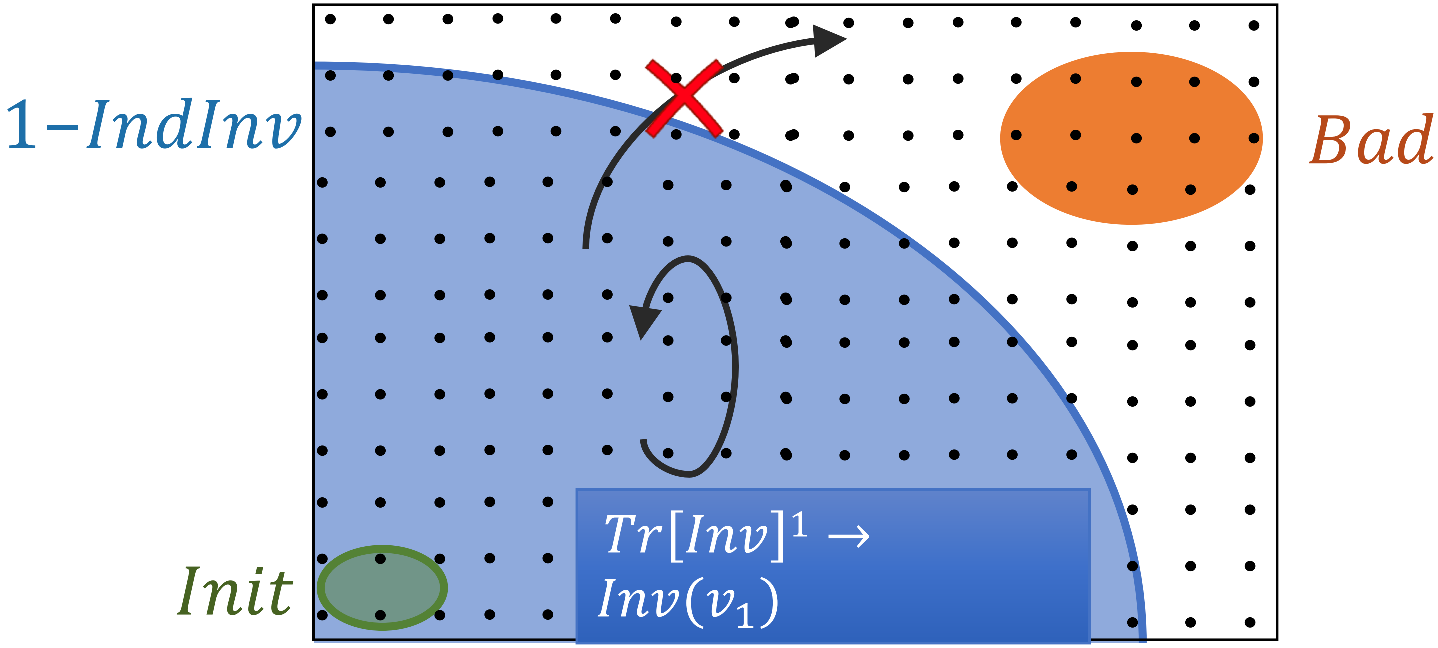

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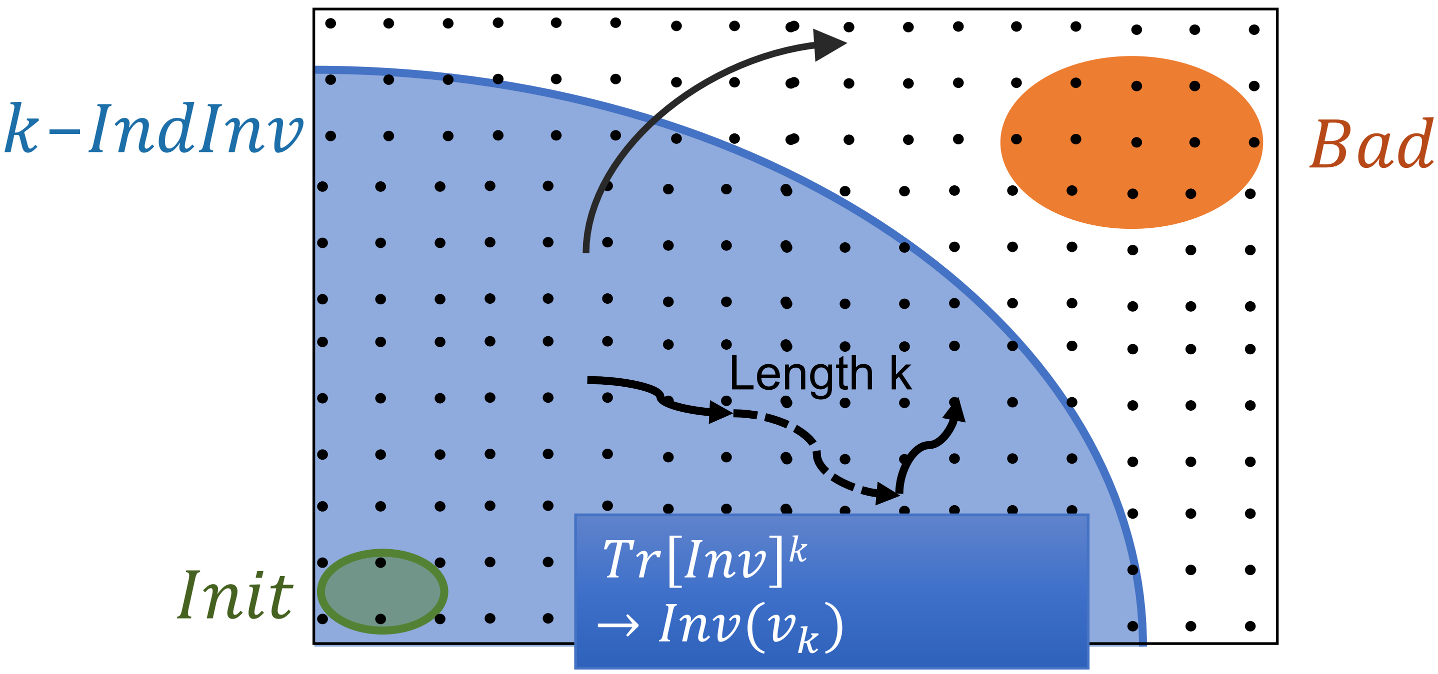

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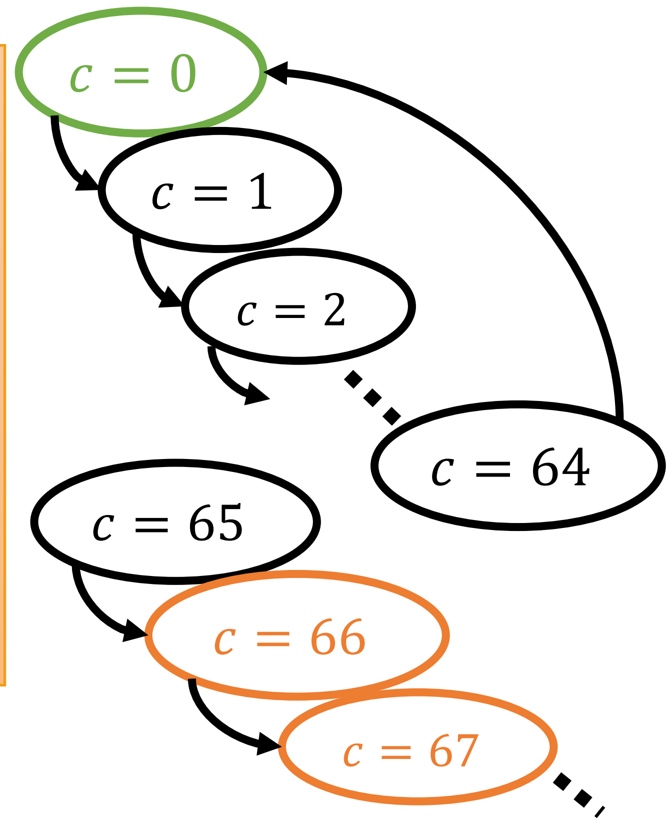
Example circuit in Verilog

```
reg [7:0] c = 0;
always
if(c == 64)
     c = 0;
else
     c = c + 1;
end
assert property(c < 66);
```



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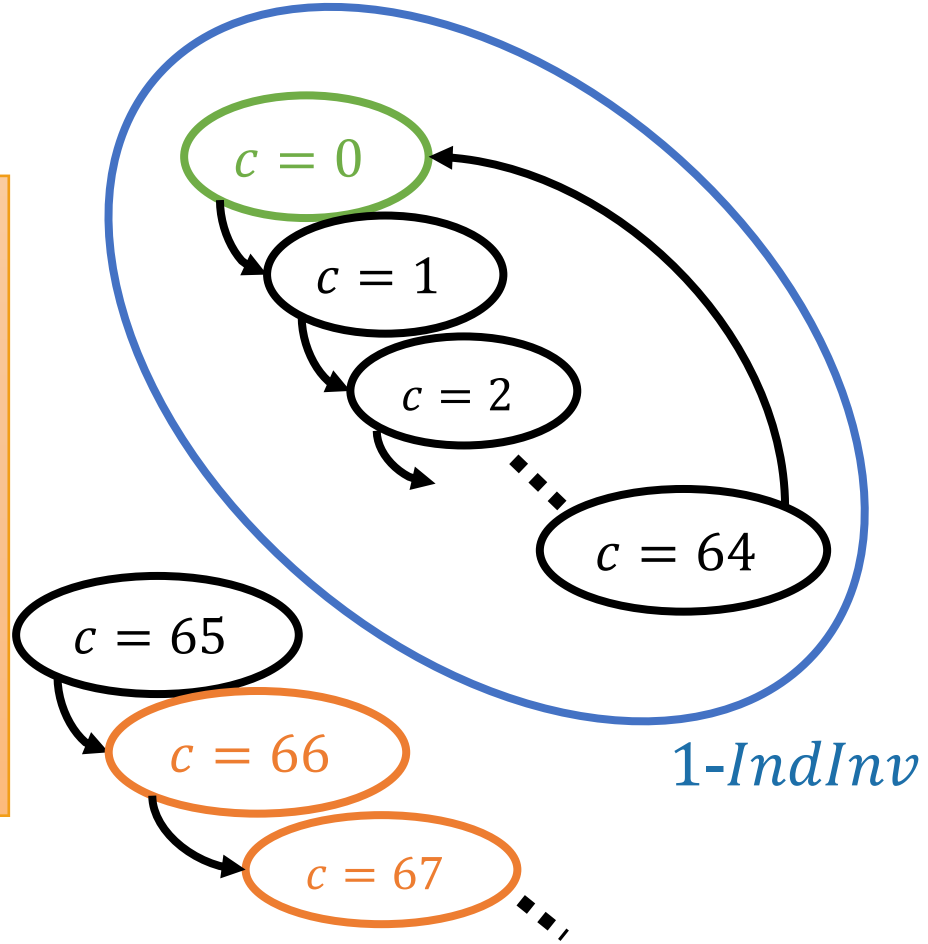
1-Inductive Invariant

```
reg [7:0] c = 0;
always
if(c == 64)
      c = 0;
else
      c = c + 1;
end
assert property(c < 66);
```



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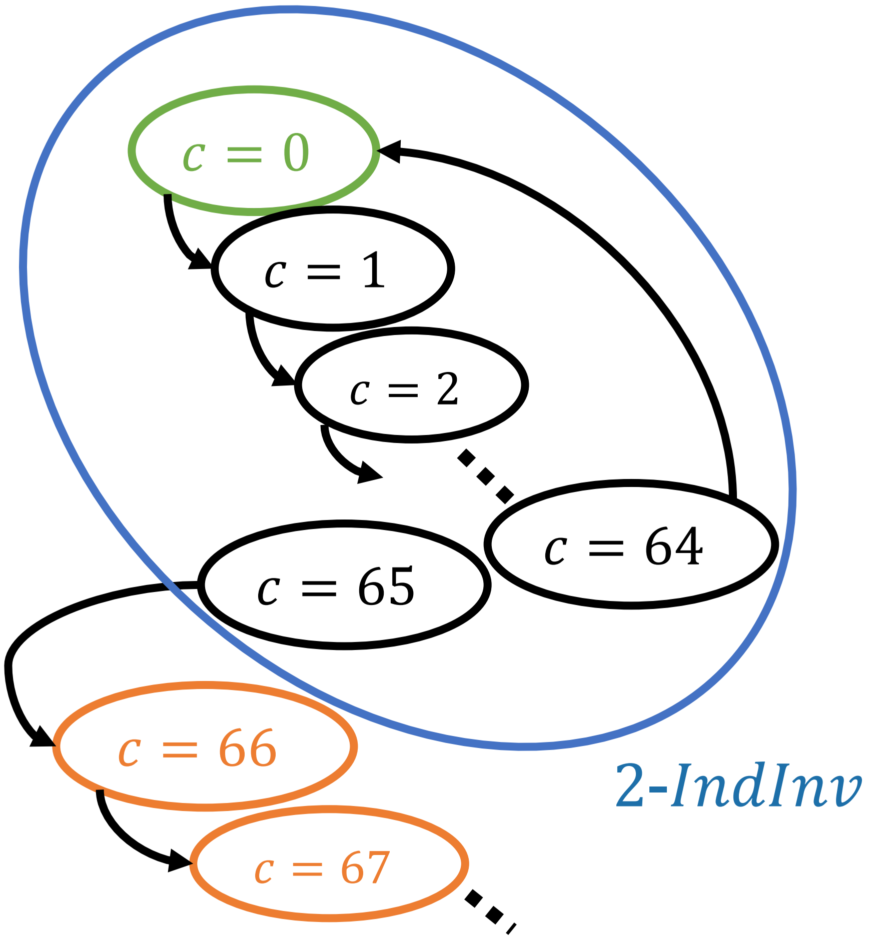
2-Inductive Invariant

```
reg [7:0] c = 0;
always
if(c == 64)
      c = 0;
else
      c = c + 1;
end
assert property(c < 66);
```



K-induction with a SAT solver (IND)

Recall:

$U_k = T^{<0>} \wedge T^{<1>} \wedge \ldots \wedge T^{<k-1>}$

Two formulas to check:

- Base case:

$$I^{<0>} \wedge U_{k-1} \Rightarrow P^{<0>} \ldots P^{<k-1>}$$

- Induction step:

$$U_k \wedge P^{<0>} \ldots P^{<k-1>} \Rightarrow P^{<k>}$$

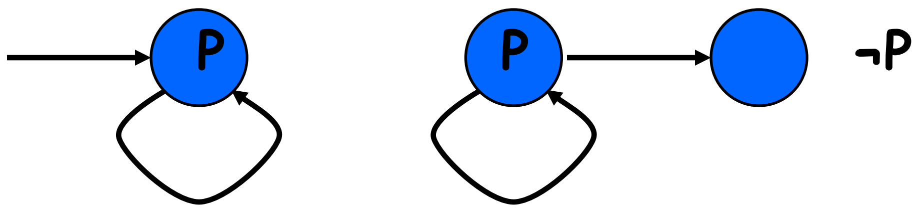
If both are valid, then *P* always holds.

If not, increase *k* and try again.

Simple path assumption

Unfortunately, k-induction is not complete.

- Some properties are not k-inductive for any k.



Simple path restriction:

- There is a path to ¬P iff there is a *simple* path to ¬P (path with no repeated states).

Induction over simple paths

Let simple($s_{0..k}$) be defined as:

- $\forall i,j$ in 0..k ¬ $(i \neq j) \Rightarrow s_i \neq s_j$

k-induction over simple paths:

$$\frac{\begin{array}{c} P(s_{0..k-1}) \\ \forall i \neg\ \text{simple}(s_{0..k}) \wedge P(s_{i..i+k-1}) \Rightarrow P(s_{i+k}) \end{array}}{\forall i \neg\ P(s_i)}$$

Must hold for k large enough, since a simple path cannot be unboundedly long. Length of longest simple path is called *recurrence diameter*.

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...with a SAT solver

For simple path restriction, let

$S_k = \forall t=0..k,\ u=t+1..k\text{:}\ \neg\forall v \text{ in } V\ \neg\ v_t = v_u$

(where V is the set of state variables).

Two formulas to check

- Base case

$$I^{<0>} \wedge U_{k-1} \Rightarrow P^{<0>}...P^{<k-1>}$$

- Induction step

$$S_k \wedge U_k \wedge P^{<0>}...P^{<k-1>} \Rightarrow P^{<k>}$$

If both are valid, then P always holds.

If not, increase k and try again.

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Termination

Termination condition

k is the length of the longest simple path of the form

$$P^*\neg P$$

This can be exponentially longer than the diameter.

- example
  - loadable mod 2^N counter where P is (count $\neq 2^N-1$)
  - diameter = 1
  - longest simple path = 2^N

Useful special cases

- P is a tautology (k=0)
- P is inductive invariant (k=1)

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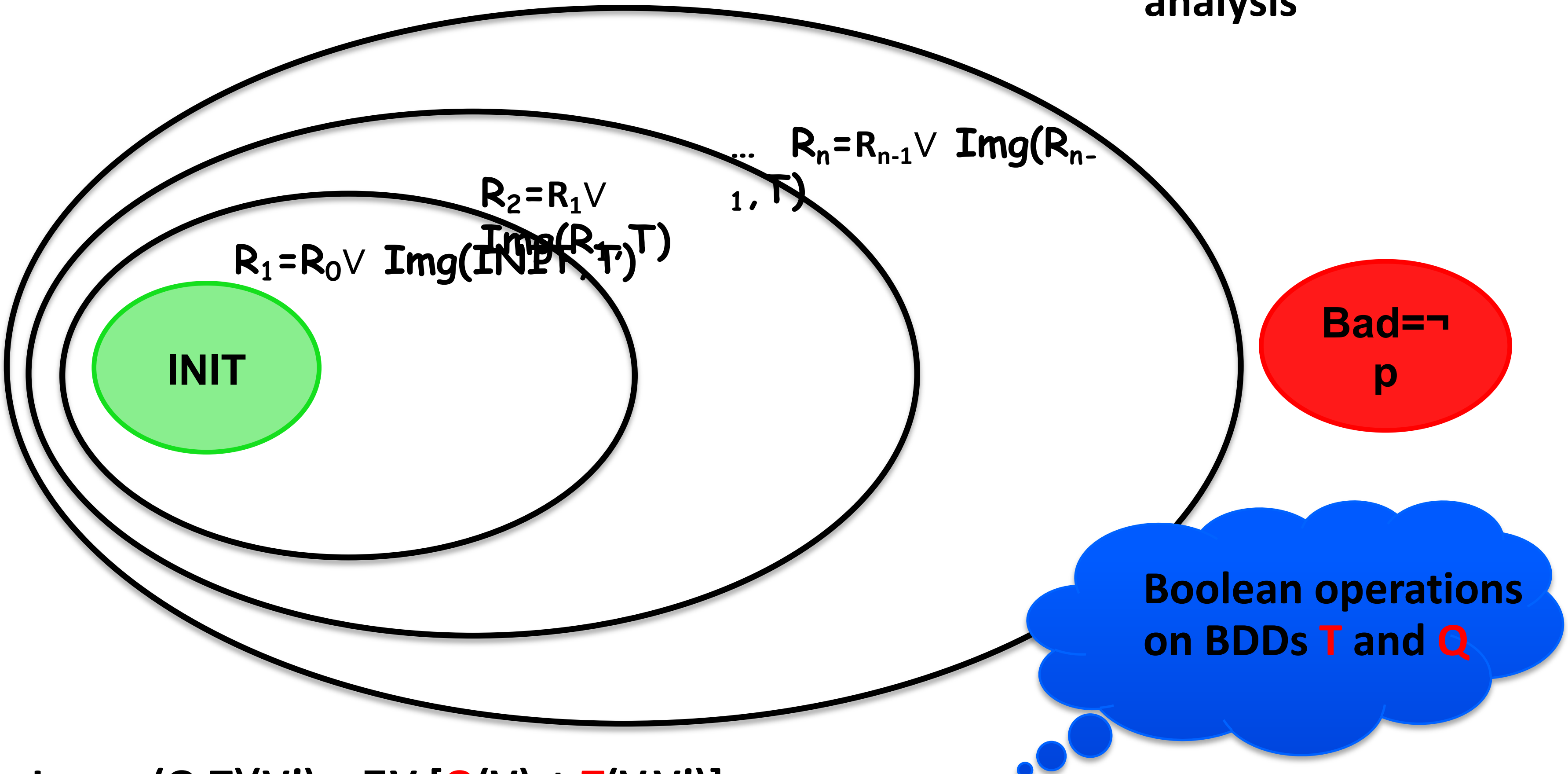
BDD-BASED SYMBOLIC REACHABILITY

Forward Reachability Analysis with BDDs

Does AG p hold?

All safety properties reduce to reachability analysis



Image(Q,T)(V') = ∃V [Q(V) ∧ T(V,V')]

Representing Sets as Prop. Formulas

$[F]$ states satisfying F, i.e. $\{\sigma \mid \sigma \vDash F\}$	F propositional formula over V
$[F_1] \cap [F_2]$	$F_1 \wedge F_2$
$[F_1] \cup [F_2]$	$F_1 \vee F_2$
$\overline{[F]}$	$\neg F$
$[F_1] \subseteq [F_2]$	$F_1 \Rightarrow F_2$ i.e. $F_1 \wedge \neg F_2$ unsatisfiable

BDDs in a nutshell

Typically mean Reduced Ordered Binary Decision Diagrams (ROBDDs)

Canonical representation of Boolean formulas

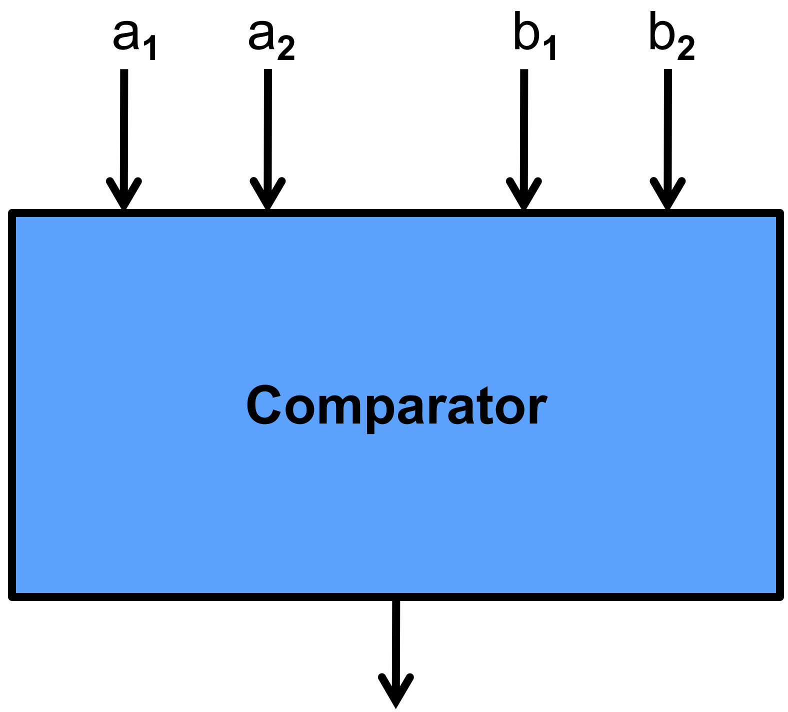
Often substantially more compact than a traditional normal form

Can be manipulated very efficiently

- Conjunction, Disjunction, Negation, Existential Quantification

R. E. Bryant. Graph-based algorithms for boolean function manipulation. *IEEE Transactions on Computers, C-35(8), 1986.*

Running Example¬ Comparator



$f = 1 \Leftrightarrow a_1 = b_1 \wedge a_2 = b_2$

Conjunctive Normal Form

$$f$$

$$a_1 = b_1 \quad \wedge \quad a_2 = b_2$$

$$a_1 \Rightarrow b_1 \wedge b_1 \Rightarrow a_1 \quad \wedge \quad a_2 \Rightarrow b_2 \wedge b_2 \Rightarrow a_2$$

$$(\neg a_1 \vee b_1) \wedge (\neg b_1 \vee a_1) \wedge (\neg a_2 \vee b_2) \wedge (\neg b_2 \vee a_2)$$

$$(\neg b_1 \vee a_1) \wedge (\neg a_1 \vee b_1) \wedge (\neg a_2 \vee b_2) \wedge (\neg b_2 \vee a_2)$$

Not Canonical

Truth Table (1)

a_1	b_1	a_2	b_2	f
0	0	0	0	1
0	0	0	1	0
0	0	1	0	0
0	0	1	1	1
0	1	0	0	0
0	1	0	1	0
0	1	1	0	0
0	1	1	1	0
1	0	0	0	0
1	0	0	1	0
1	0	1	0	0
1	0	1	1	0
1	1	0	0	1
1	1	0	1	0
1	1	1	0	0
1	1	1	1	1

Still Not Canonical

Truth Table (2)

a_1	a_2	b_1	b_2	f
0	0	0	0	1
0	0	0	1	0
0	0	1	0	0
0	0	1	1	0
0	1	0	0	0
0	1	0	1	1
0	1	1	0	0
0	1	1	1	0
1	0	0	0	0
1	0	0	1	0
1	0	1	0	1
1	0	1	1	0
1	1	0	0	0
1	1	0	1	0
1	1	1	0	0
1	1	1	1	1

Canonical if you fix variable order.

But always exponential in # of variables. Let's try to fix this.

Shannon's / Boole's Expansion

Every Boolean formula $f(a_0, a_1, \ldots, a_n)$ can be written as

$$(a_0 \land f(true, a_1, \ldots, a_n)) \lor (\neg a_0 \land f(false, a_1, \ldots, a_n))$$

or, simply,

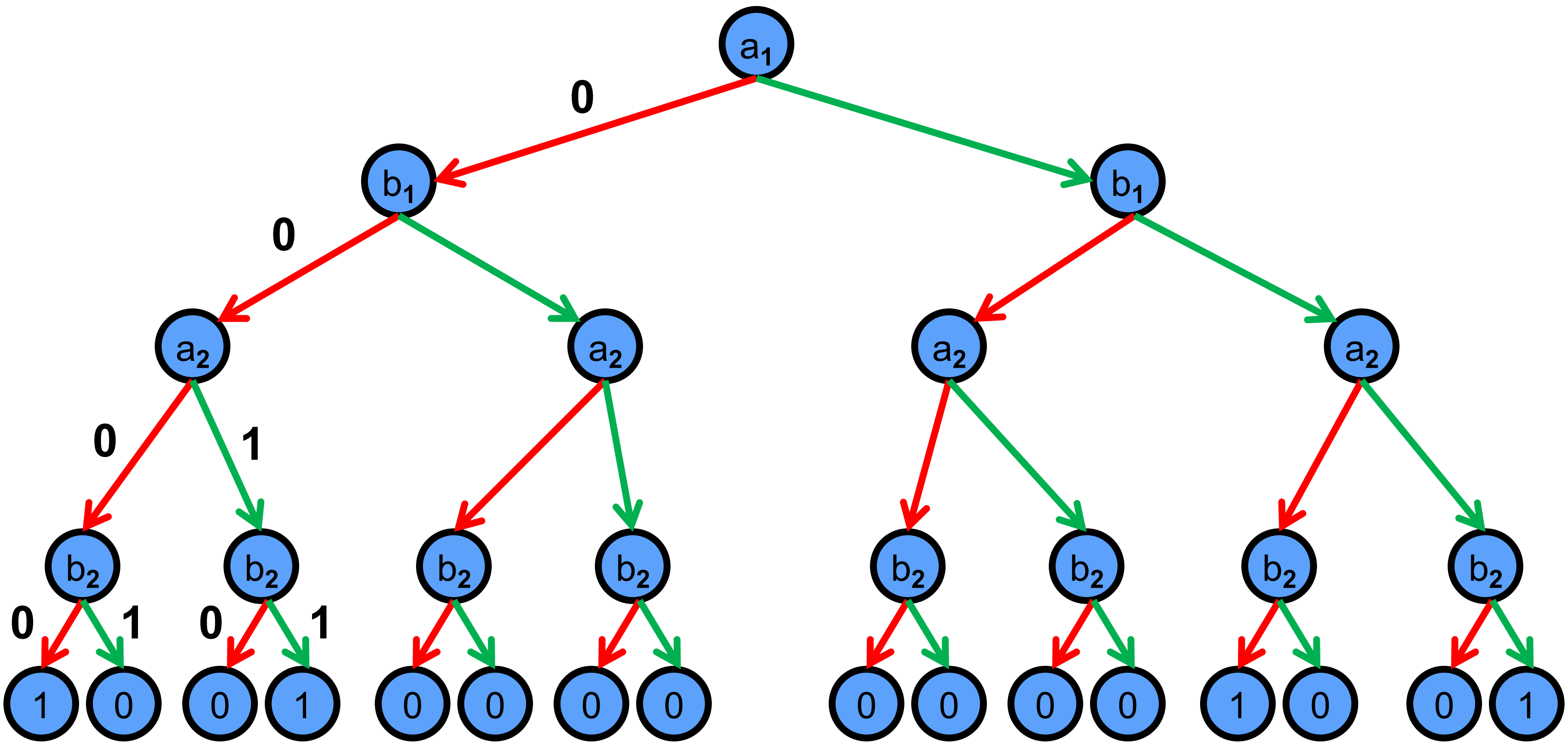
$$ITE\ (a_0, f(true, a_1, \ldots, a_n), f(false, a_1, \ldots, a_n))$$

where ITE stands for If-Then-Else

The formula $f(true, a_1, \ldots, a_n)$ is called the *cofactor* of f w.r.t. a_0

The formula $f(false, a_1, \ldots, a_n)$ is called the *cofactor* of f w.r.t. $\neg a_0$

Representing a Truth Table using a Graph



Binary Decision Tree (in this case ordered)

Binary Decision Tree: Formal Definition

Balanced binary tree. Length of each path = # of variables

Leaf nodes labeled with either 0 or 1

Internal node v labeled with a Boolean variable var(v)

- Every node on a path labeled with a different variable

Internal node v has two children¬ low(v) and high(v)

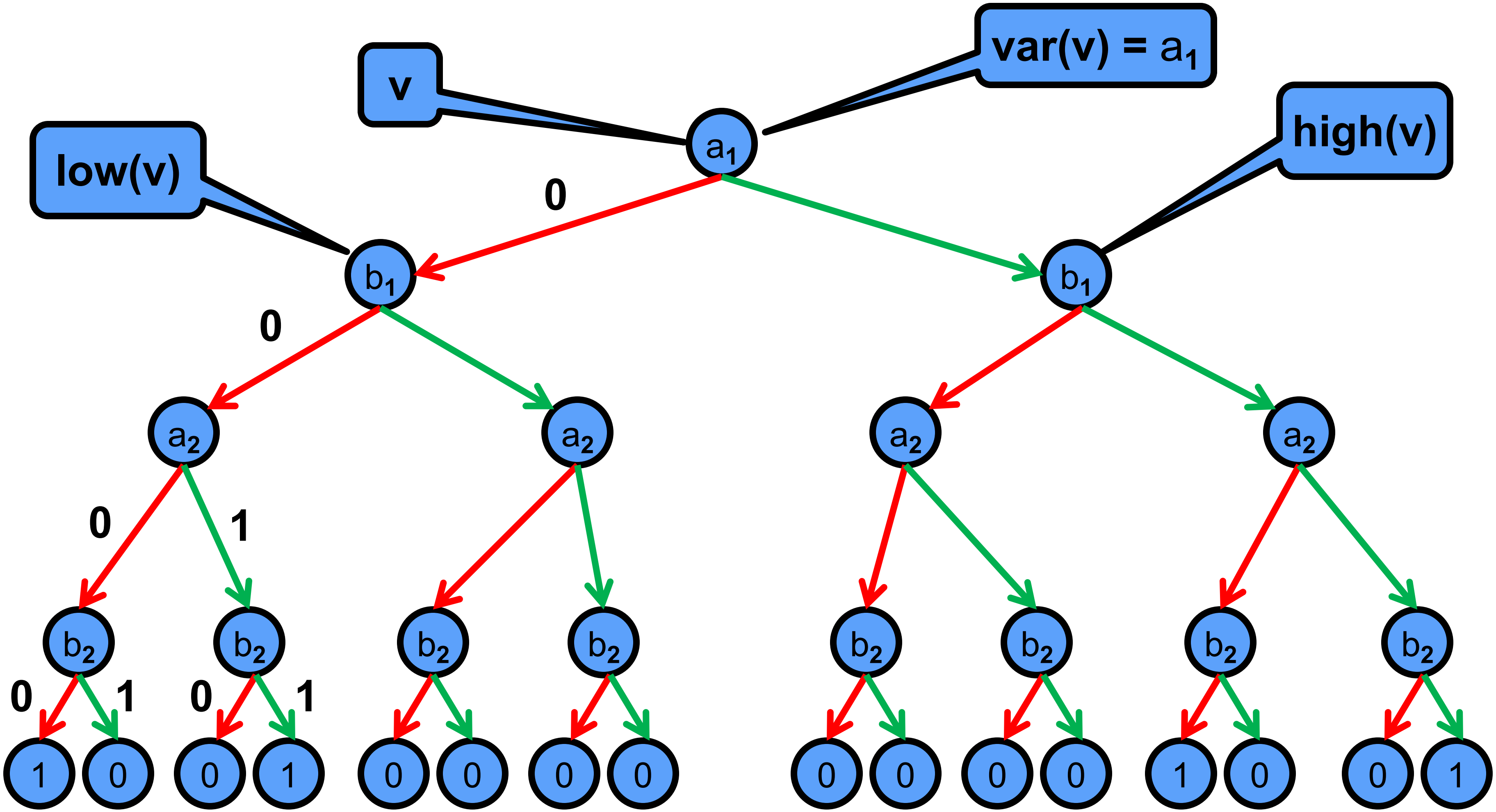
Each path corresponds to a (partial) truth assignment to variables

- Assign 0 to var(v) if low(v) is in the path, and 1 if high(v) is in the path

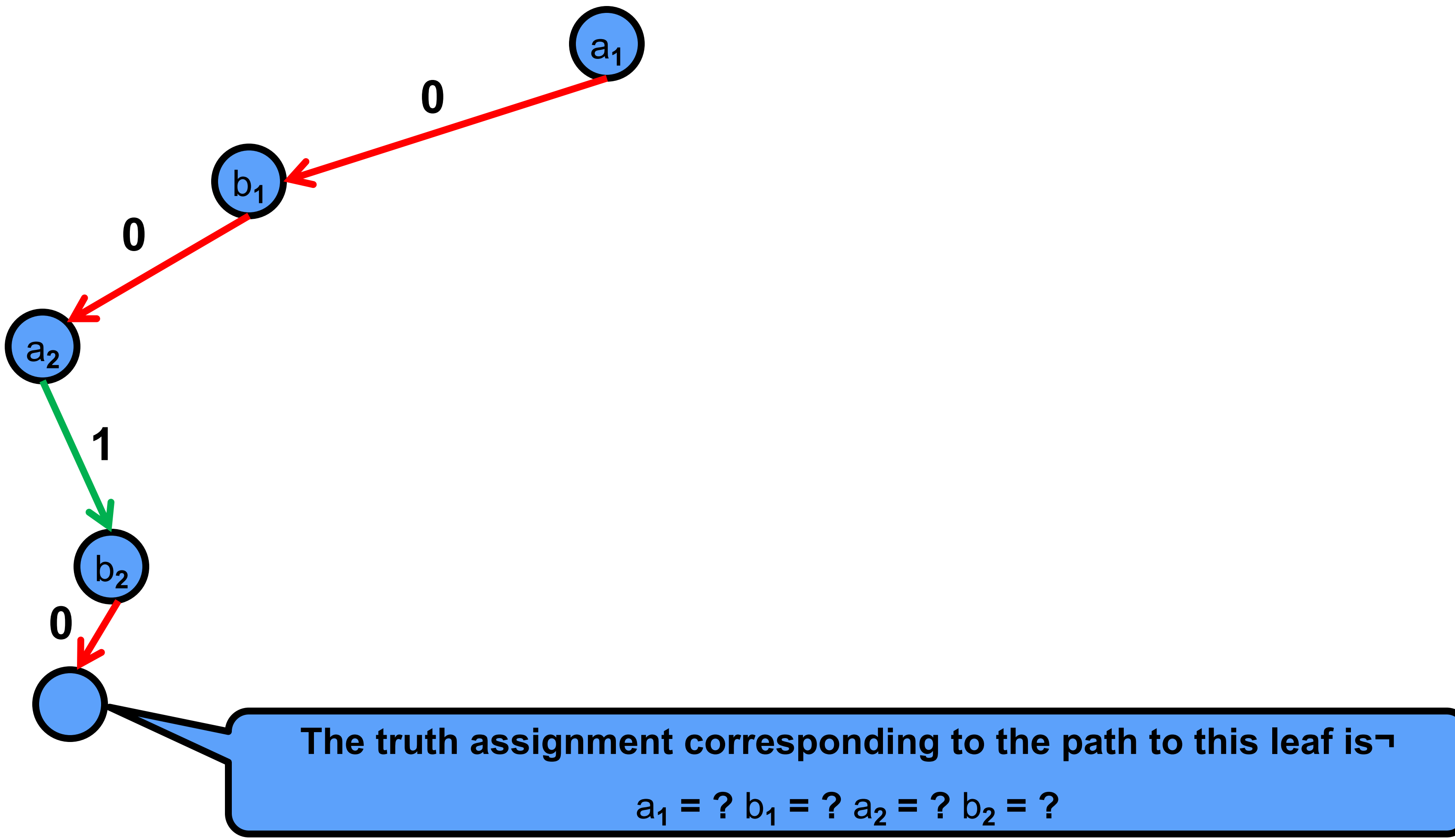
Value of a leaf is determined by:

- Constructing the truth assignment for the path leading to it from the root
- Looking up the truth table with this truth assignment

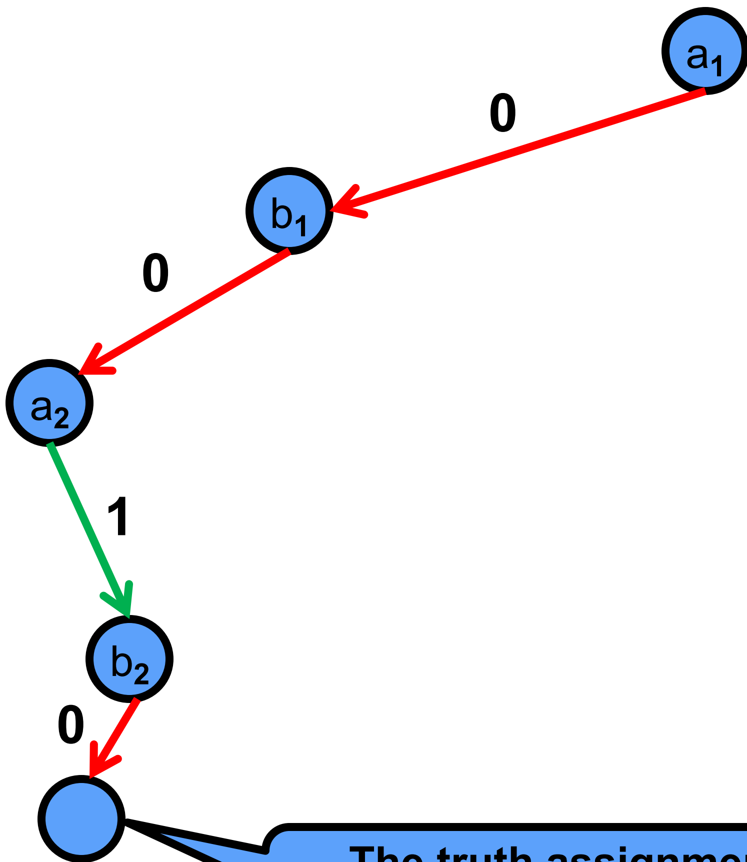
Binary Decision Tree



Binary Decision Tree



Binary Decision Tree

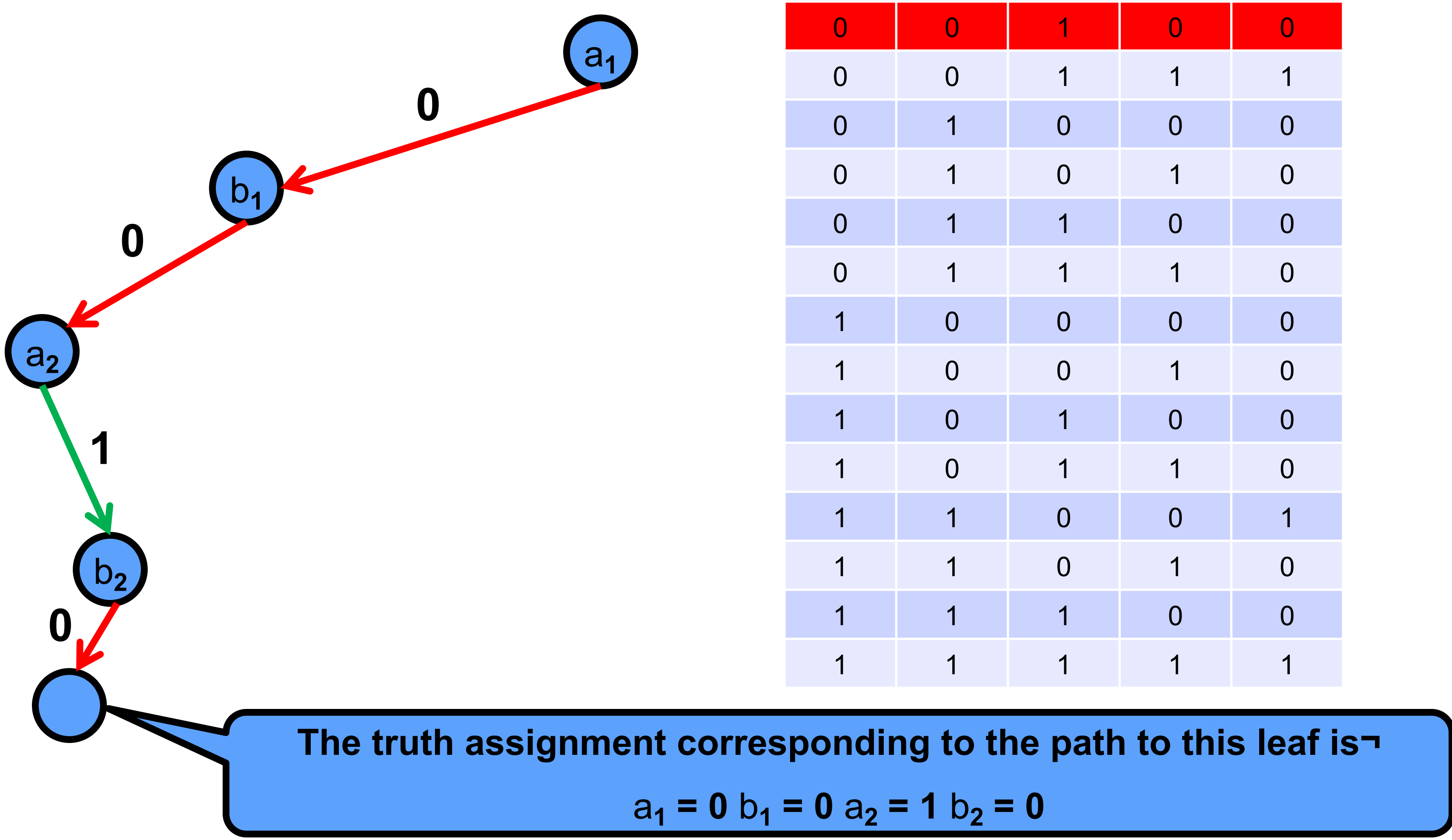


a_1	b_1	a_2	b_2	f
0	0	0	0	1
0	0	0	1	0
0	0	1	0	0
0	0	1	1	1
0	1	0	0	0
0	1	0	1	0
0	1	1	0	0
0	1	1	1	0
1	0	0	0	0
1	0	0	1	0
1	0	1	0	0
1	0	1	1	0
1	1	0	0	1
1	1	0	1	0
1	1	1	0	0
1	1	1	1	1

The truth assignment corresponding to the path to this leaf is¬

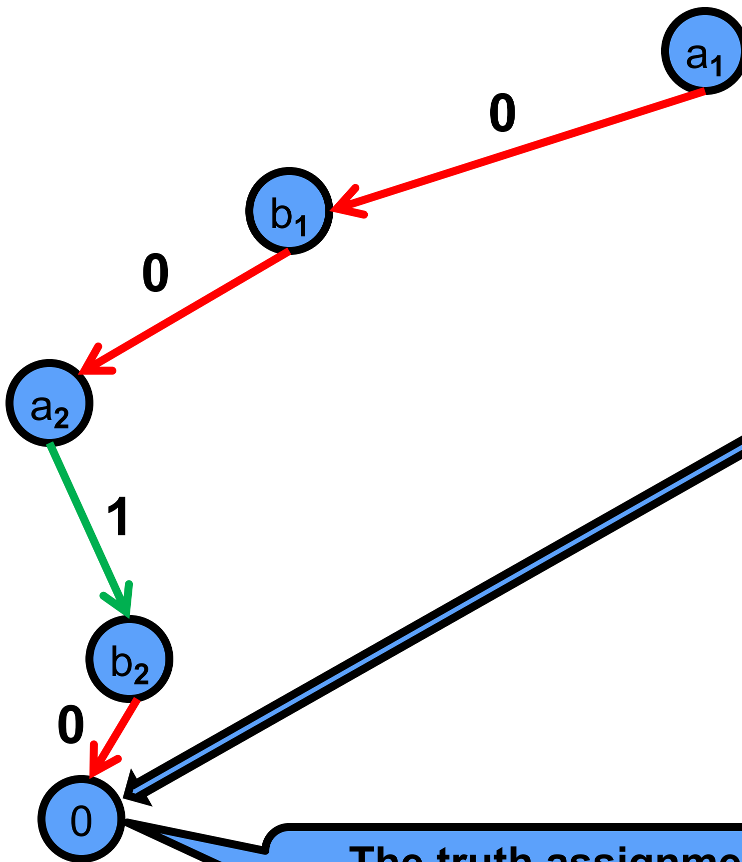
$a_1 = 0$ $b_1 = 0$ $a_2 = 1$ $b_2 = 0$

Binary Decision Tree



a_1	b_1	a_2	b_2	f
0	0	0	0	1
0	0	0	1	0
0	0	1	0	0
0	0	1	1	1
0	1	0	0	0
0	1	0	1	0
0	1	1	0	0
0	1	1	1	0
1	0	0	0	0
1	0	0	1	0
1	0	1	0	0
1	0	1	1	0
1	1	0	0	1
1	1	0	1	0
1	1	1	0	0
1	1	1	1	1

Binary Decision Tree

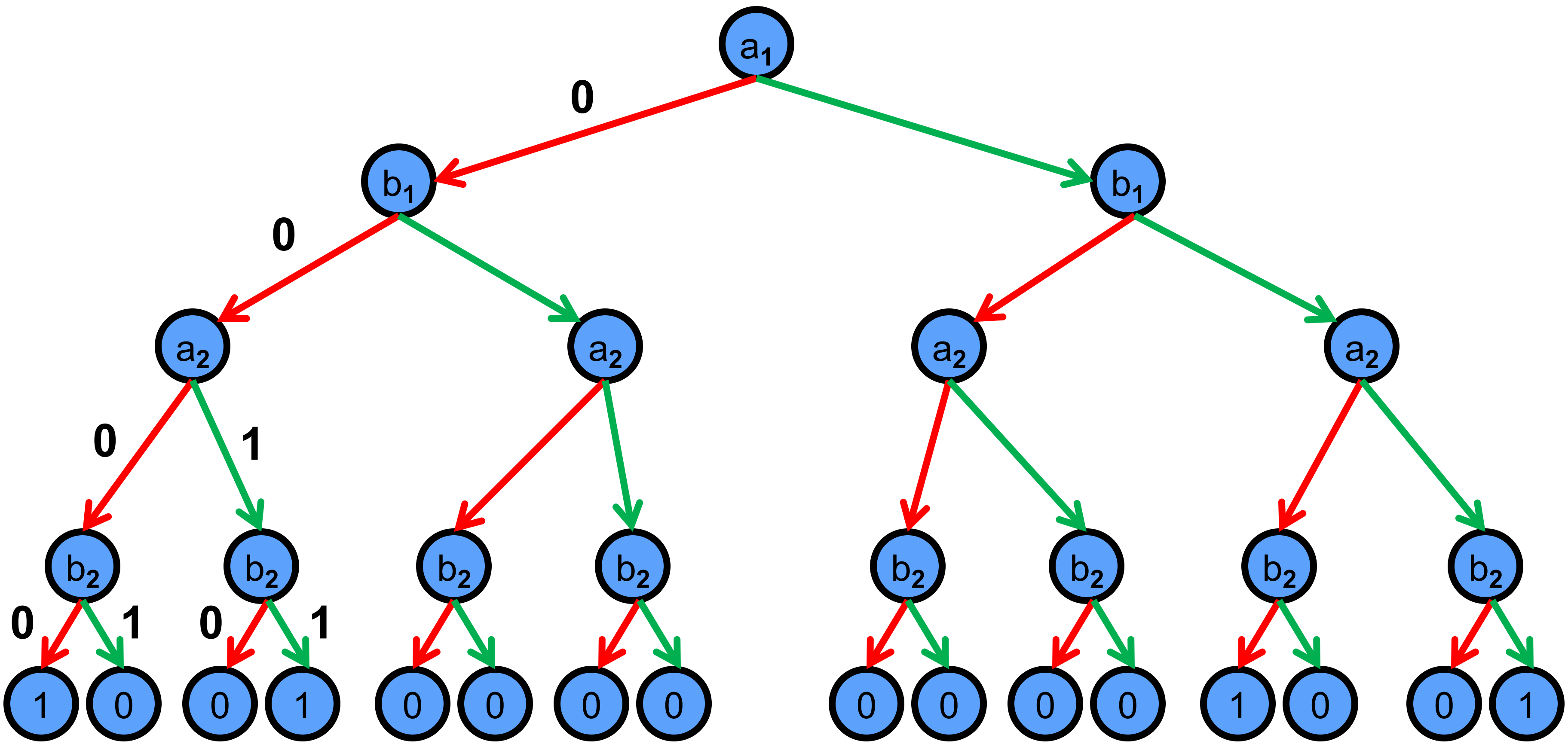


a_1	b_1	a_2	b_2	f
0	0	0	0	1
0	0	0	1	0
0	0	1	0	0
0	0	1	1	1
0	1	0	0	0
0	1	0	1	0
0	1	1	0	0
0	1	1	1	0
1	0	0	0	0
1	0	0	1	0
1	0	1	0	0
1	0	1	1	0
1	1	0	0	1
1	1	0	1	0
1	1	1	0	0
1	1	1	1	1

The truth assignment corresponding to the path to this leaf is¬

a_1 **= 0** b_1 **= 0** a_2 **= 1** b_2 **= 0**

Binary Decision Tree (BDT)



Canonical if you fix variable order (i.e., use ordered BDT)

But still exponential in # of variables. Let's try to fix this.

Reduced Ordered BDD

Conceptually, a ROBDD is obtained from an ordered BDT (OBDT) by eliminating redundant sub-diagrams and nodes

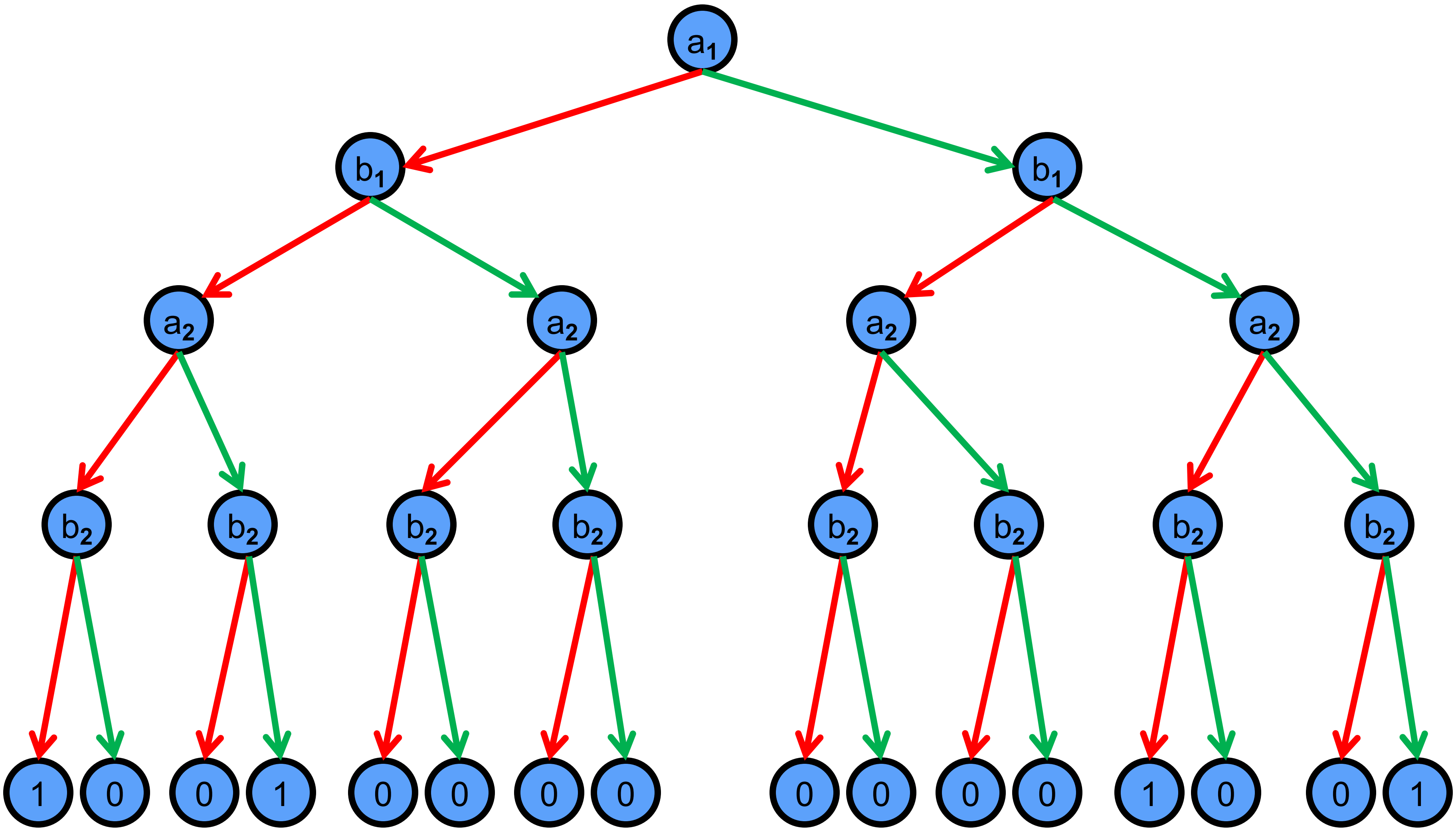
Start with OBDT and repeatedly apply the following two operations as long as possible¬

1. Eliminate duplicate sub-diagrams. Keep a single copy. Redirect edges into the eliminated duplicates into this single copy.
2. Eliminate redundant nodes. Whenever low(v) = high(v), remove v and redirect edges into v to low(v).

- Why does this terminate?

ROBDD is often exponentially smaller than the corresponding OBDT

OBDT to ROBDD



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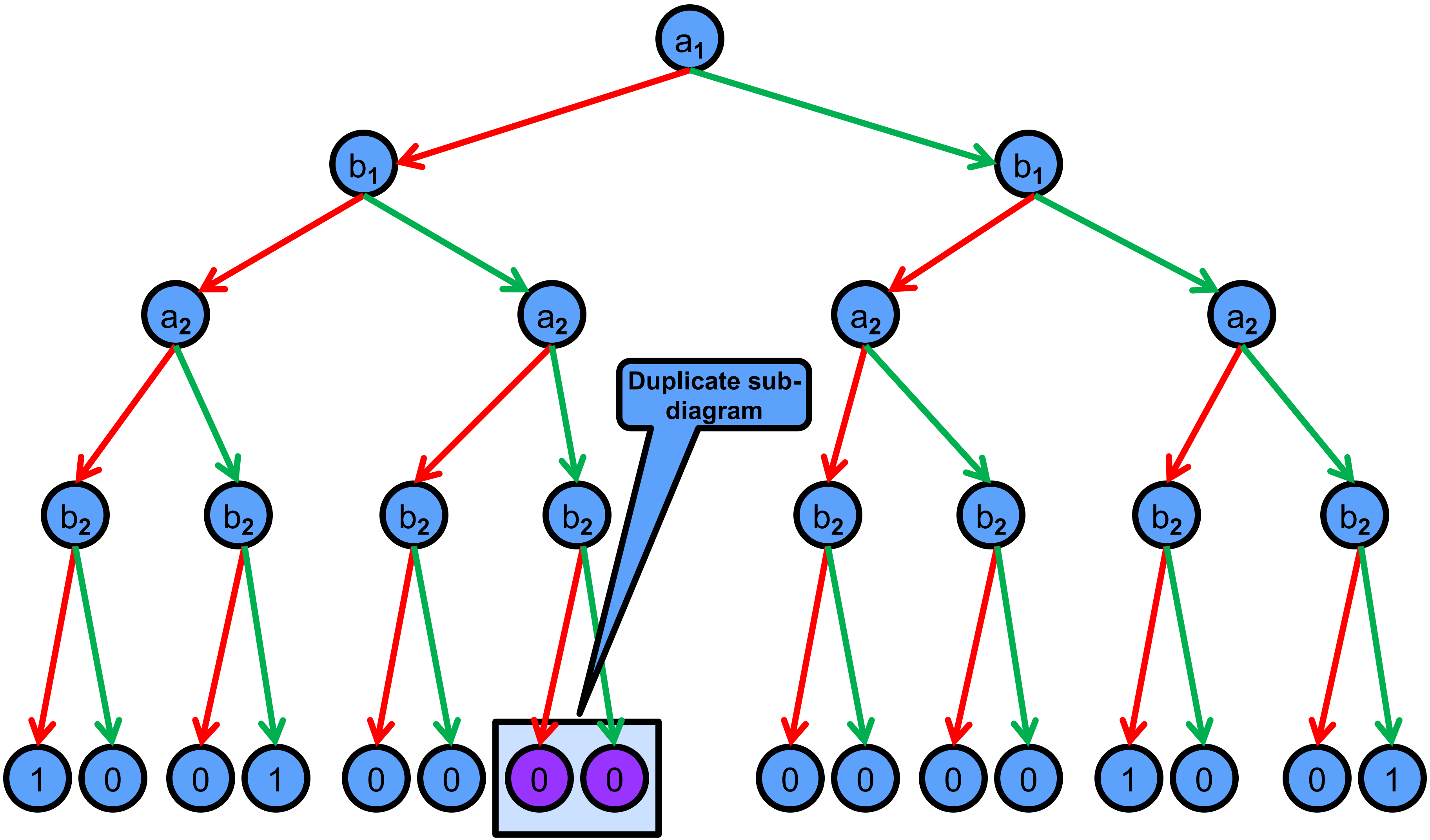
OBDT to ROBDD



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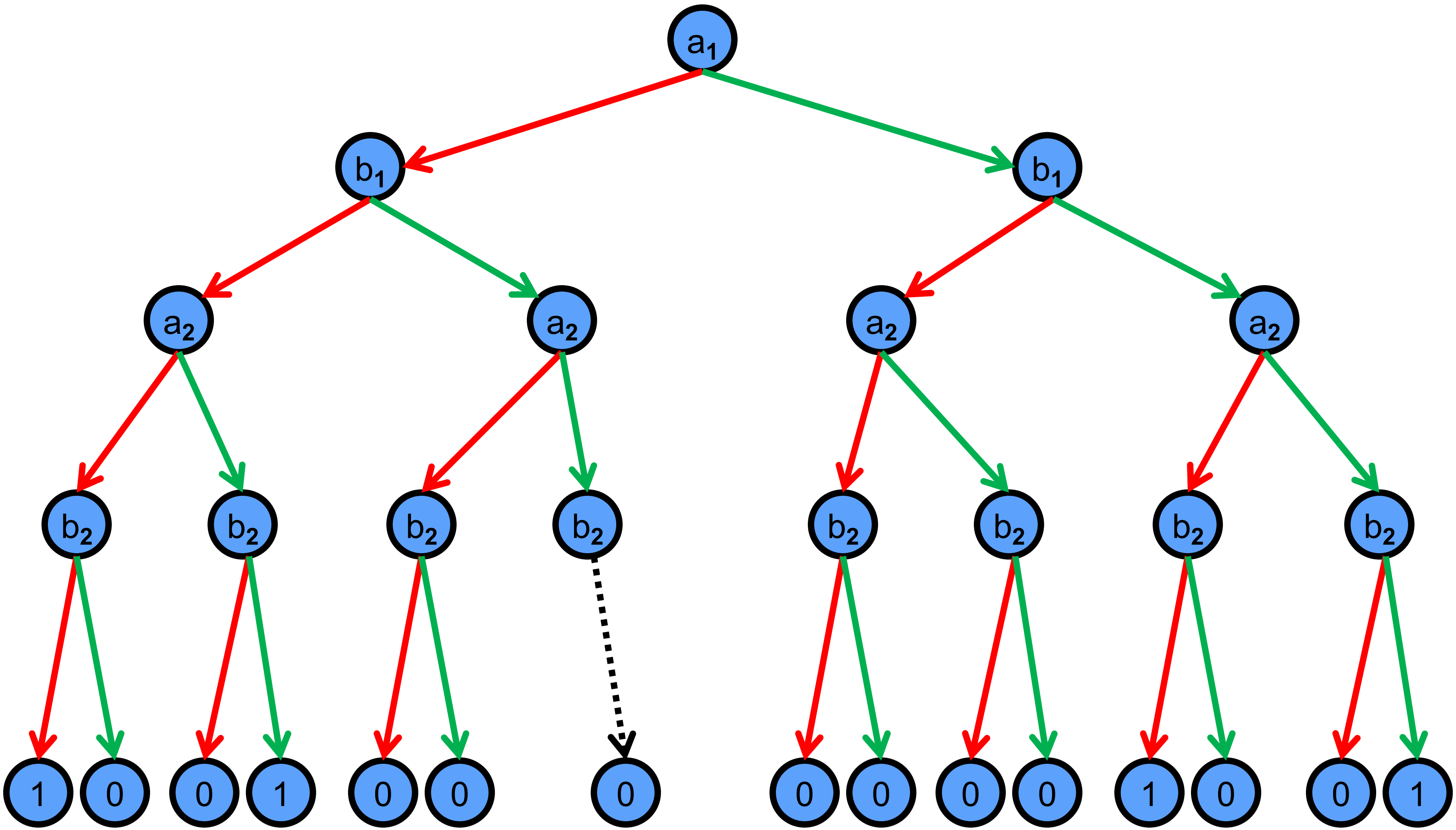
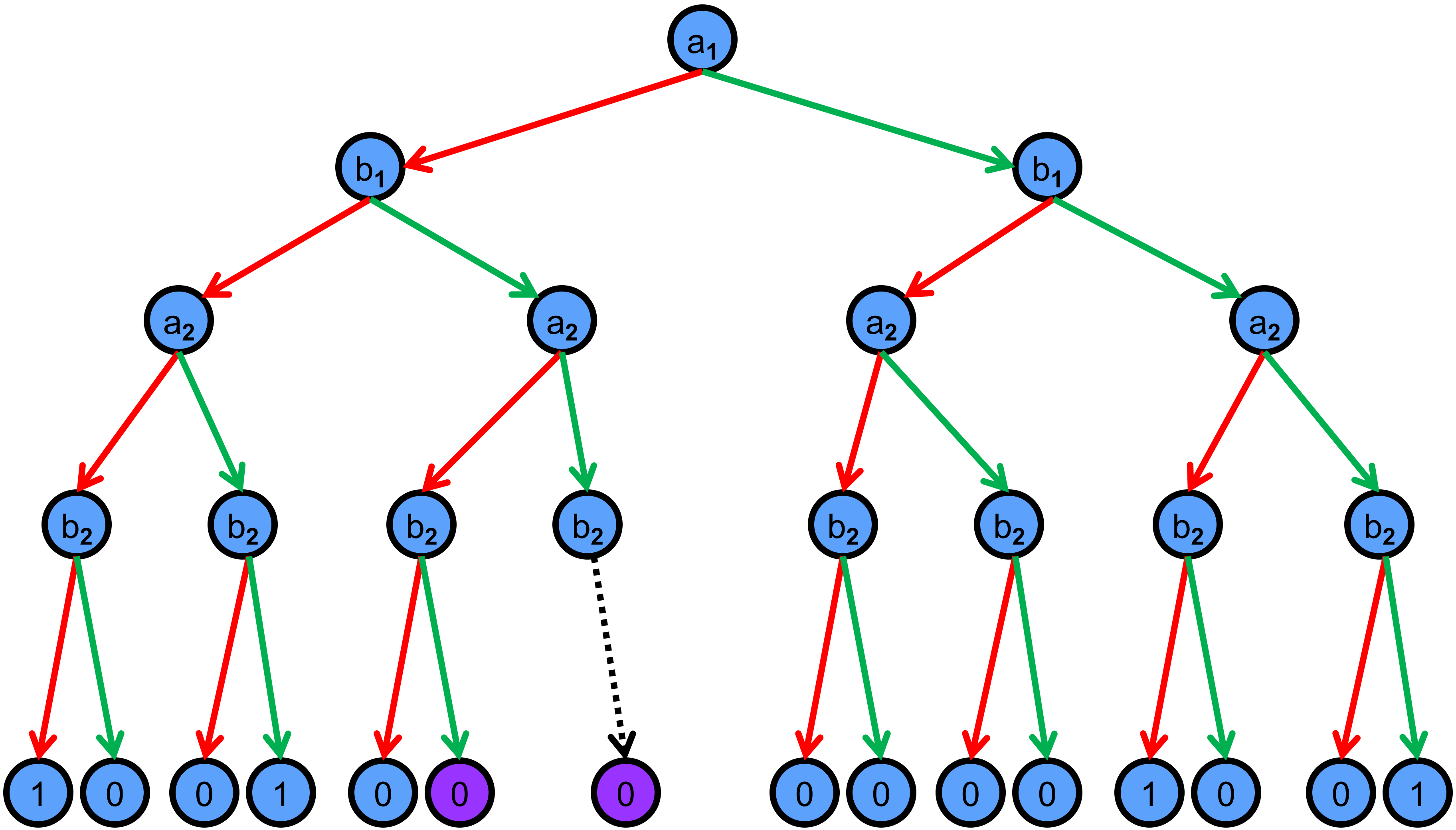
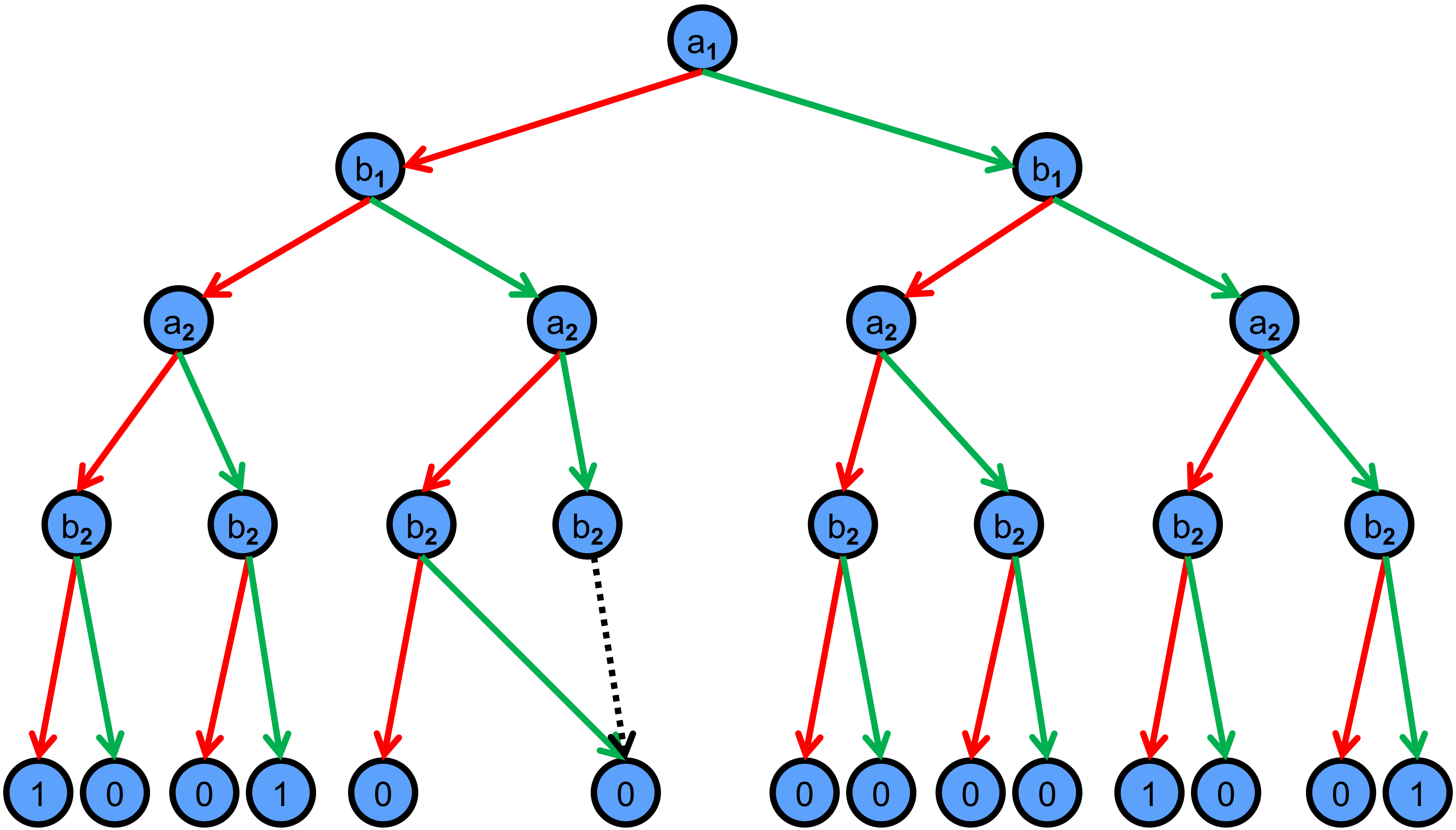
OBDT to ROBDD



OBDT to ROBDD



OBDT to ROBDD



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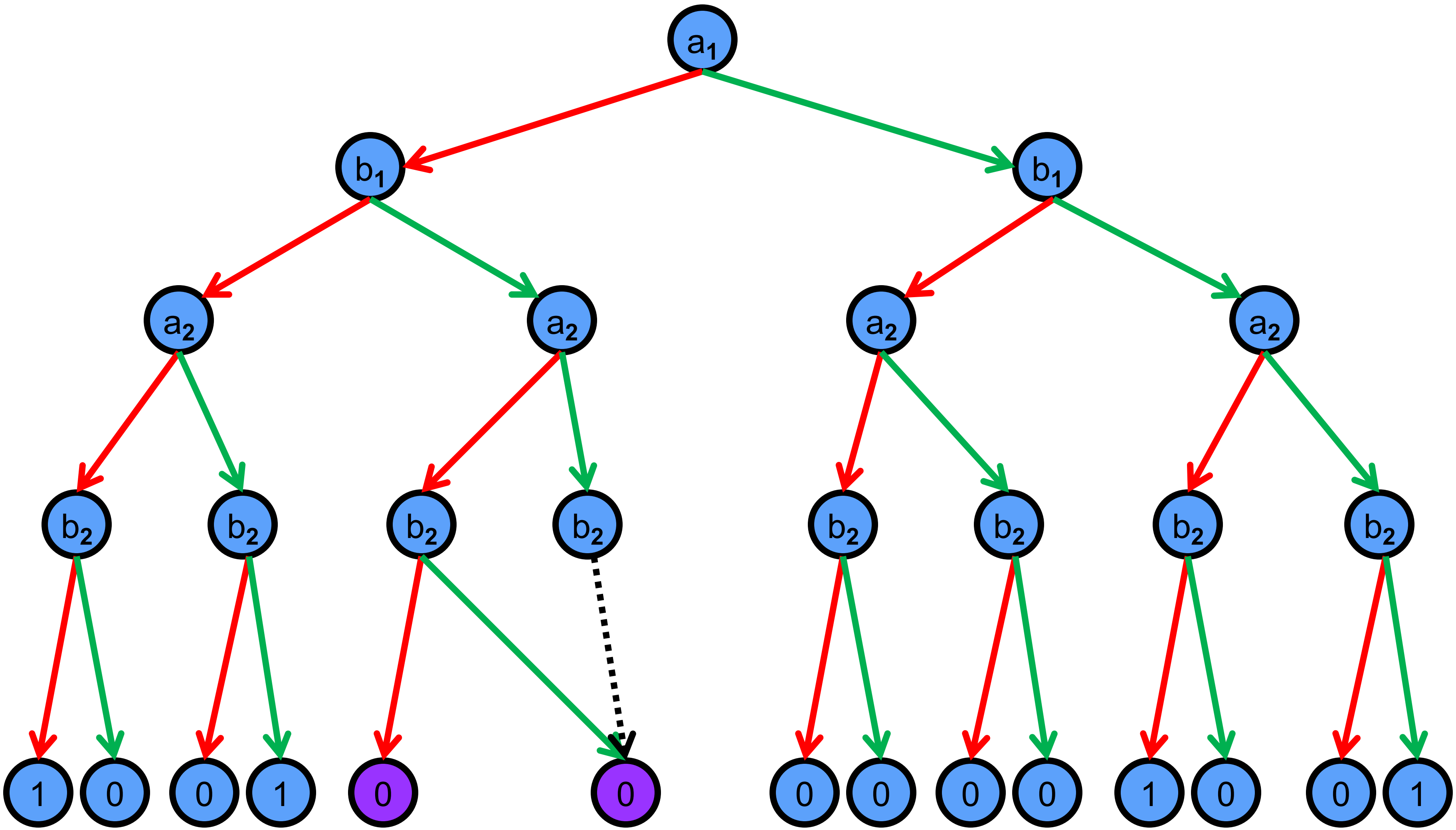
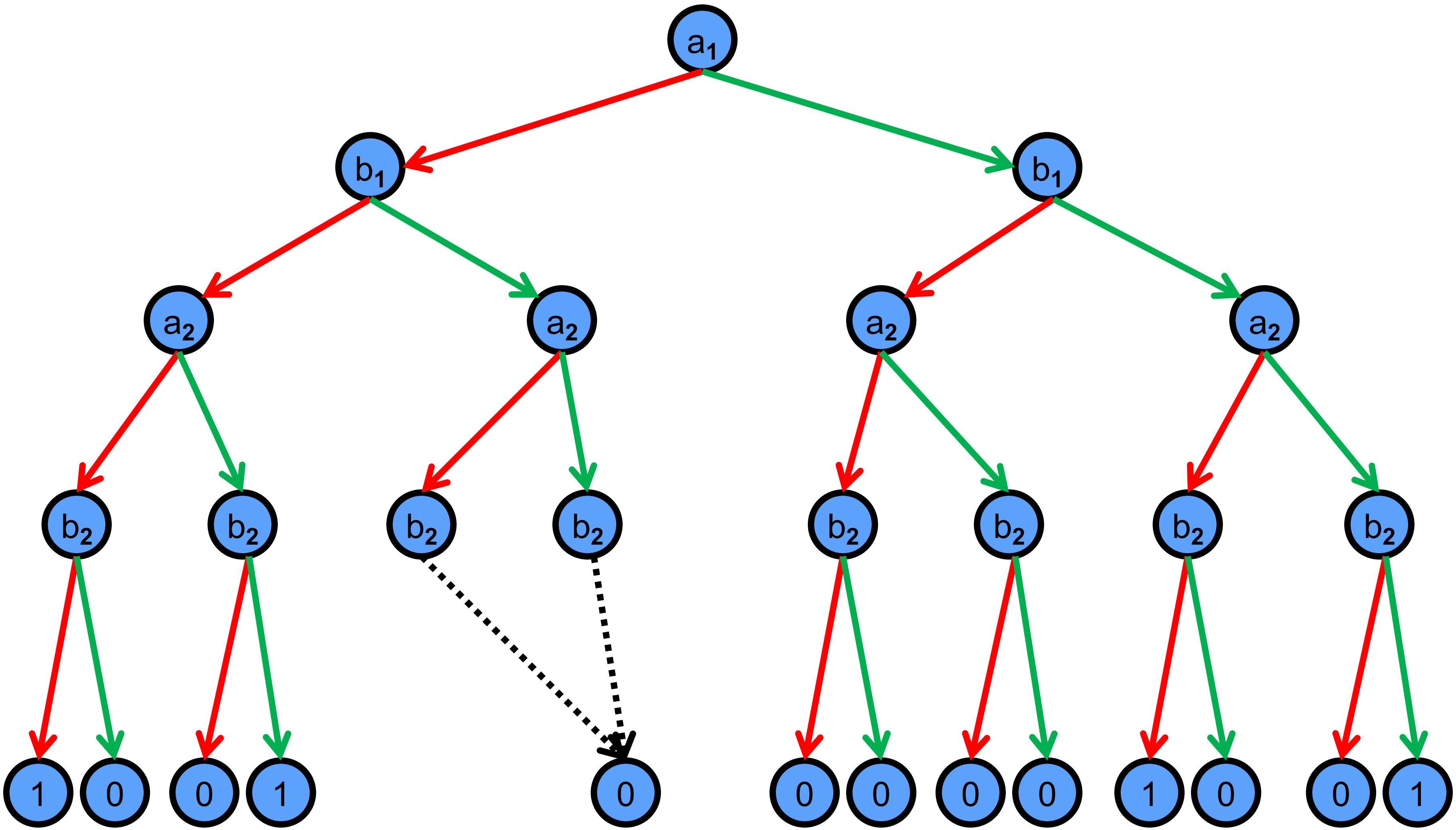
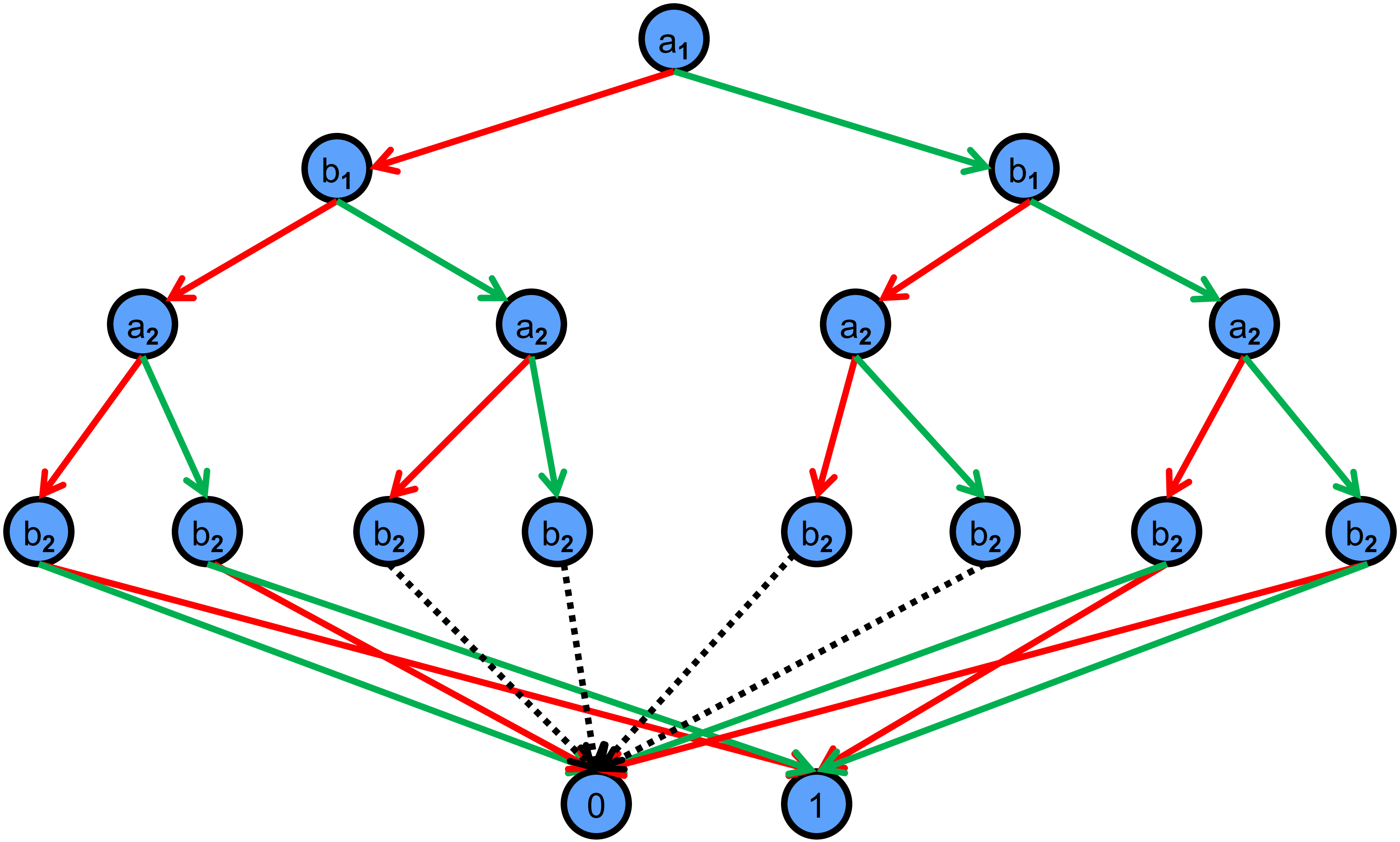
OBDT to ROBDD



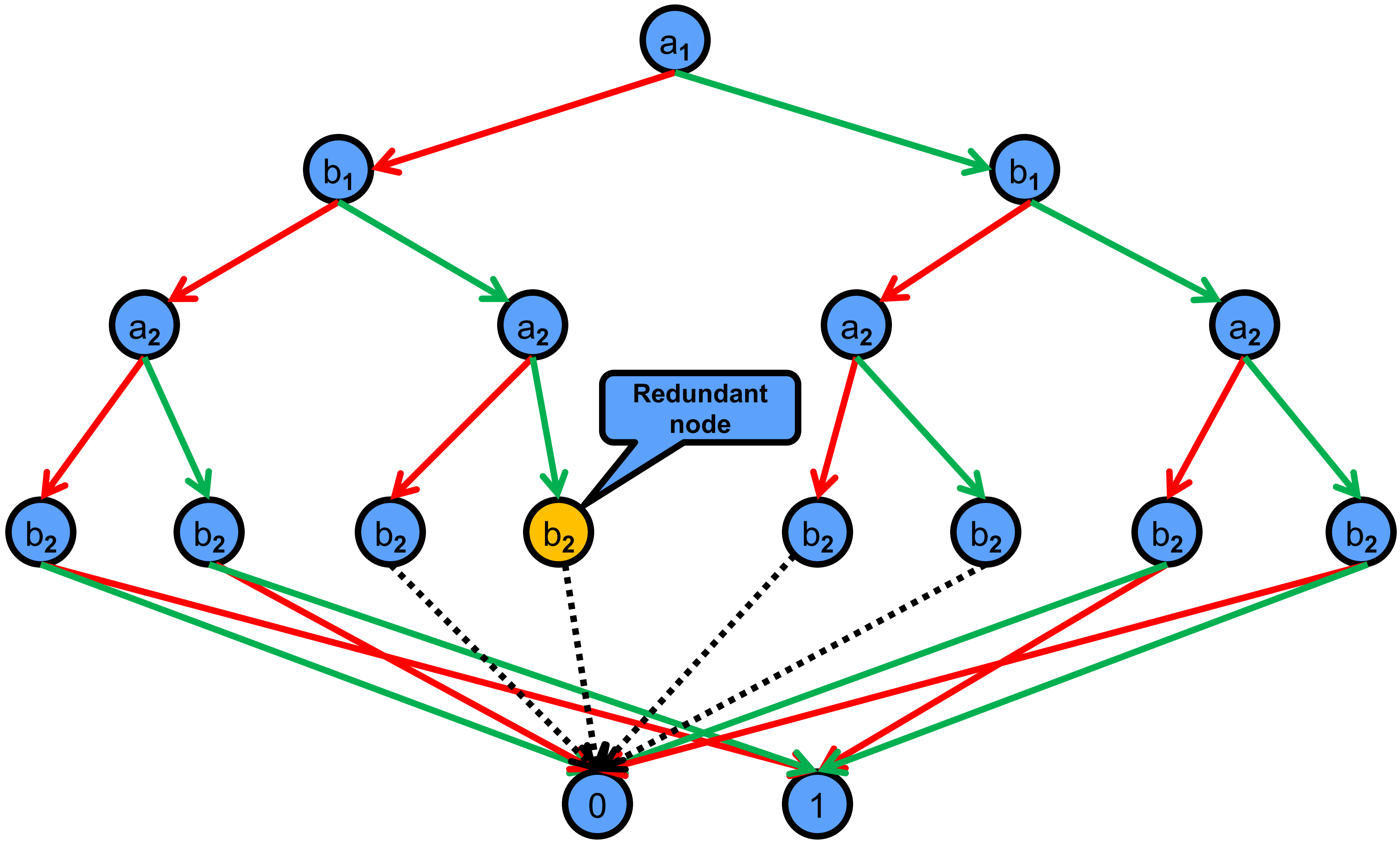
OBDT to ROBDD



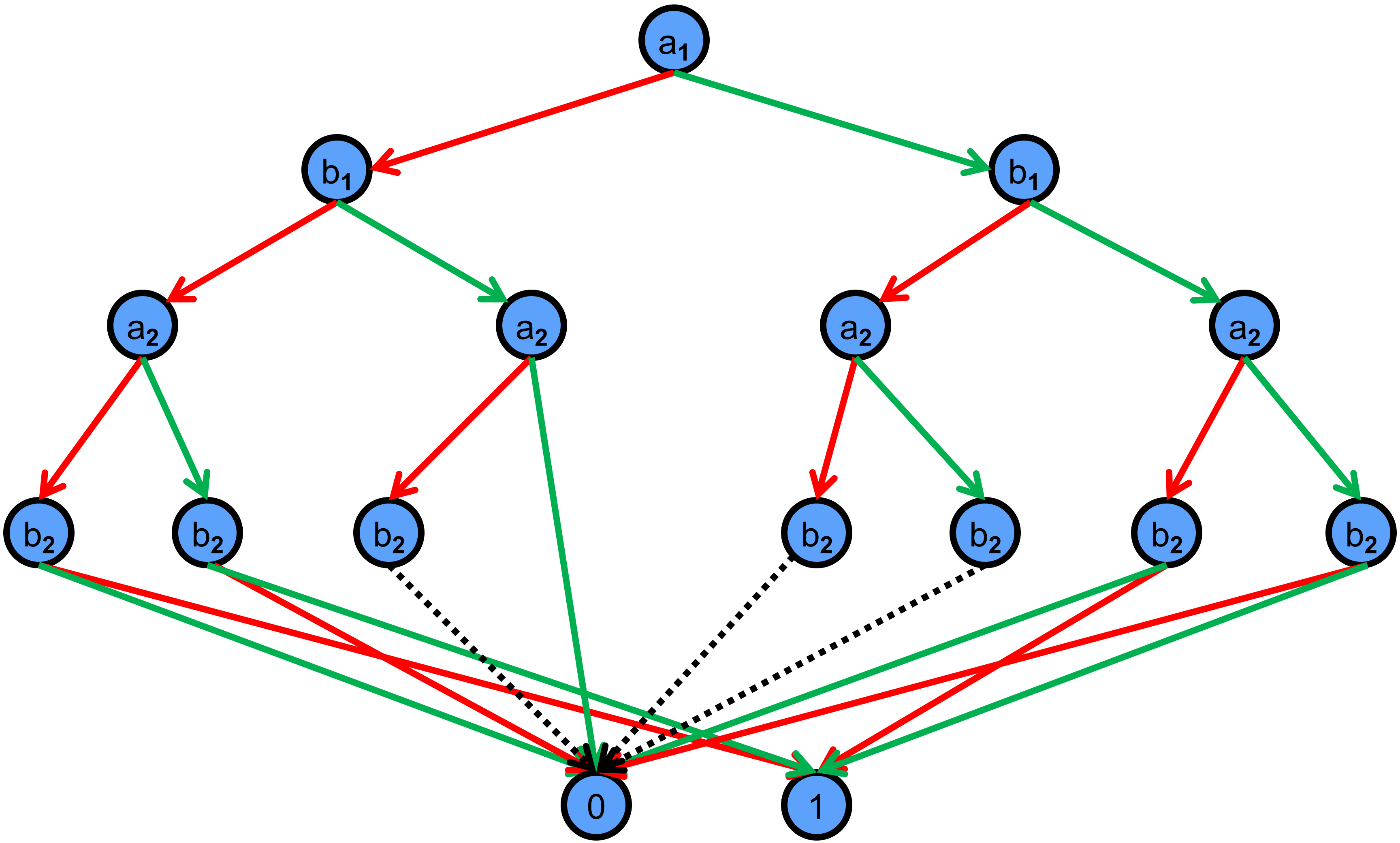
OBDT to ROBDD



OBDT to ROBDD



OBDT to ROBDD



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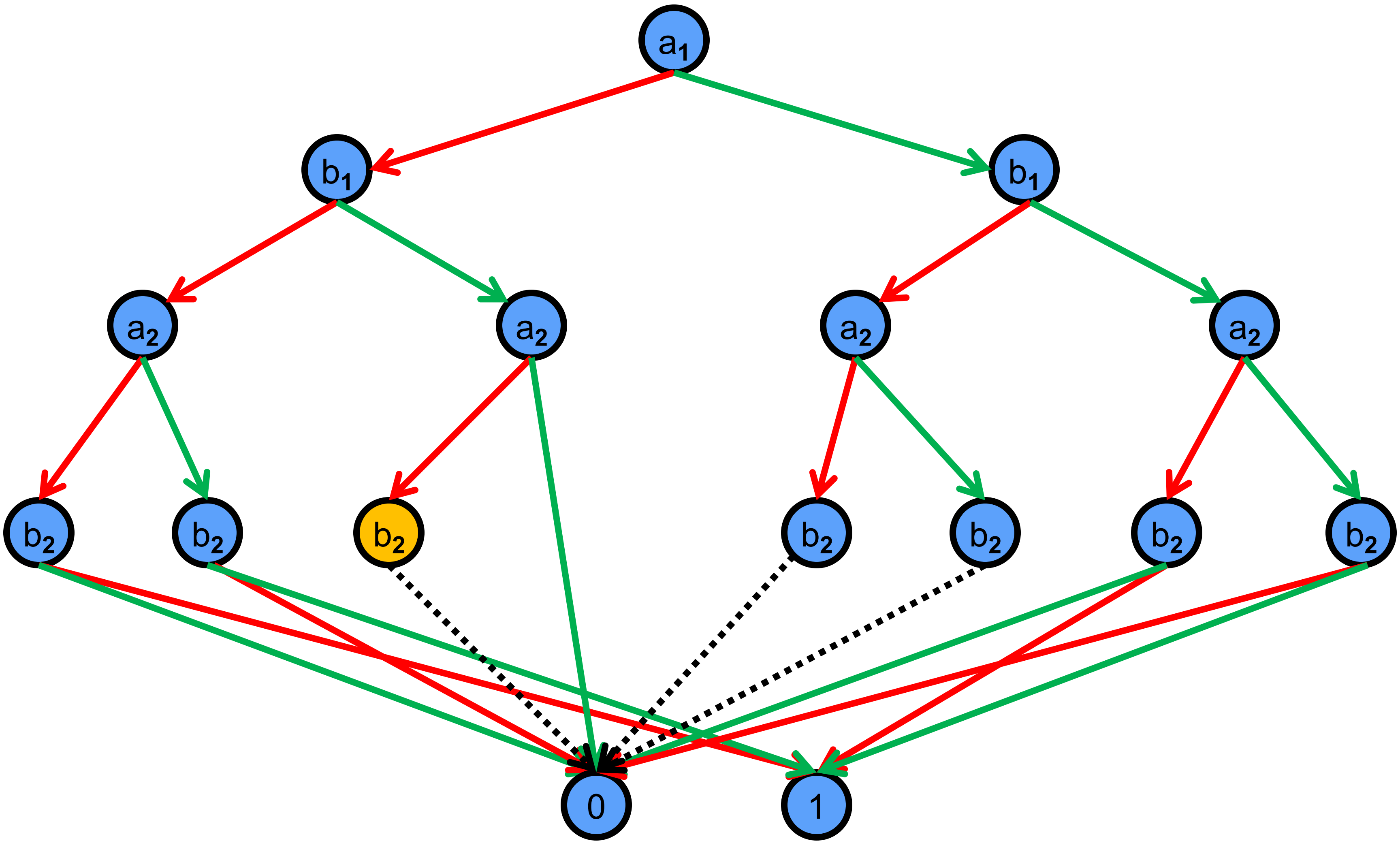
OBDT to ROBDD



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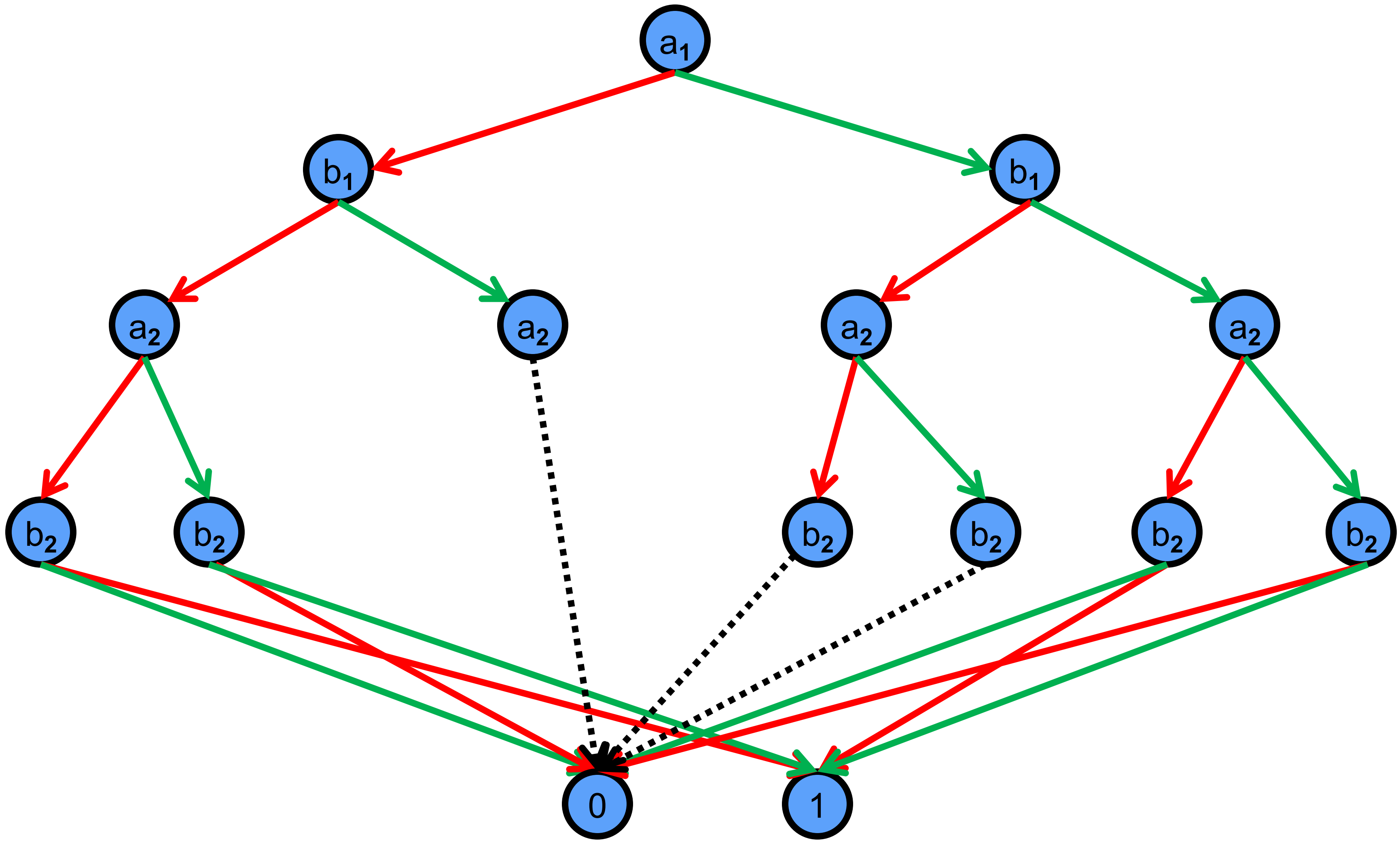
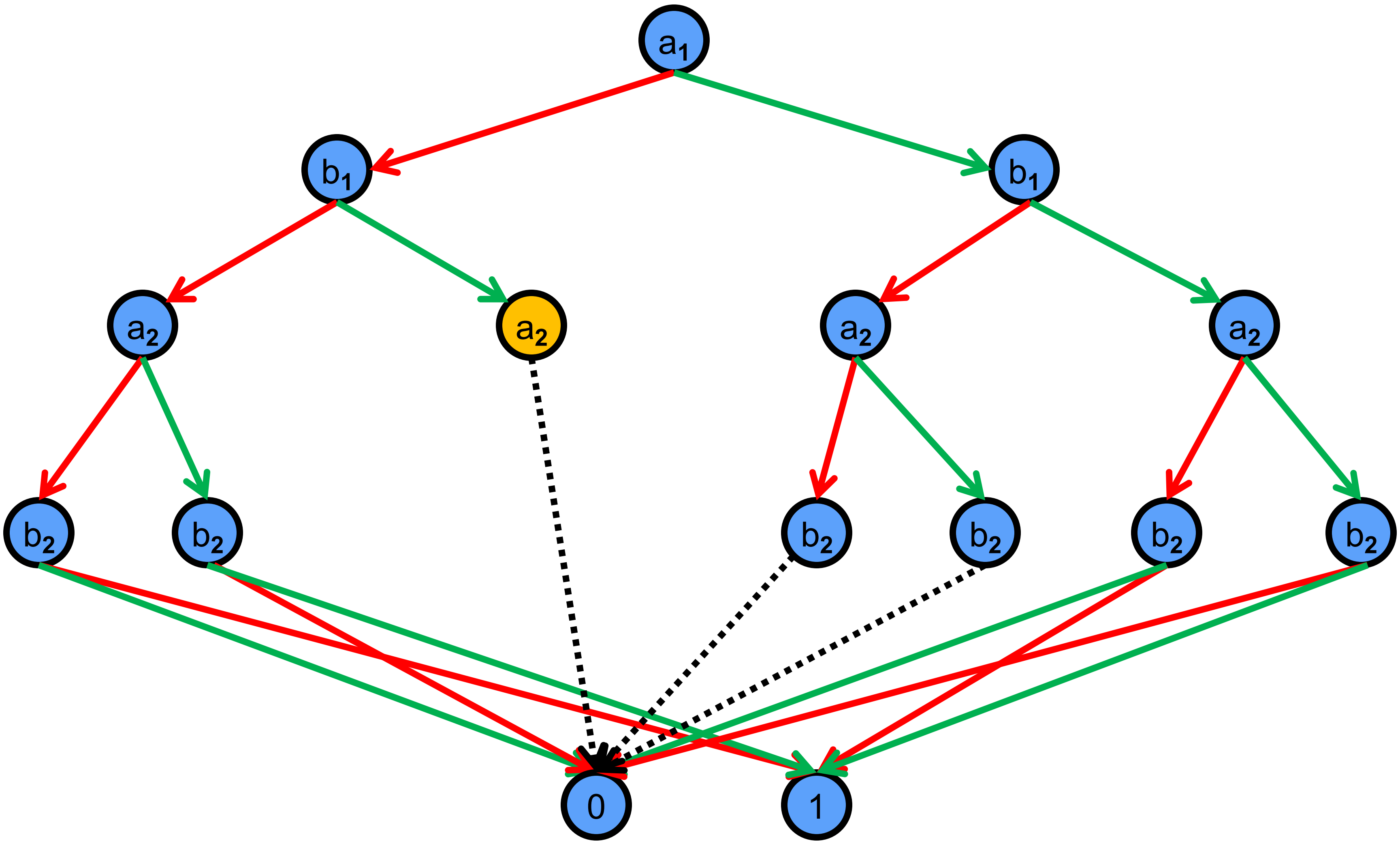
OBDT to ROBDD



OBDT to ROBDD



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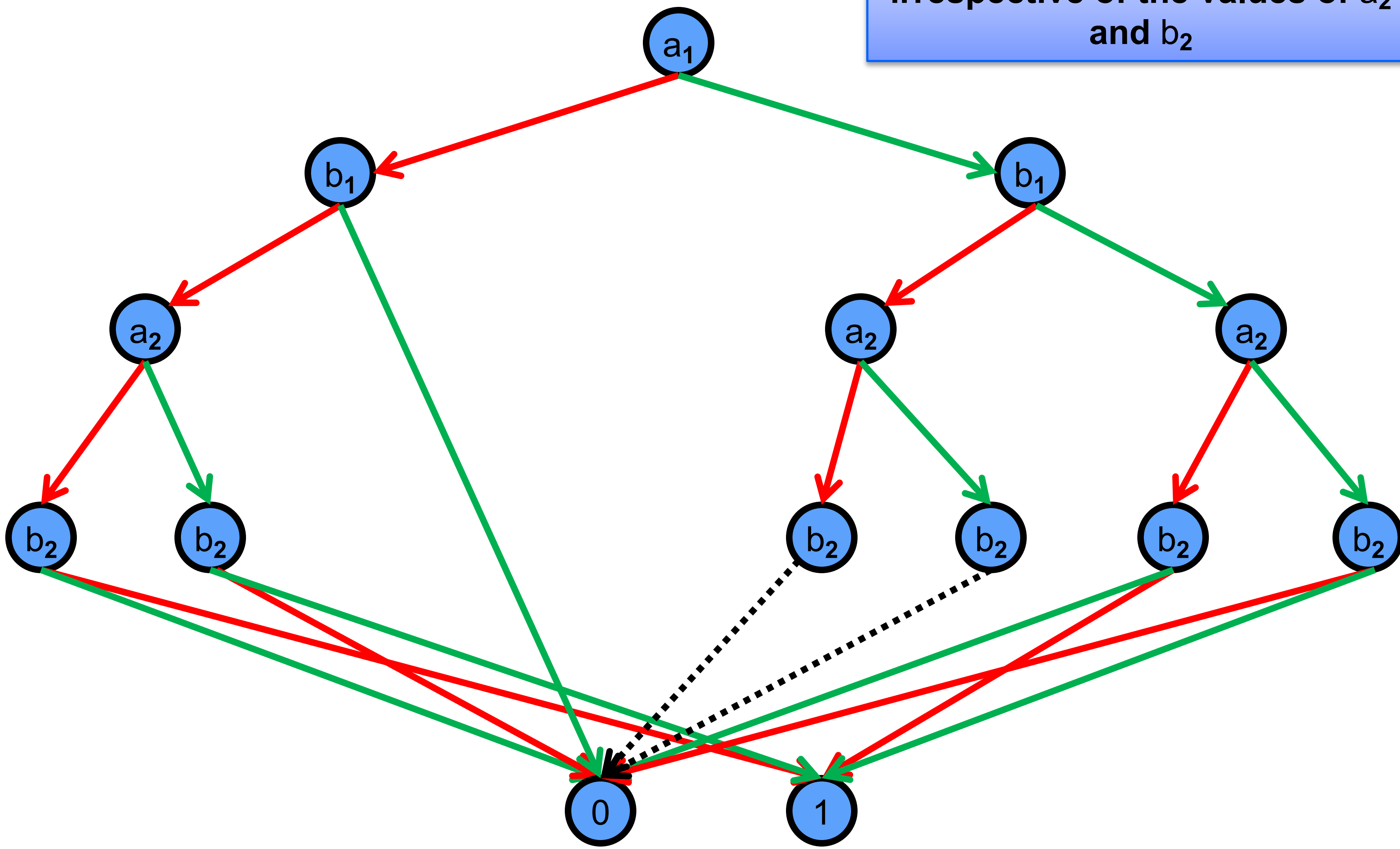
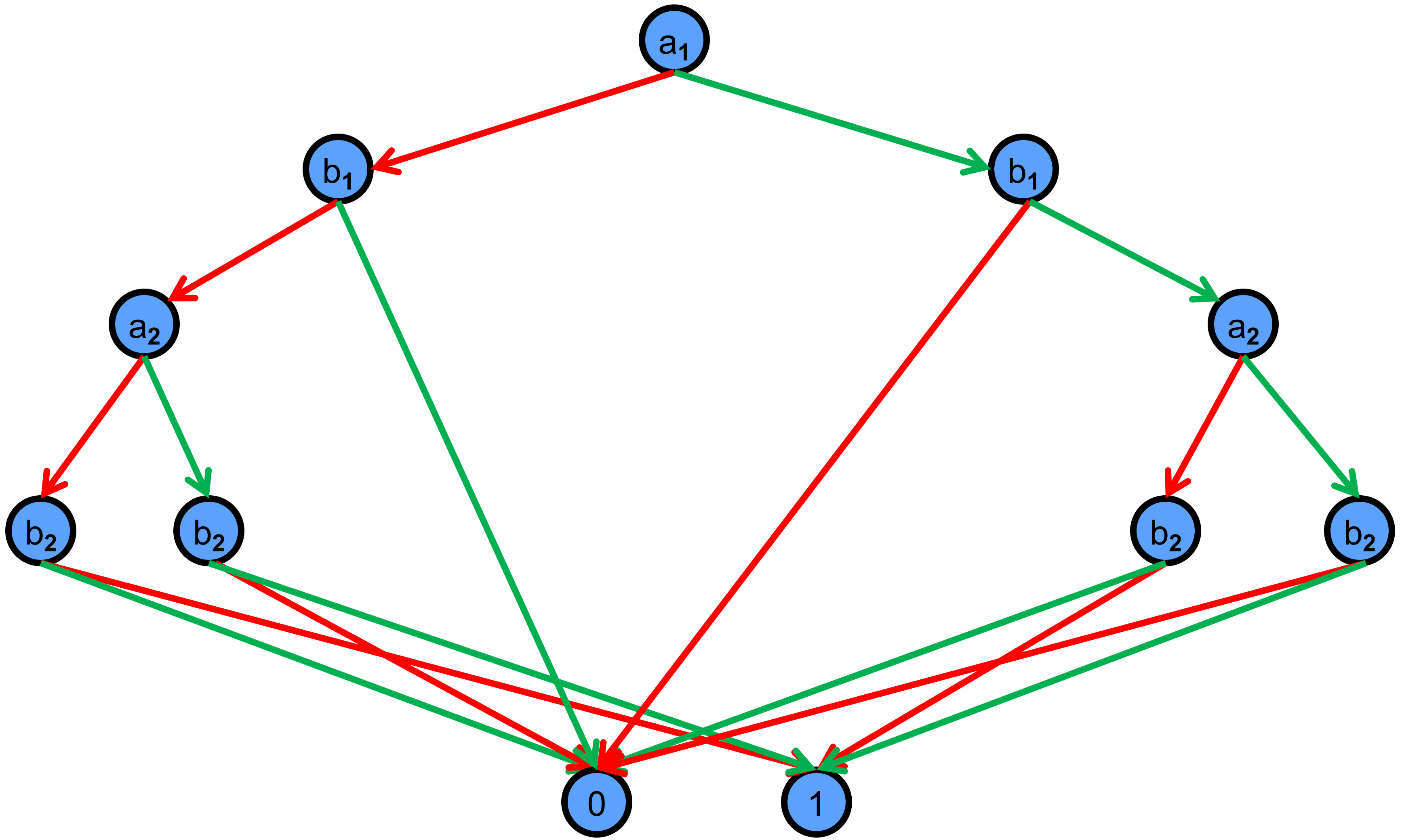
OBDT to ROBDD

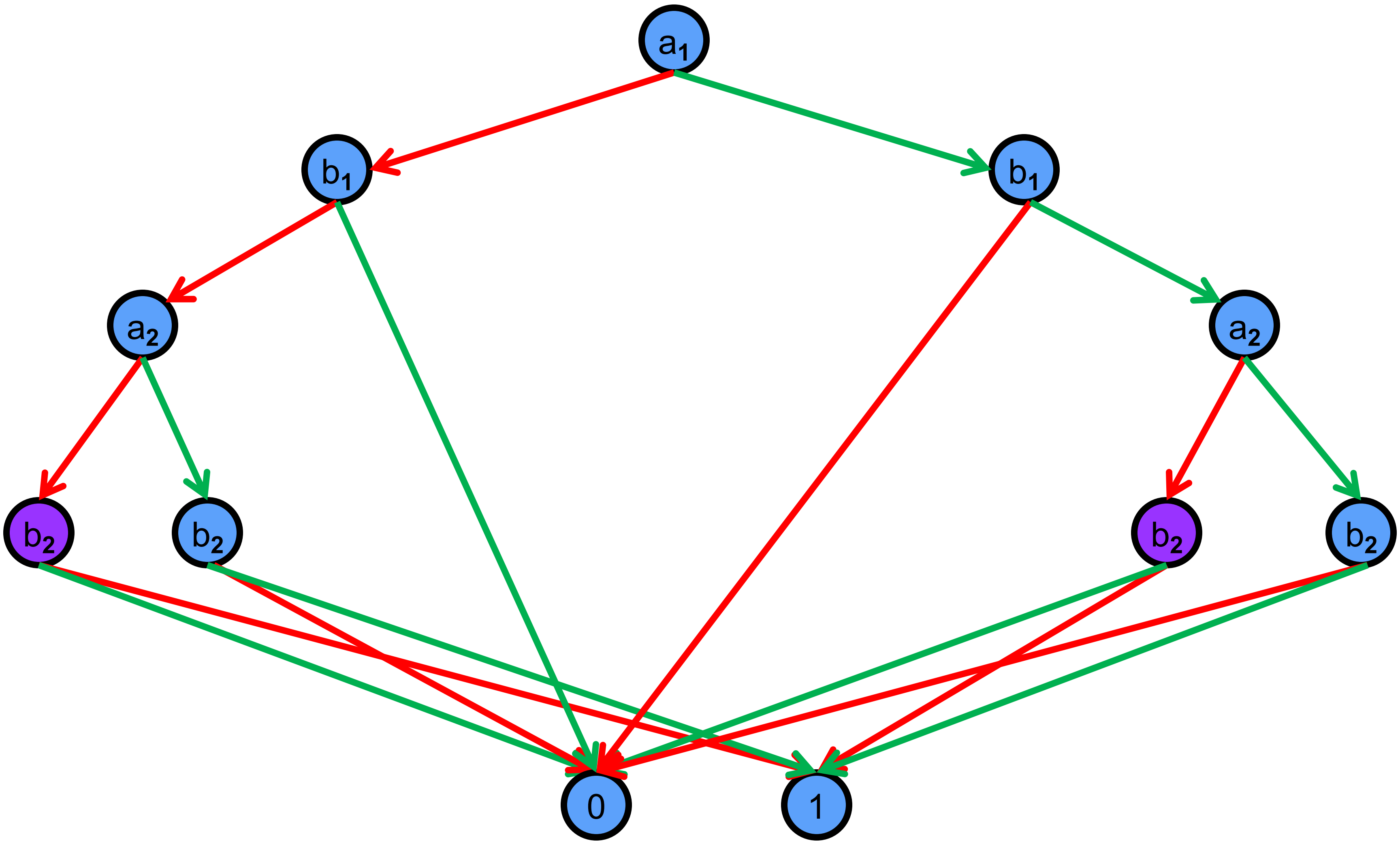
If a_1 = 0 and b_1 = 1 then f = 0 irrespective of the values of a_2 and b_2



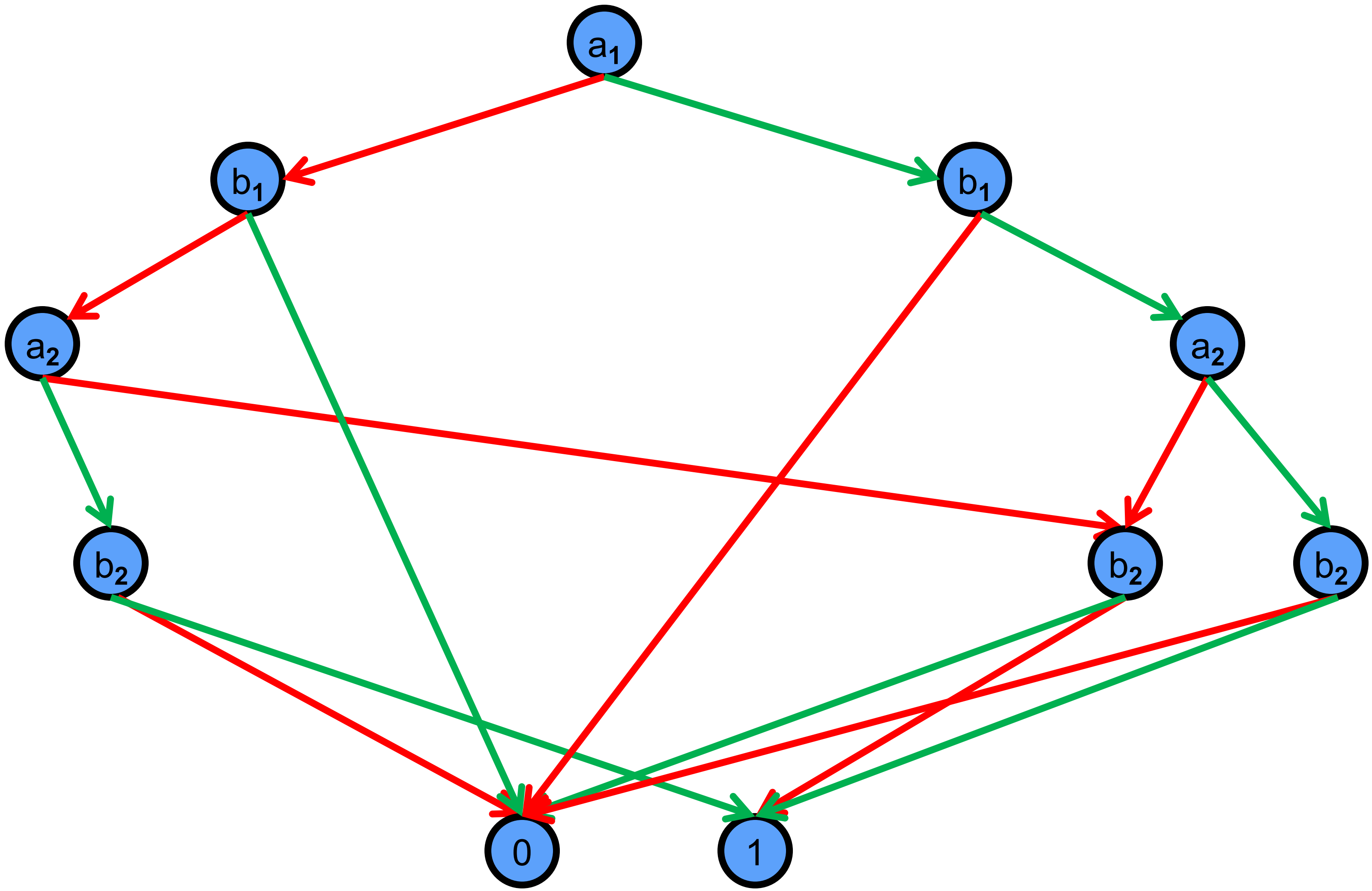
OBDT to ROBDD



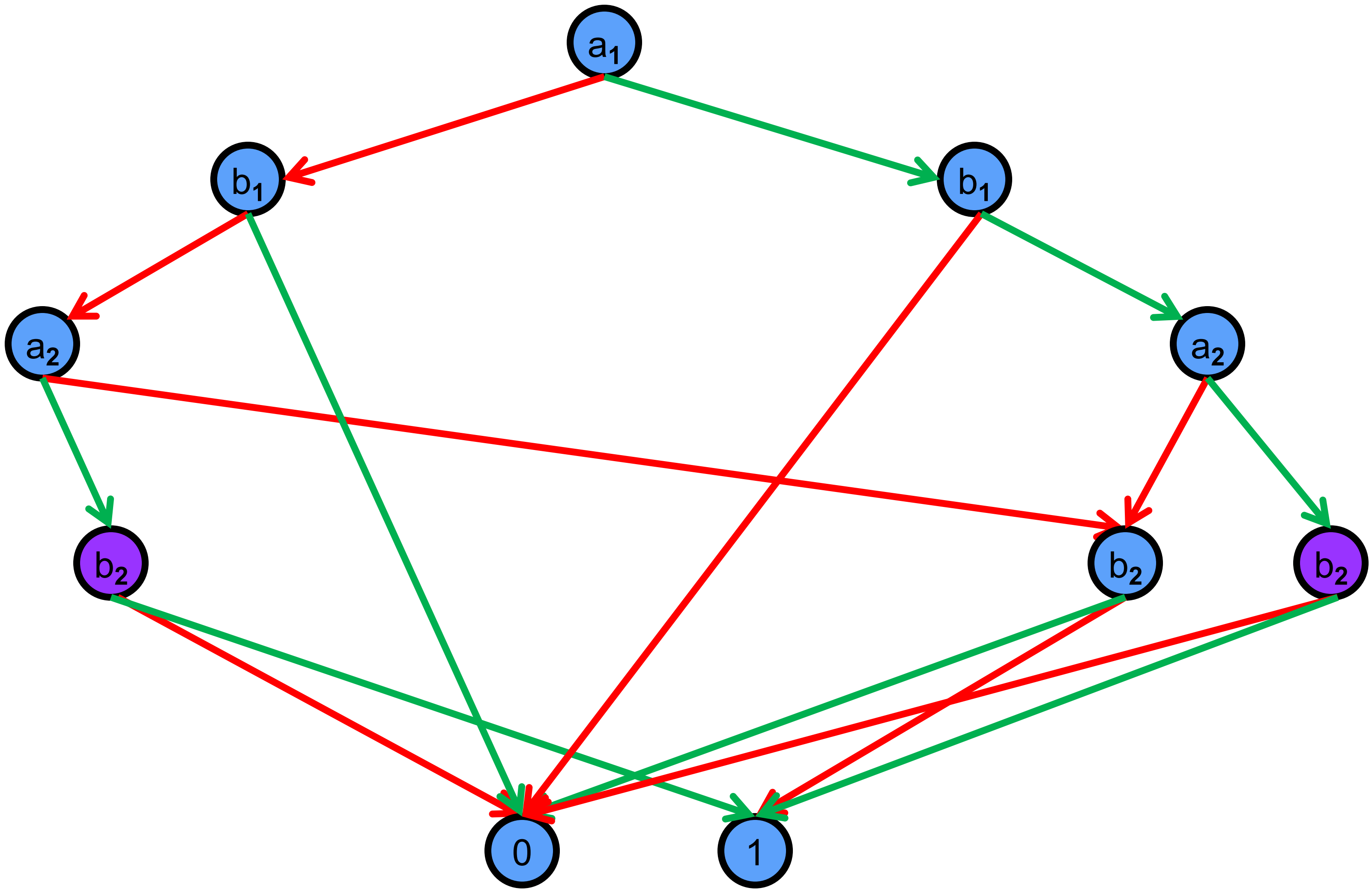
OBDT to ROBDD



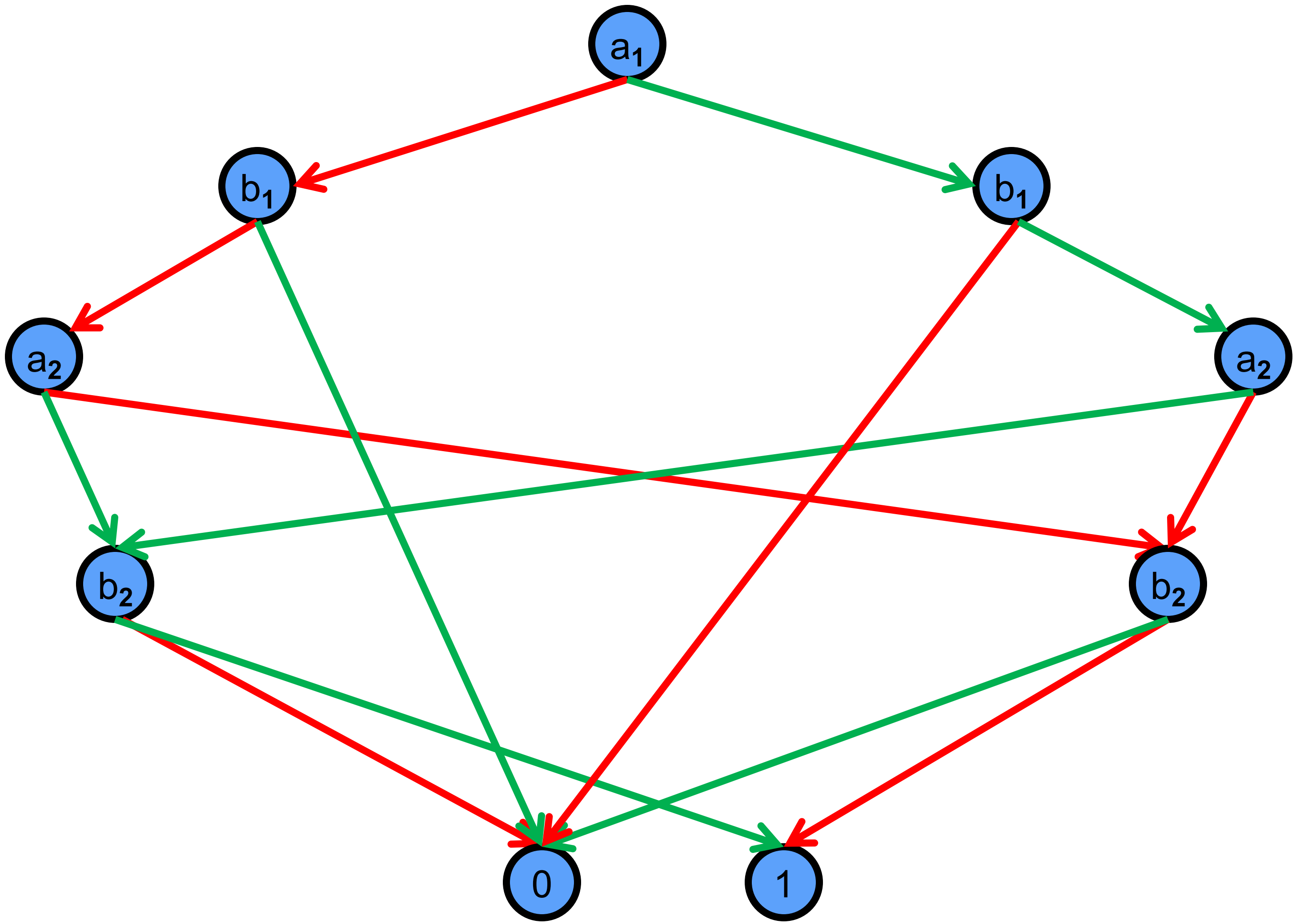
OBDT to ROBDD



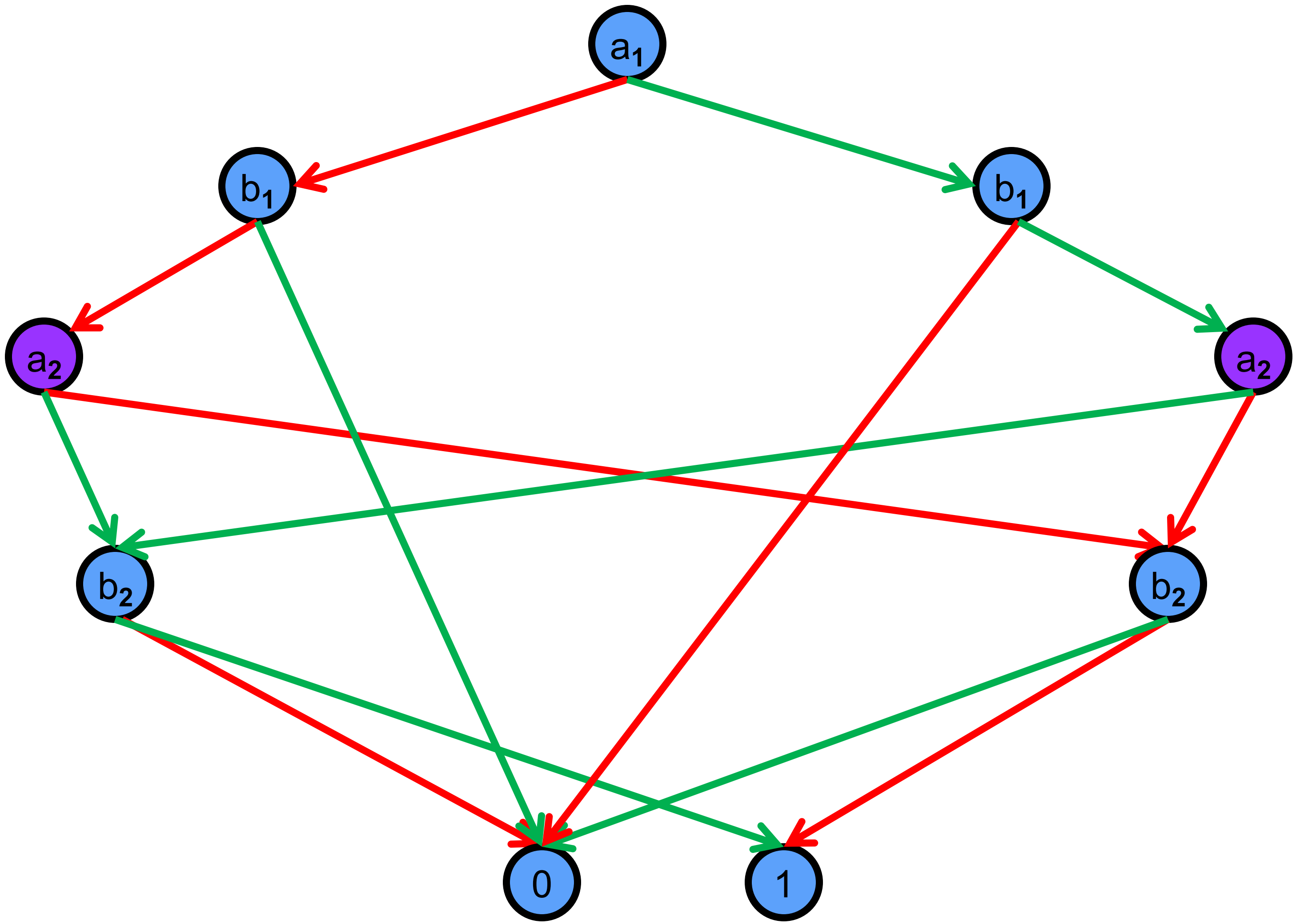
OBDT to ROBDD



OBDT to ROBDD

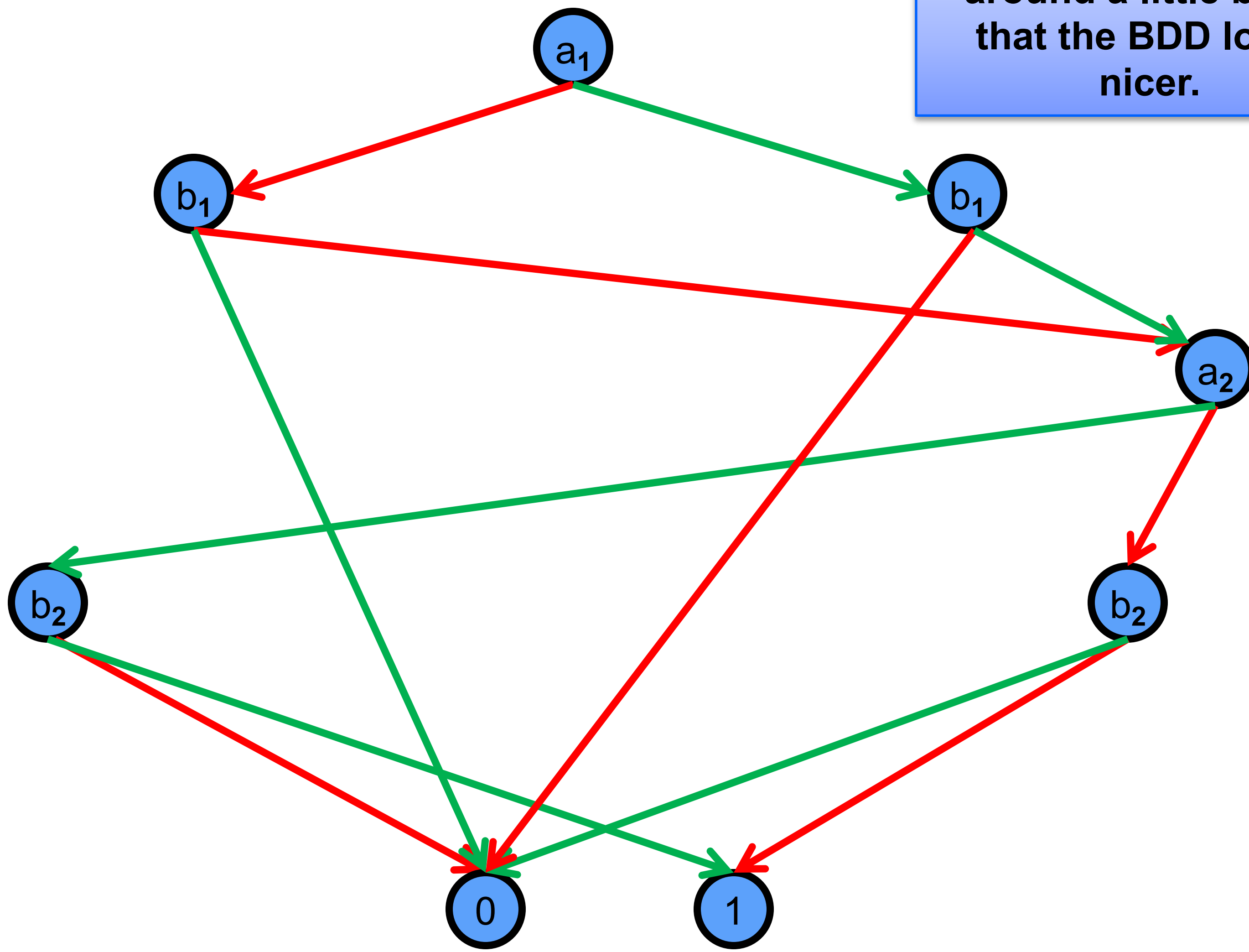


OBDT to ROBDD

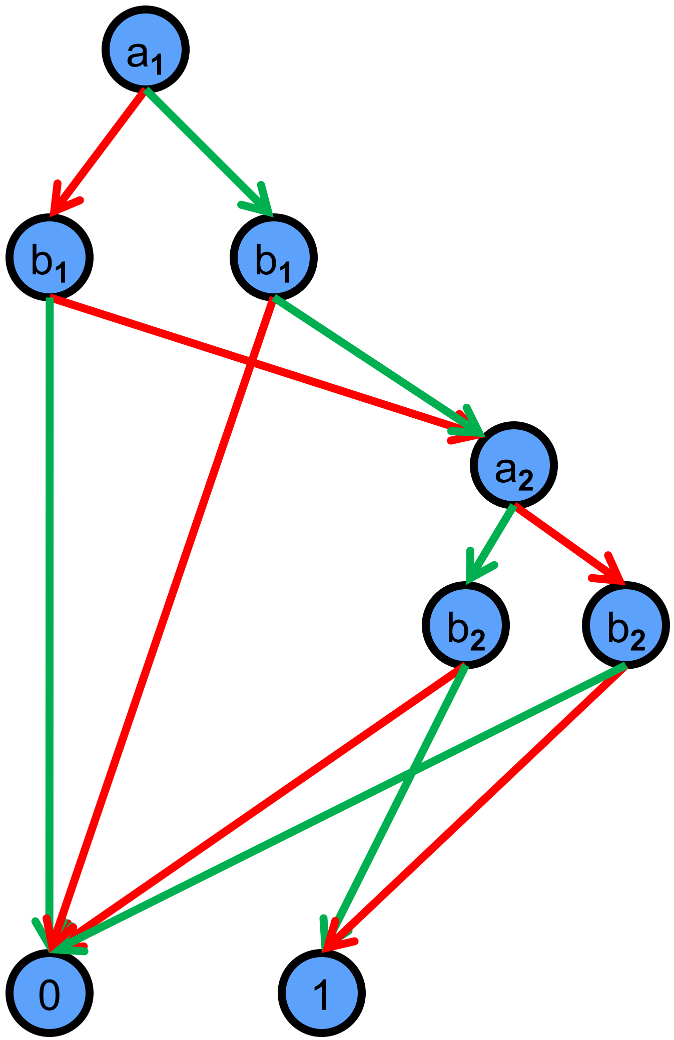


OBDT to ROBDD

Let's move things around a little bit so that the BDD looks nicer.



OBDT to ROBDD



Bryant gave a linear-time algorithm (called Reduce) to convert OBDT to ROBDD.

In practice, BDD packages don't use Reduce directly. They apply the two reductions on-the-fly as new BDDs are constructed from existing ones. Why?

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ROBDD (a.k.a. BDD) Summary

BDDs are canonical representations of Boolean formulas

- $f_1 = f_2 \Leftrightarrow$?

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- $f_1 = f_2 \Leftrightarrow$ BDD(f_1) and BDD(f_2) are isomorphic
- f is unsatisfiable $\Leftrightarrow$?

ROBDD (a.k.a. BDD) Summary

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- $f_1 = f_2 \Leftrightarrow$ BDD(f_1) and BDD(f_2) are isomorphic
- f is unsatisfiable $\Leftrightarrow$ BDD(f) is the leaf node "0"
- f is valid $\Leftrightarrow$?

ROBDD (a.k.a. BDD) Summary

BDDs are canonical representations of Boolean formulas

- $f_1 = f_2 \Leftrightarrow$ BDD(f_1) and BDD(f_2) are isomorphic
- f is unsatisfiable $\Leftrightarrow$ BDD(f) is the leaf node “0”
- f is valid $\Leftrightarrow$ BDD(f) is the leaf node “1”
- BDD packages do these operations in constant time

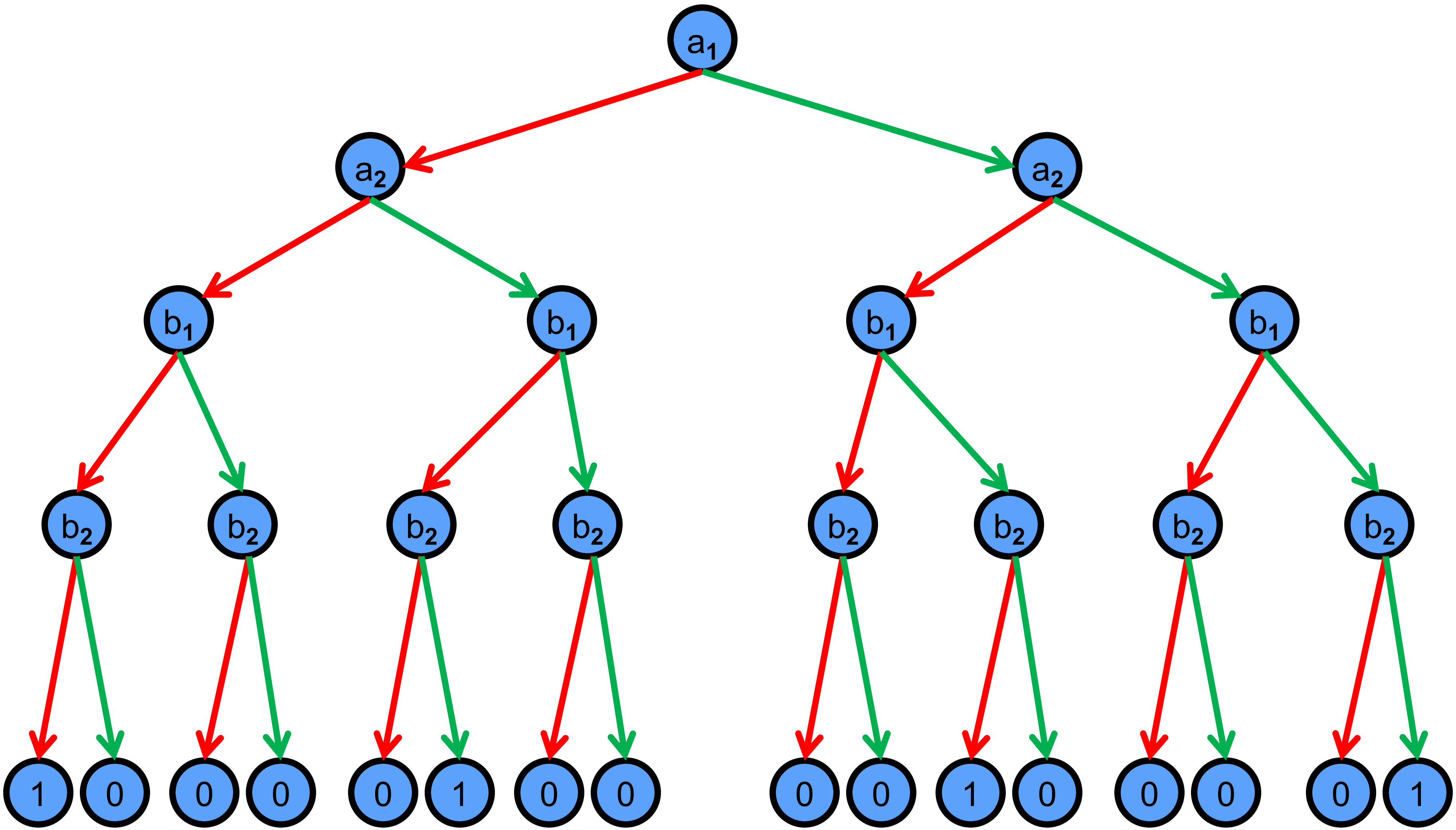
Logical operations can be performed efficiently on BDDs

- Polynomial in argument size

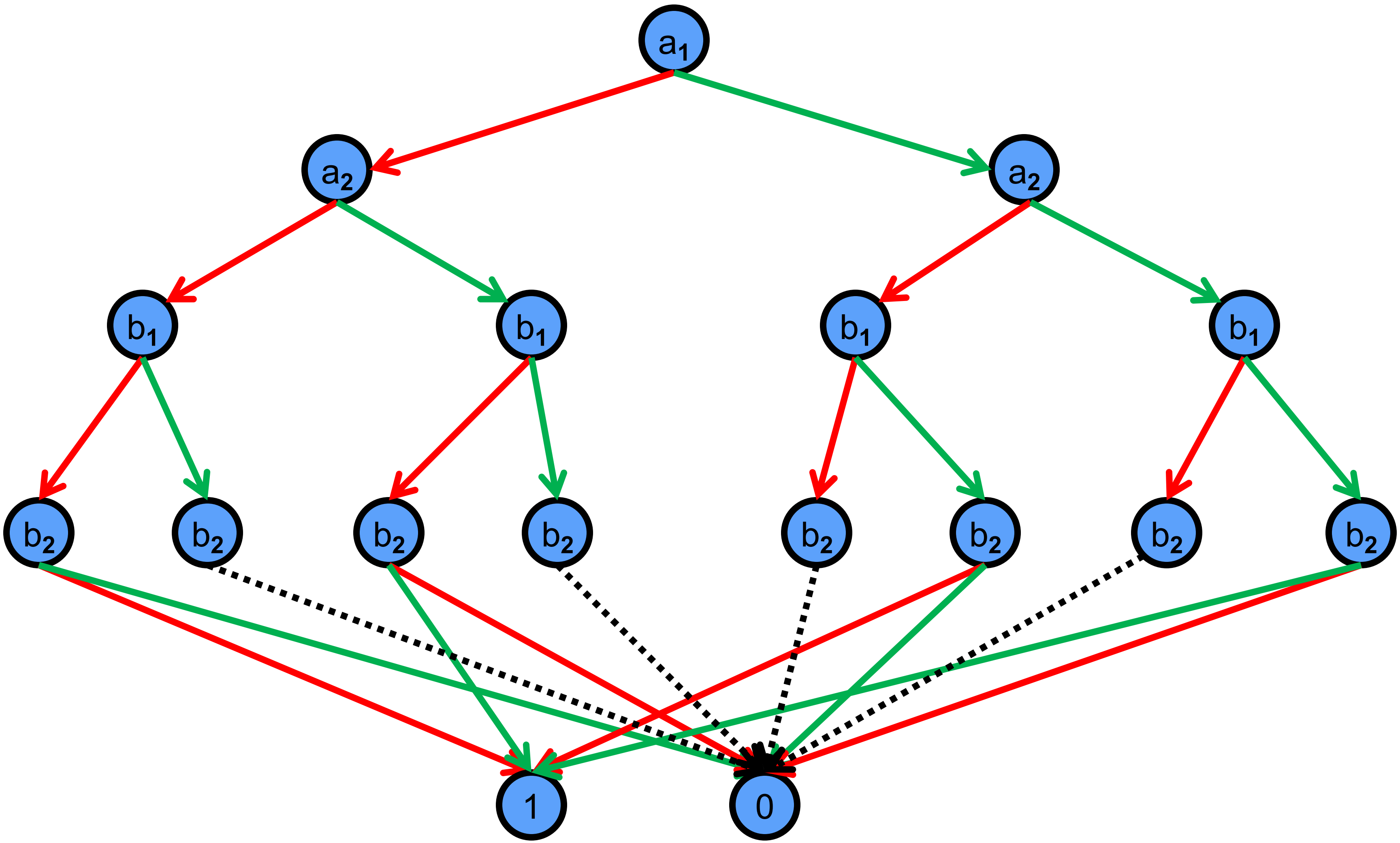
BDD size depends critically on the variable ordering

- Some formulas have exponentially large sizes for all ordering
- Others are polynomial for some ordering and exponential for others

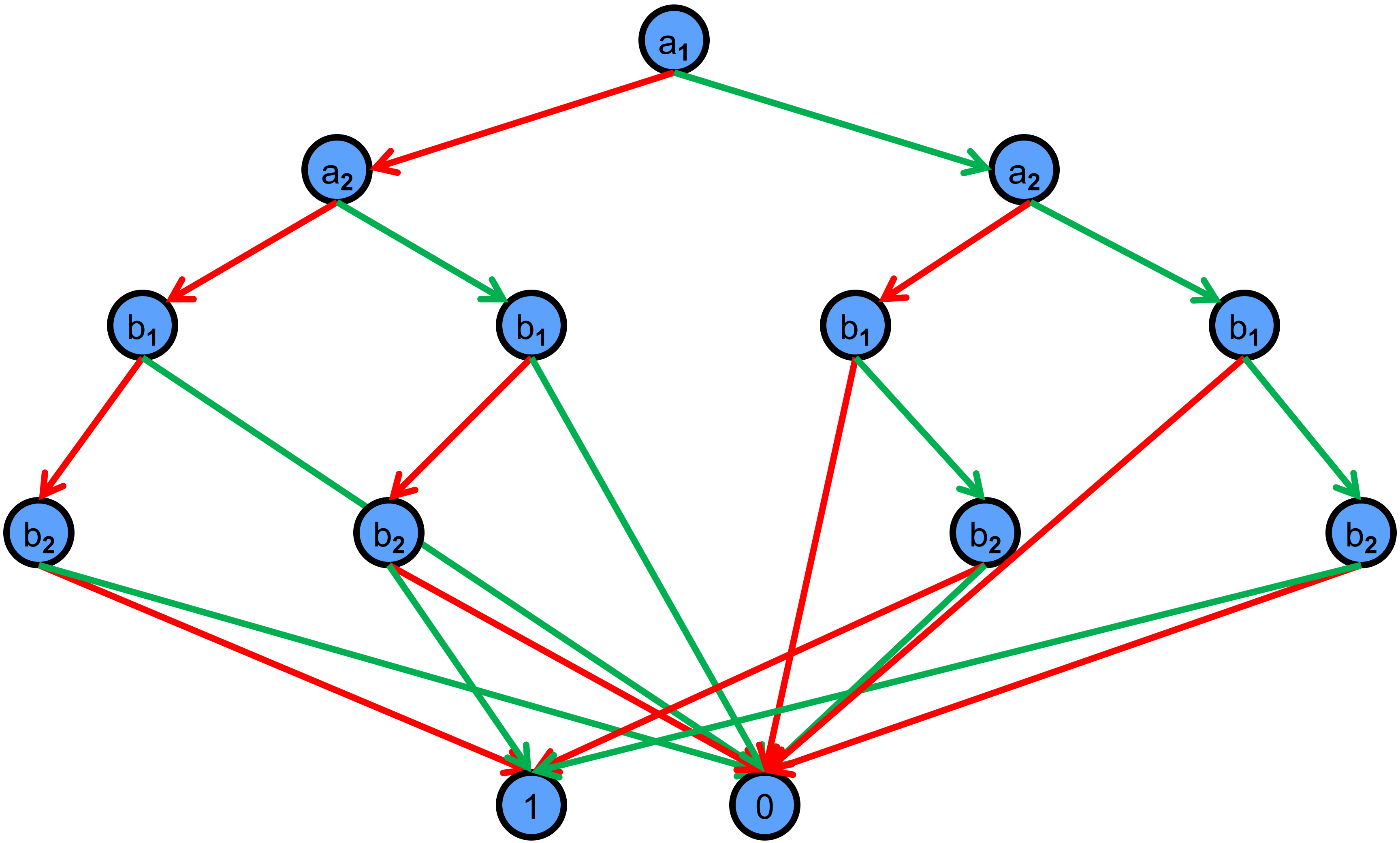
ROBDD and variable ordering



ROBDD and variable ordering



ROBDD and variable ordering



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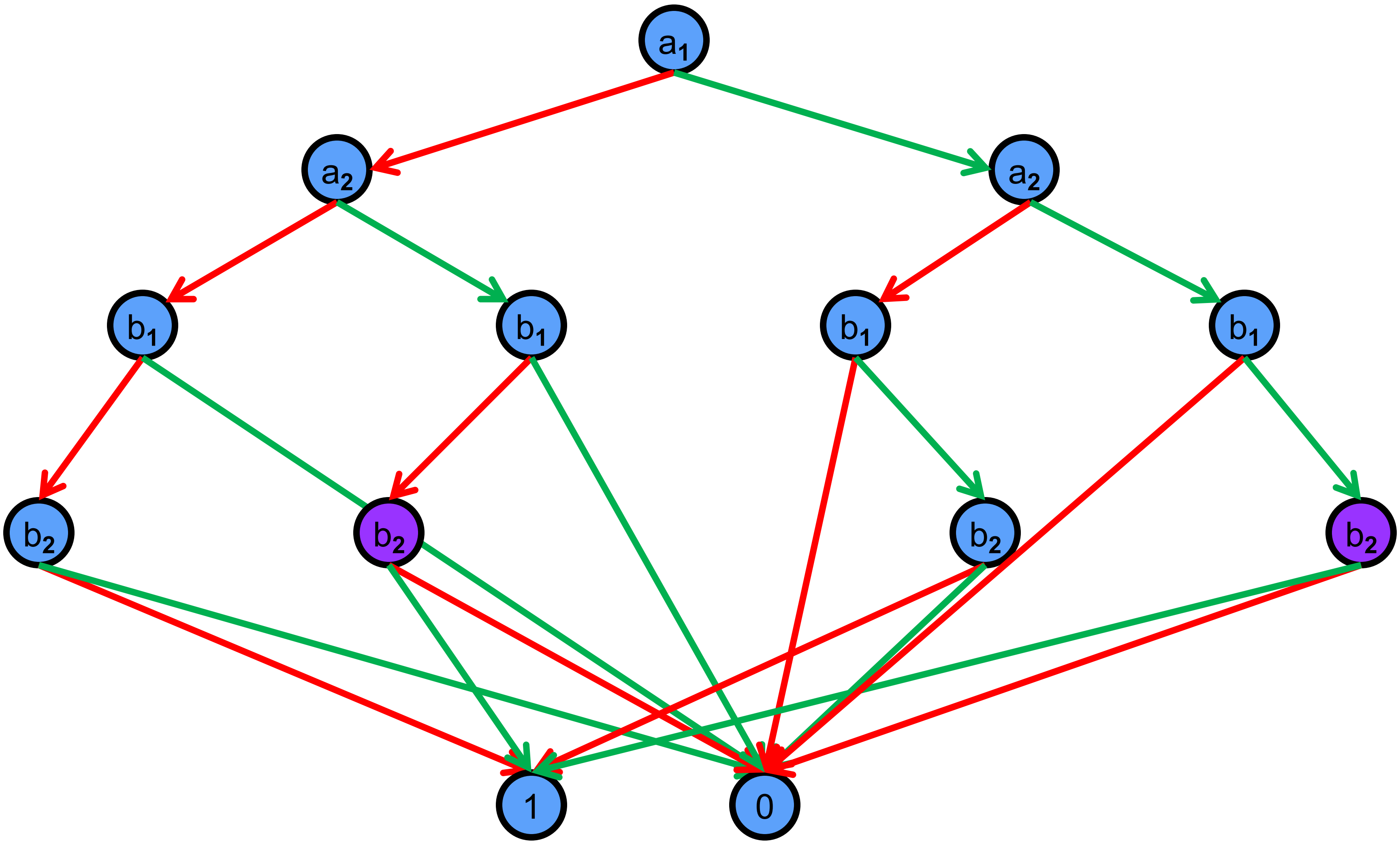
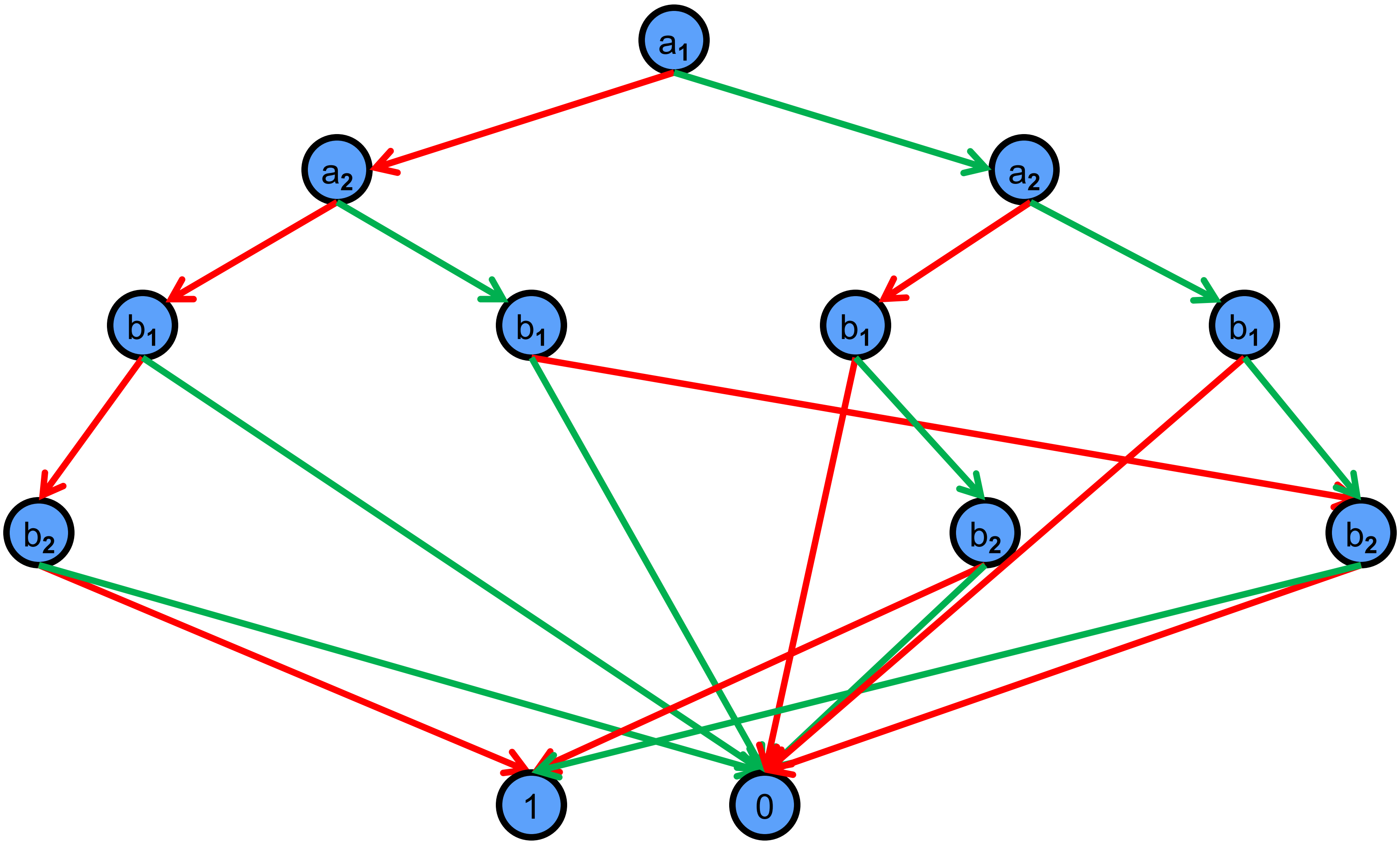
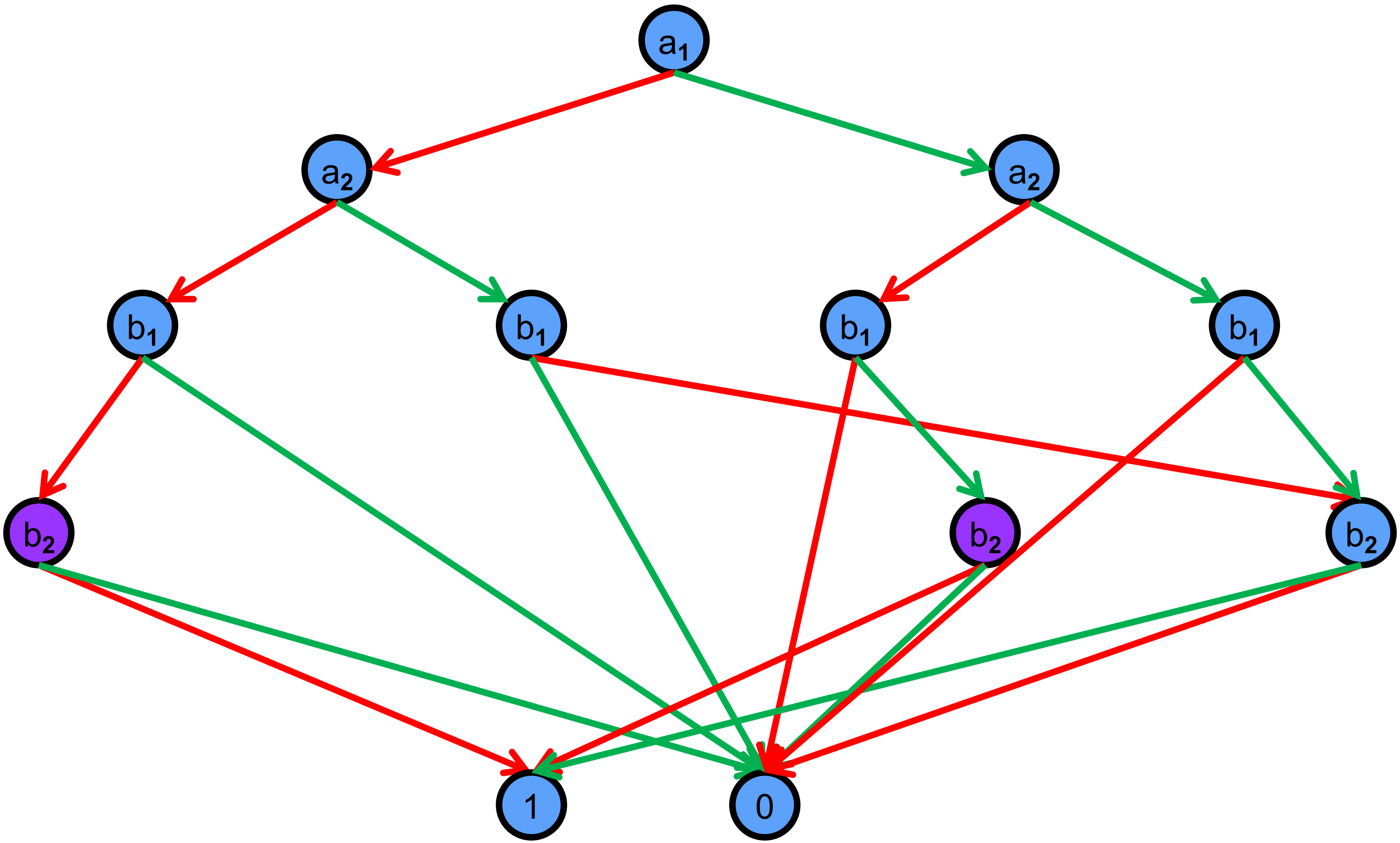
ROBDD and variable ordering



ROBDD and variable ordering

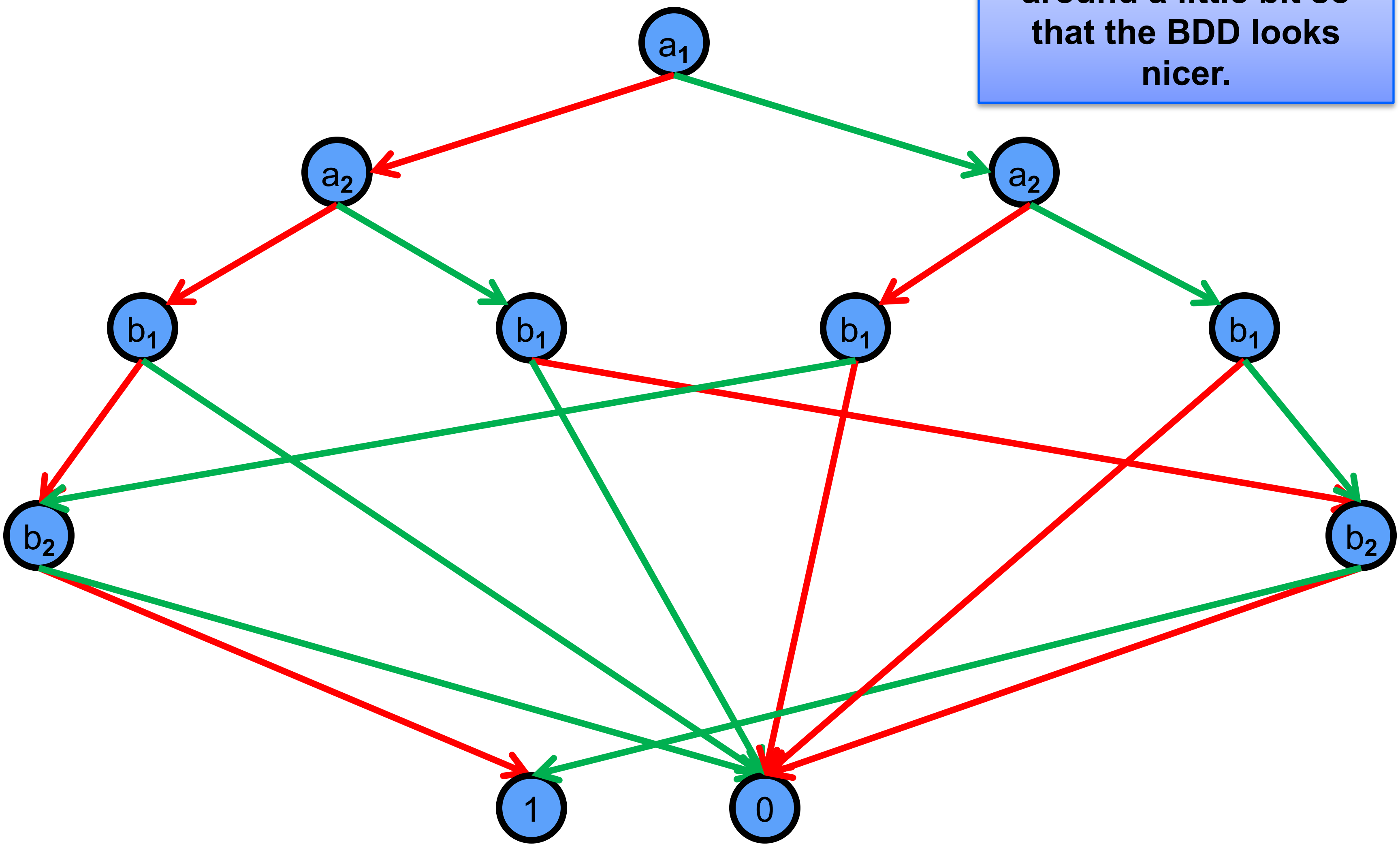


ROBDD and variable ordering



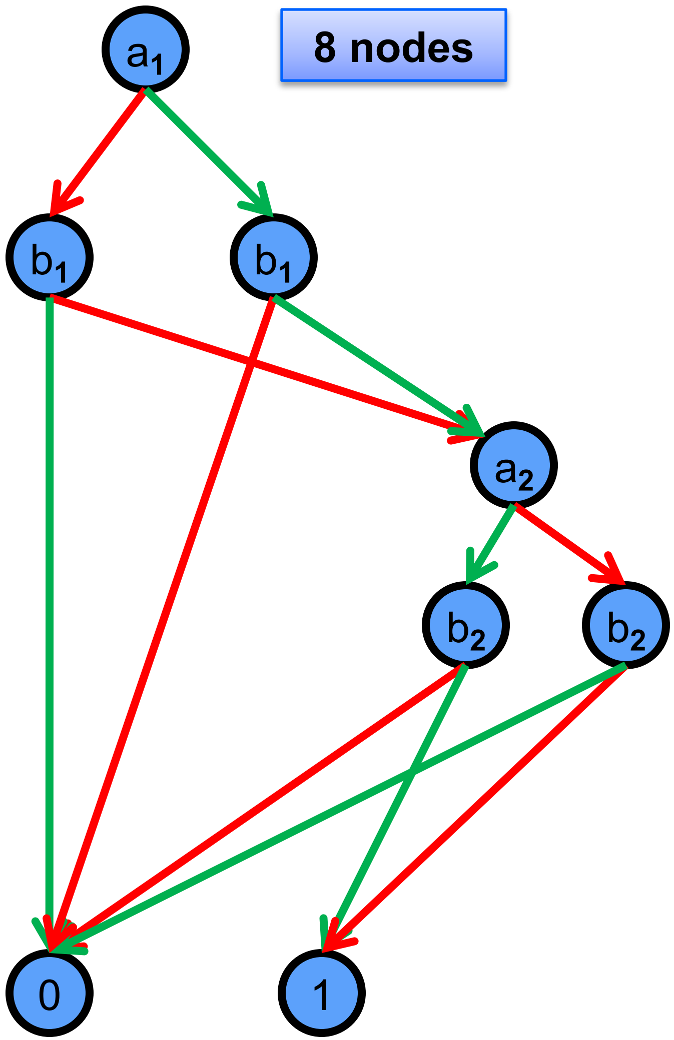
ROBDD and variable ordering

Let's move things around a little bit so that the BDD looks nicer.



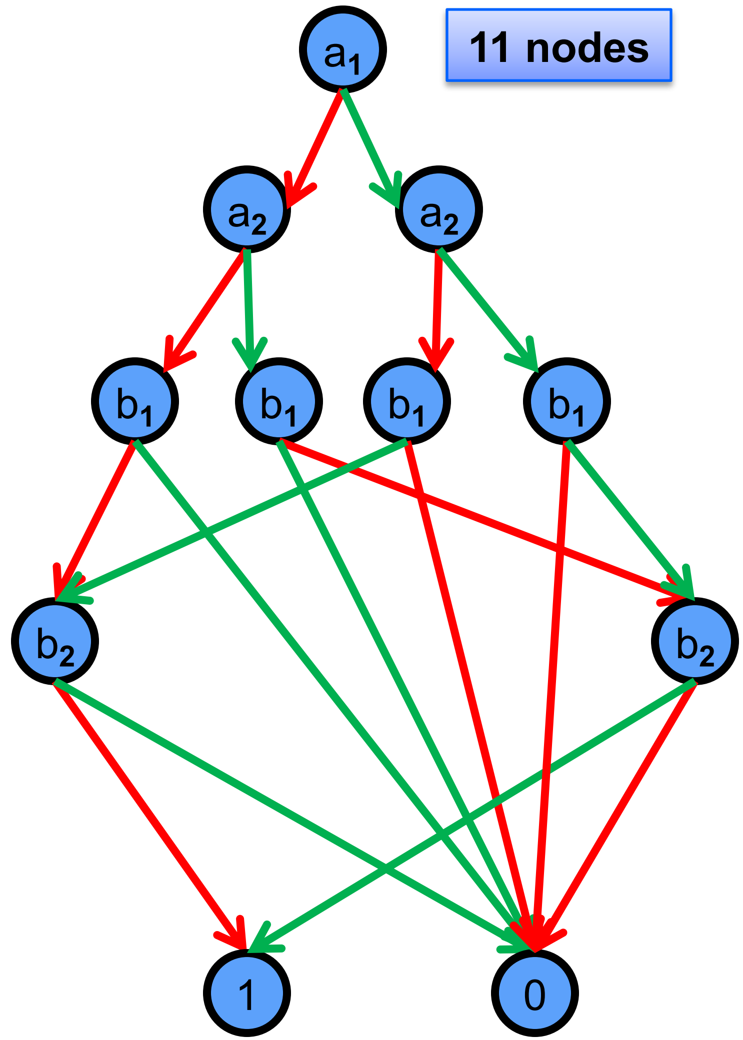
ROBDD and variable ordering

8 nodes



$a_1 < b_1 < a_2 < b_2$

11 nodes



$a_1 < a_2 < b_1 < b_2$

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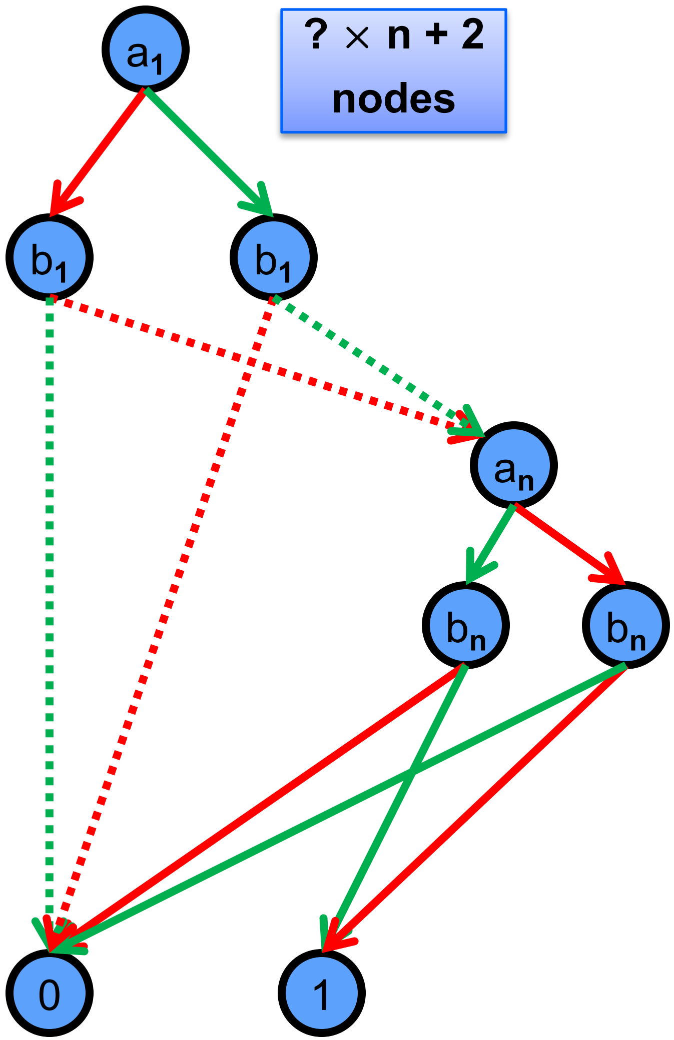
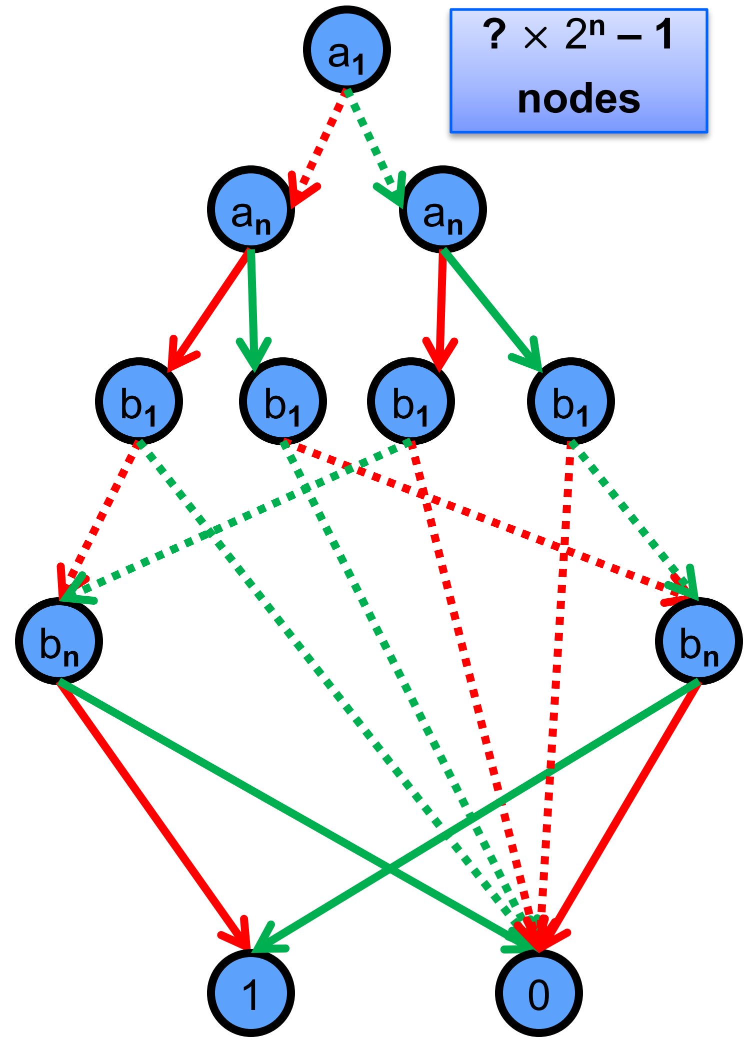
ROBDD and variable ordering



? × n + 2 nodes

$a_1 < b_1 < \ldots < a_n < b_n$



? × 2^n – 1 nodes

$a_1 < \ldots < a_n < b_1 < \ldots < b_n$

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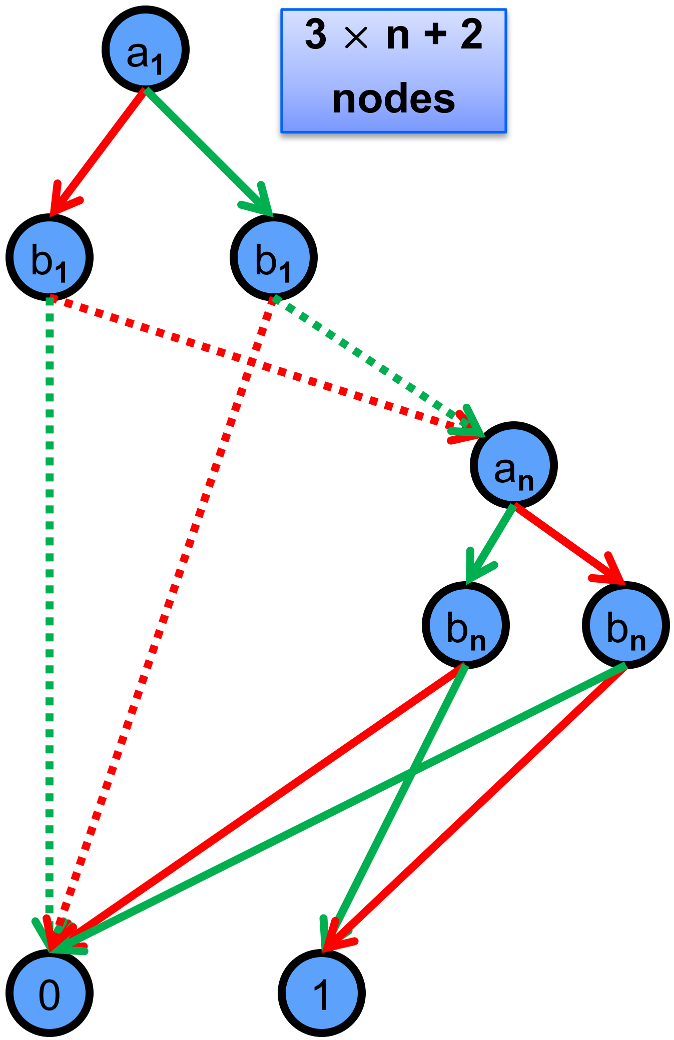
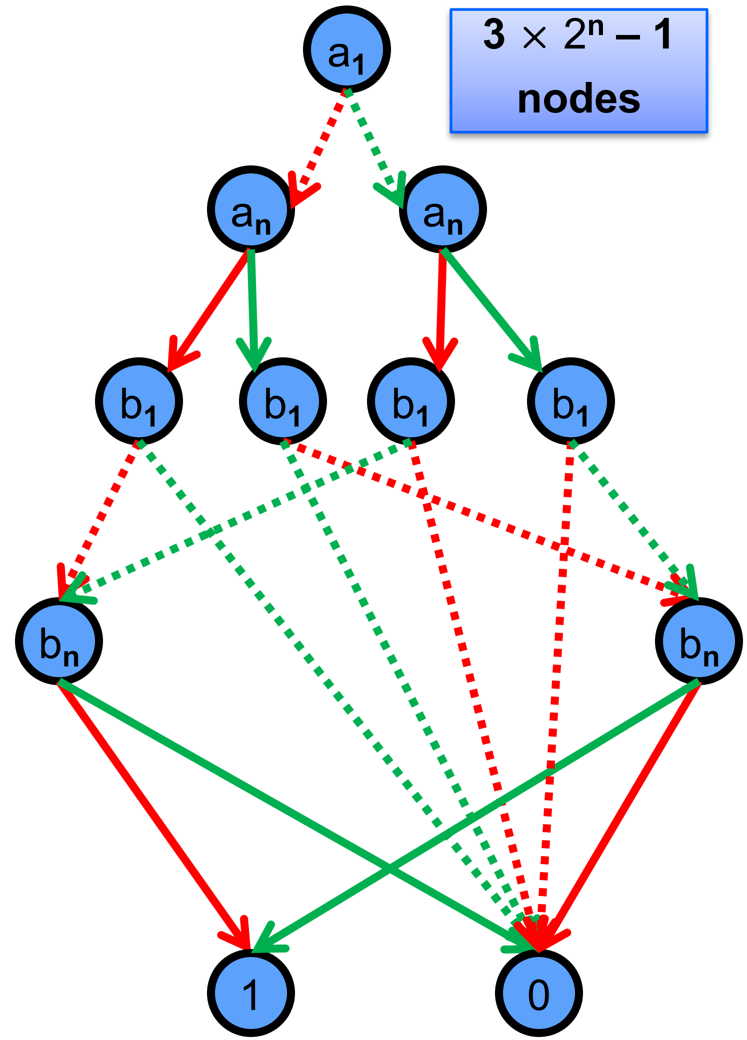
ROBDD and variable ordering

$3 \times n + 2$ nodes



$a_1 < b_1 < \ldots < a_n < b_n$

$3 \times 2^n - 1$ nodes



$a_1 < \ldots < a_n < b_1 < \ldots < b_n$

BDD Operations

True ¬ BDD(TRUE)

False¬ BDD(FALSE)

Var ¬ v $\mapsto$ BDD(v)

Not ¬ BDD(f) $\mapsto$ BDD(¬f)

And ¬ BDD(f_1) $\times$ BDD(f_2) $\mapsto$ BDD($f_1 \wedge f_2$)

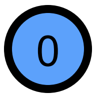
Or ¬ BDD(f_1) $\times$ BDD(f_2) $\mapsto$ BDD($f_1 \vee f_2$)

Exists ¬ BDD(f) $\times$ v $\mapsto$ BDD($\exists$ v. f)

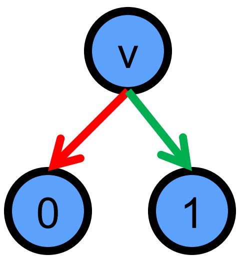
Basic BDD Operations

True

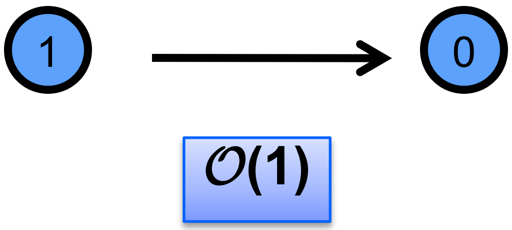
False

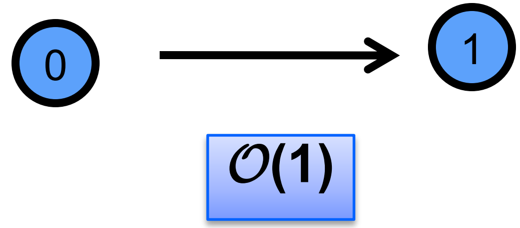


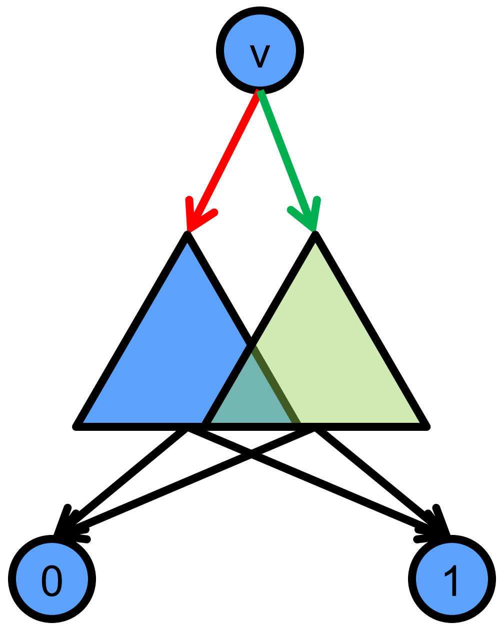
Var(v)



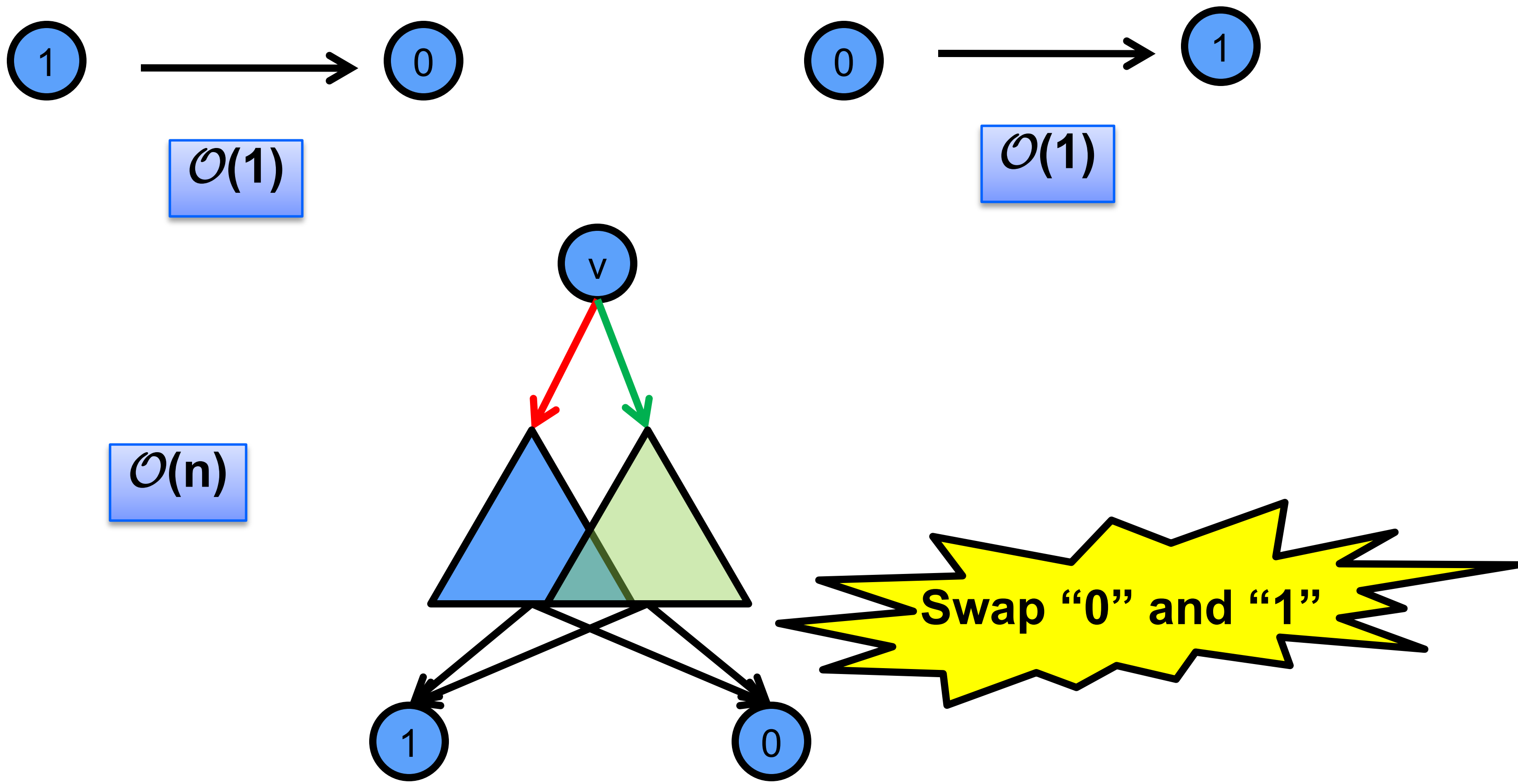
BDD Operations: Not



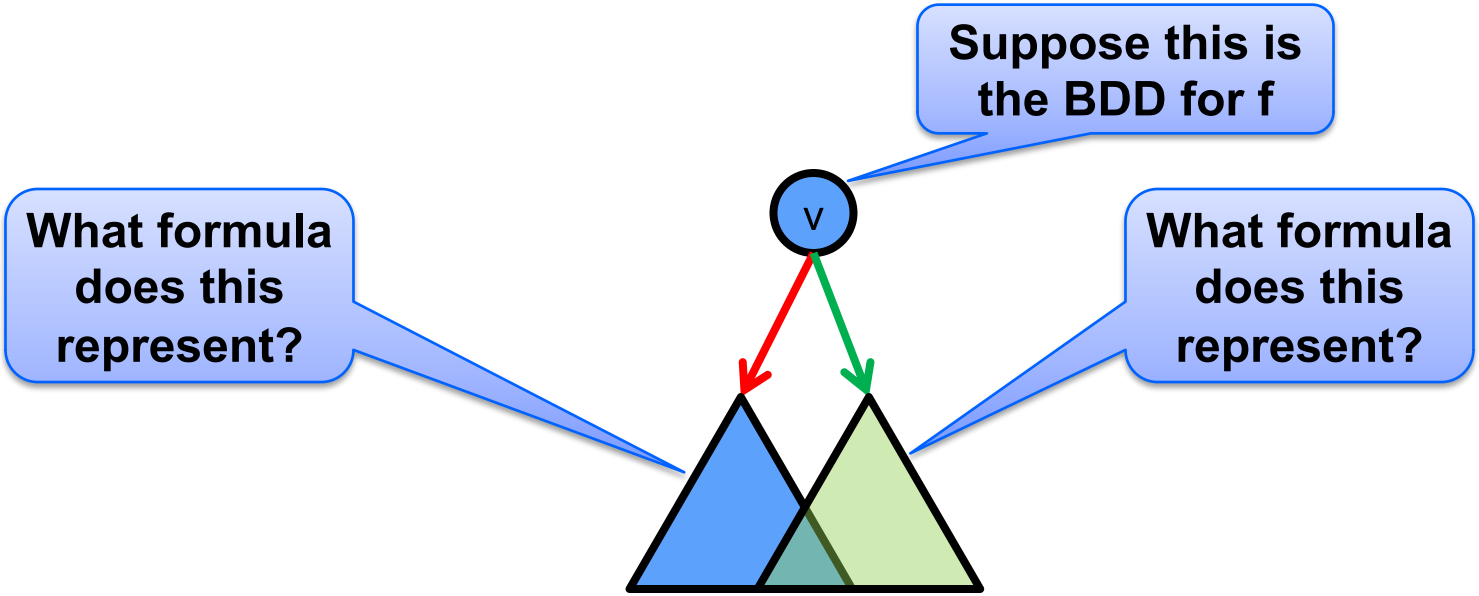




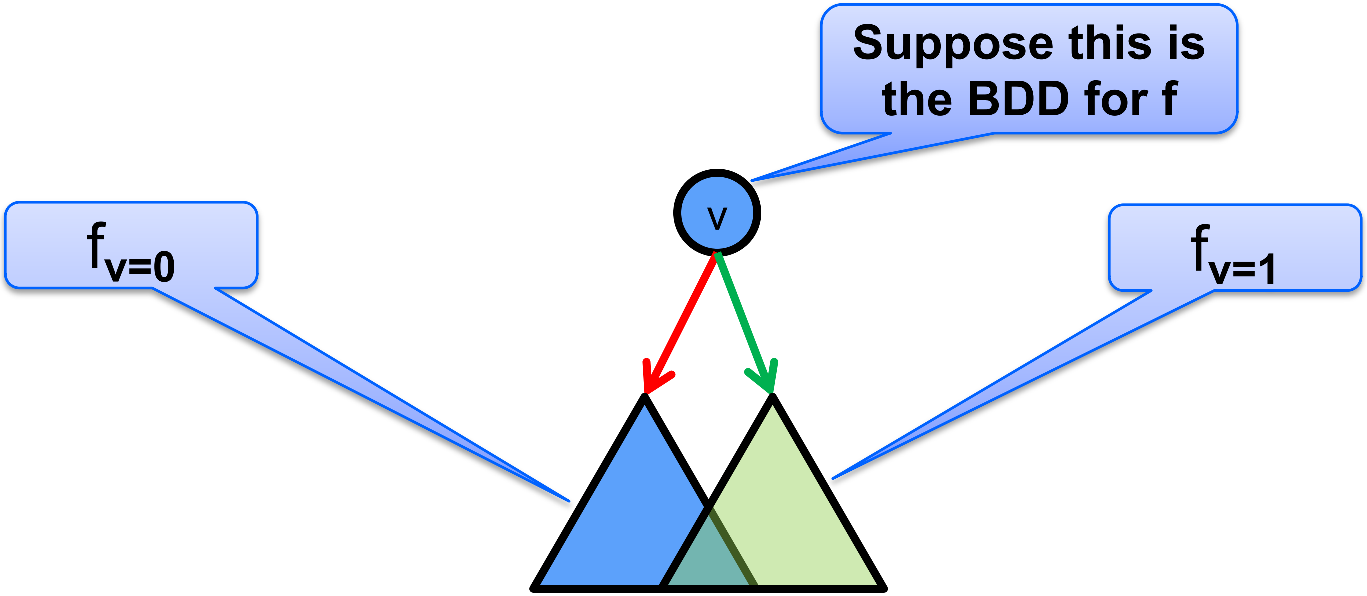
BDD Operations: Not



BDD Operations: And



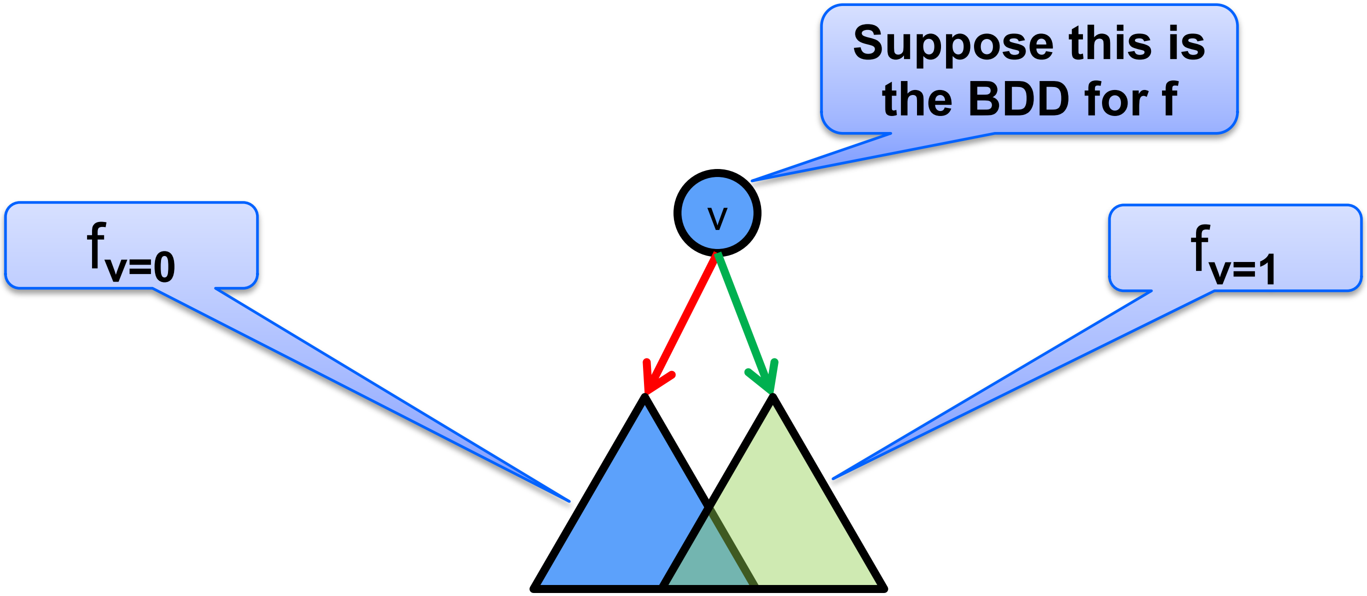
BDD Operations: And



$f_{v=0}$ and $f_{v=1}$ are known as the co-factors of f w.r.t. v

$f = (X \wedge f_{v=0}) \vee (Y \wedge f_{v=1})$

BDD Operations: And



$f_{v=0}$ and $f_{v=1}$ are known as the co-factors of f w.r.t. v

$f = (\neg v \wedge f_{v=0}) \vee (v \wedge f_{v=1})$

BDD Operations: And (Simple Cases)

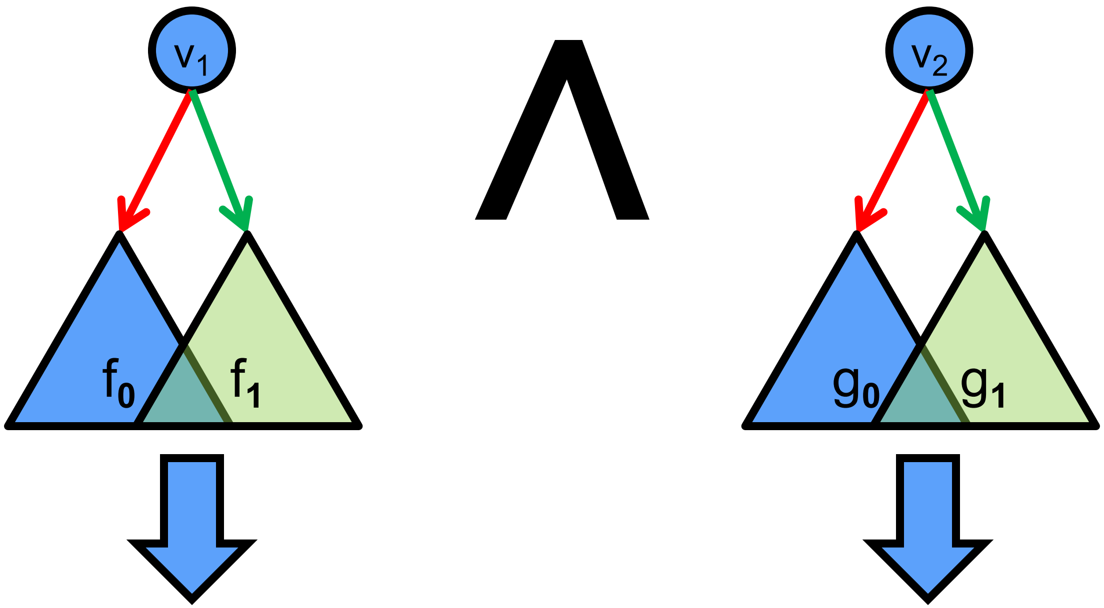
And (f, 0) = 0

And (f, 1) = f

And (1 ,f) = f

And (0 ,f) = 0

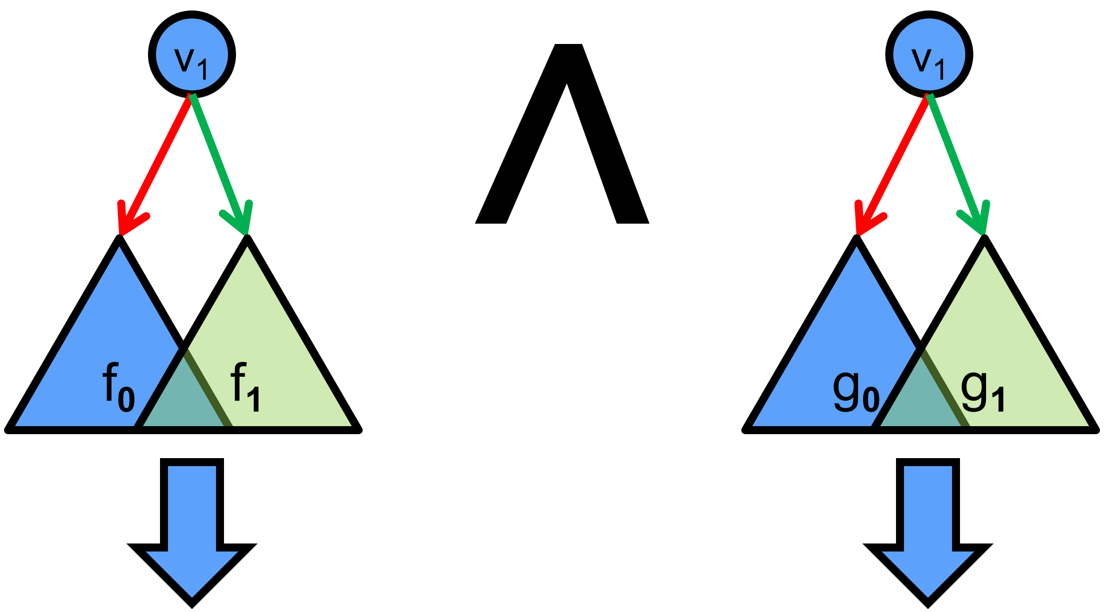
BDD Operations: And (Complex Case)



$(\neg v_1 \land f_0) \lor (v_1 \land f_1)$ $\land$ $(\neg v_2 \land g_0) \lor (v_2 \land g_1)$

BDD Operations: And (Complex Case 1)

$v_1 = v_2$



$(\neg v_1 \wedge f_0) \vee (v_1 \wedge f_1)$ $\wedge$ $(\neg v_1 \wedge g_0) \vee (v_1 \wedge g_1)$

BDD Operations: And (Complex Case 1)

$v_1 = v_2$

$(\neg v_1 \wedge X) \vee (v_1 \wedge Y)$

$(\neg v_1 \wedge f_0) \vee (v_1 \wedge f_1)$ $\wedge$ $(\neg v_1 \wedge g_0) \vee (v_1 \wedge g_1)$

BDD Operations: And (Complex Case 1)

$v_1 = v_2$

Compute recursively

$(\neg v_1 \wedge (f_0 \wedge g_0)) \vee (v_1 \wedge (f_1 \wedge g_1))$

↑

$(\neg v_1 \wedge f_0) \vee (v_1 \wedge f_1)$ $\wedge$ $(\neg v_1 \wedge g_0) \vee (v_1 \wedge g_1)$

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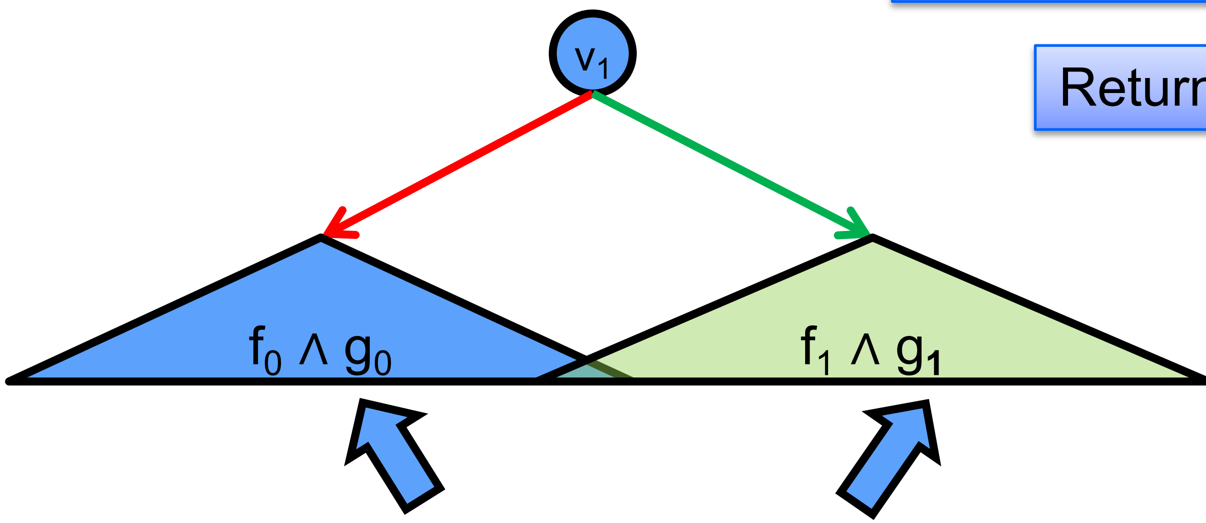
BDD Operations: And (Complex Case 1)

$v_1 = v_2$

What if $f_0 \wedge g_0 = f_1 \wedge g_1$?

Return $f_0 \wedge g_0$



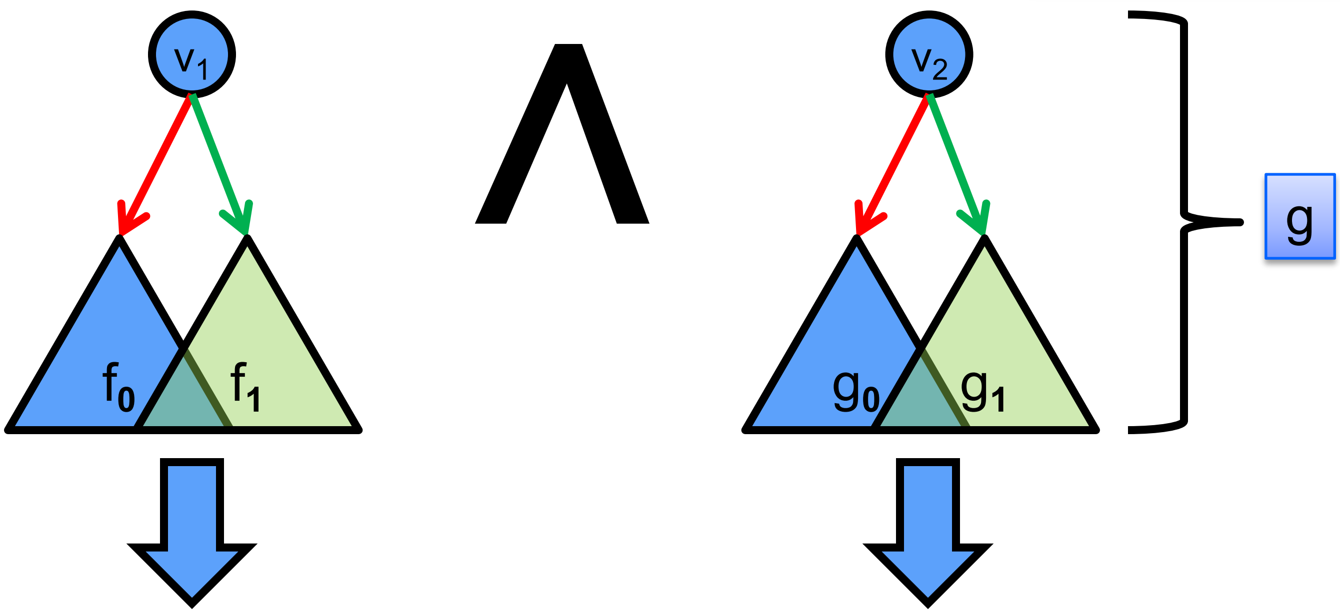
$(\neg v_1 \wedge (f_0 \wedge g_0)) \vee (v_1 \wedge (f_1 \wedge g_1))$

$(\neg v_1 \wedge f_0) \vee (v_1 \wedge f_1)$ $\wedge$ $(\neg v_1 \wedge g_0) \vee (v_1 \wedge g_1)$

BDD Operations: And (Complex Case 2)

$v_1 < v_2$

v_1 appears before v_2 in the variable ordering



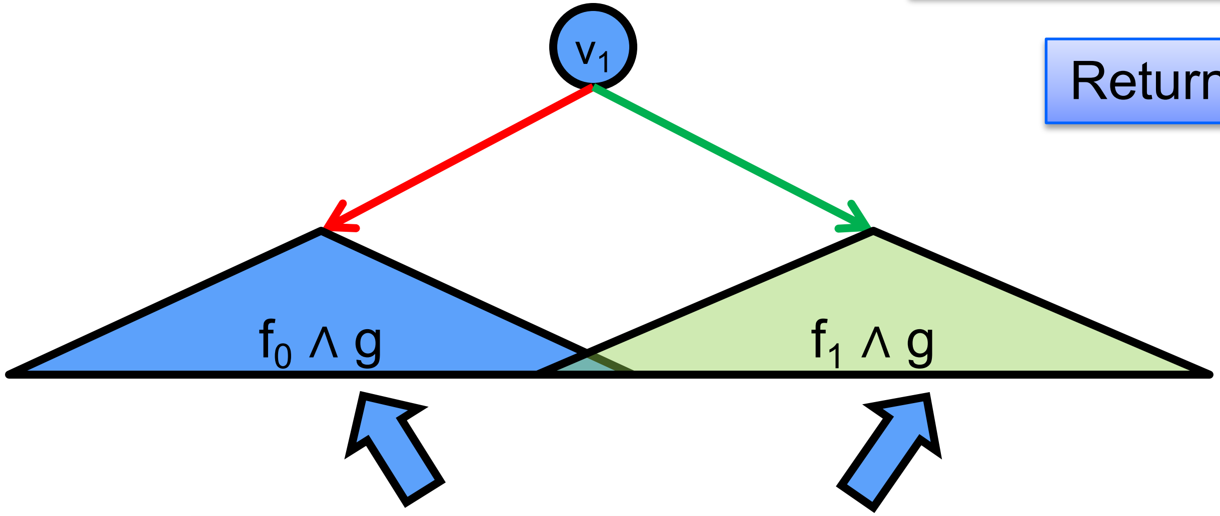
$(\neg v_1 \land f_0) \lor (v_1 \land f_1) \quad \land \quad (\neg v_2 \land g_0) \lor (v_2 \land g_1)$

BDD Operations: And (Complex Case 2)

$v_1 < v_2$

What if $f_0 \wedge g = f_1 \wedge g$?

Return $f_0 \wedge g$



$(\neg v_1 \wedge (f_0 \wedge g)) \vee (v_1 \wedge (f_1 \wedge g))$

$(\neg v_1 \wedge f_0) \vee (v_1 \wedge f_1)$ $\wedge$ g

$\mathcal{O}(n_1 \times n_2)$

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BDD Operations: And

```
BDD bddAnd (BDD f, BDD g)
  if (f == g || f == True) return g
  if (g == True) return f
  if (f == False || g == False) return False

  v = (var(f) < var(g)) ? var(f) ¬ var(g)
  f0 = (v == var(f)) ? low(f) ¬ f
  f1 = (v == var(f)) ? high(f) ¬ f

  g0 = (v == var(g)) ? low (g) ¬ g
  g1 = (v == var(g)) ? high (g) ¬ g

  T = bddAnd (f1, g1); E = bddAnd (f0, g0)
  if (T == E) return T

  return mkUnique (v, T, E)
```

returns unique BDD for `ite(v,T,E)`

BDD Operations: Or

Or(f,g)

=

Not (And (Not(f**),** Not(g**)))**

$\mathcal{O}(n_1 \times n_2)$

BDD Operations: Exists

Exists(“0”,v) = ?

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BDD Operations: Exists

Exists("0",v) = "0"

Exists("1",v) = ?

BDD Operations: Exists

$$\text{Exists}(\text{“0”}, v) = \text{“0”}$$

$$\text{Exists}(\text{“1”}, v) = \text{“1”}$$

$$\text{Exists}((\neg v \land f) \lor (v \land g), v) = ?$$

BDD Operations: Exists

$$\text{Exists}(``0", v) = ``0"$$

$$\text{Exists}(``1", v) = ``1"$$

$$\text{Exists}((\neg v \wedge f) \vee (v \wedge g)\ ,\ v) = \text{Or}(f,g)$$

$$\text{Exists}((\neg v' \wedge f) \vee (v' \wedge g)\ ,\ v) = ?$$

BDD Operations: Exists

$\mathcal{O}(n^2)$

$$\text{Exists}(\text{“0”},v) = \text{“0”}$$

$$\text{Exists}(\text{“1”},v) = \text{“1”}$$

$$\text{Exists}((\neg v \wedge f) \vee (v \wedge g), v) = \text{Or}(f,g)$$

$$\text{Exists}((\neg v' \wedge f) \vee (v' \wedge g), v) =$$

$$(\neg v' \wedge \text{Exists}(f,v)) \vee (v' \wedge \text{Exists}(g,v))$$

But f is SAT iff ∃ V. f is not “0”. So why doesn’t this imply P = NP?

Because the BDD size changes!

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BDD Applications

SAT is great if you are interested to know if a solution exists

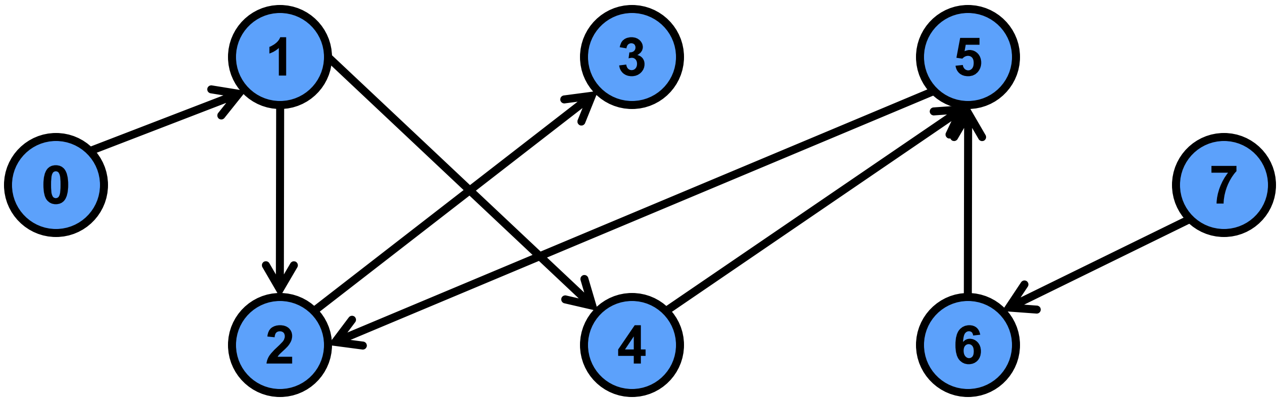
BDDs are great if you are interested in the set of all solutions

- How many solutions are there?
- How do you do this on a BDD?

BDDs are great for computing a fixed points

- Set of nodes reachable from a given node in a graph

Graph Reachability

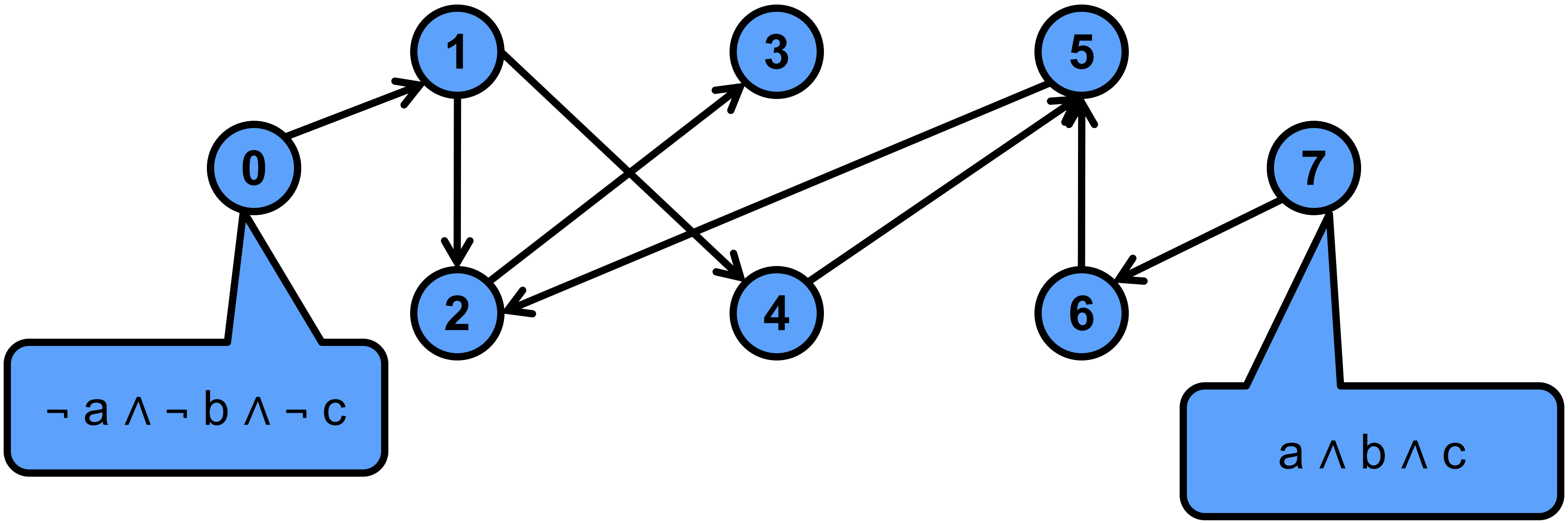


Which nodes are reachable from “7”?

{2,3,5,6,7}

But what if the graph has trillions of nodes?

Graph Reachability



Use three Boolean variables (a,b,c) to encode each node?

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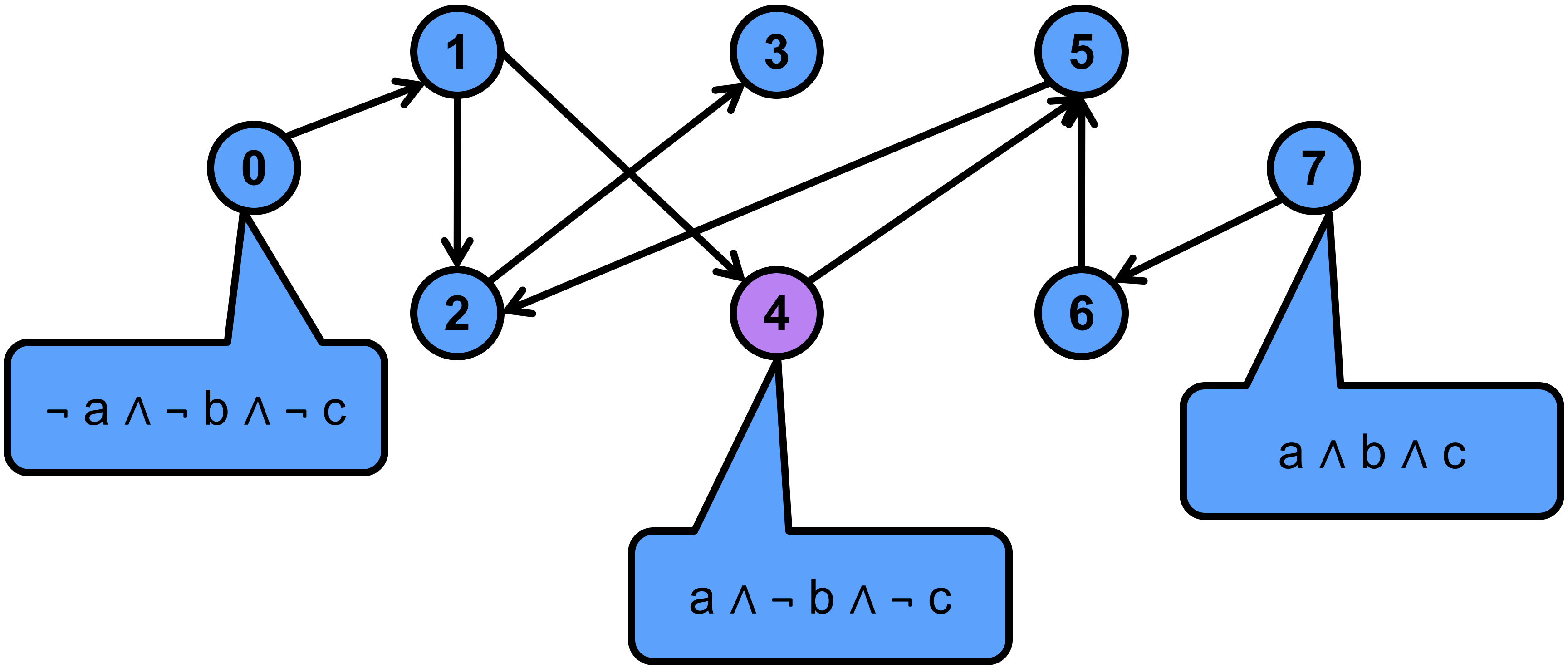
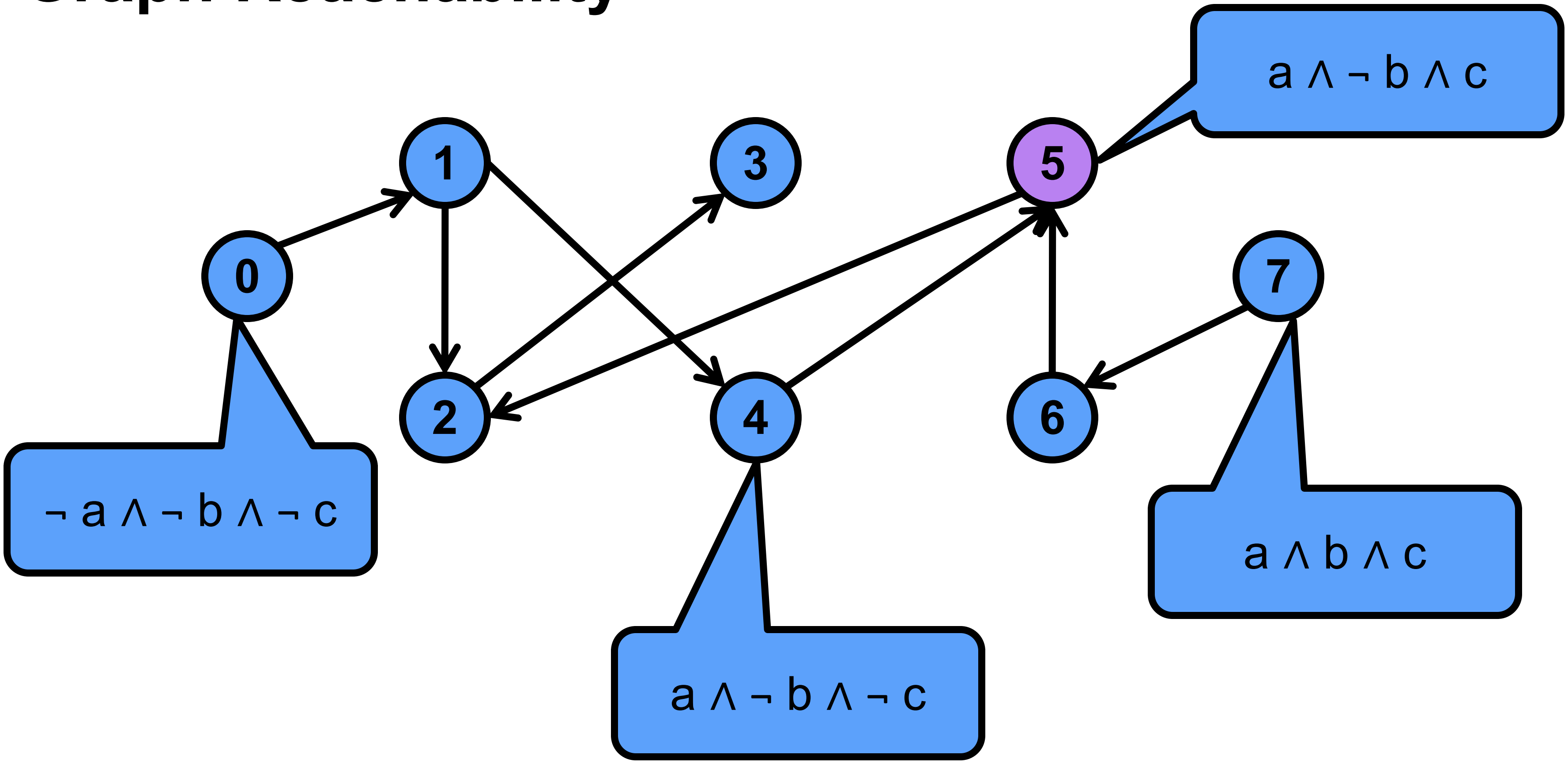
Graph Reachability



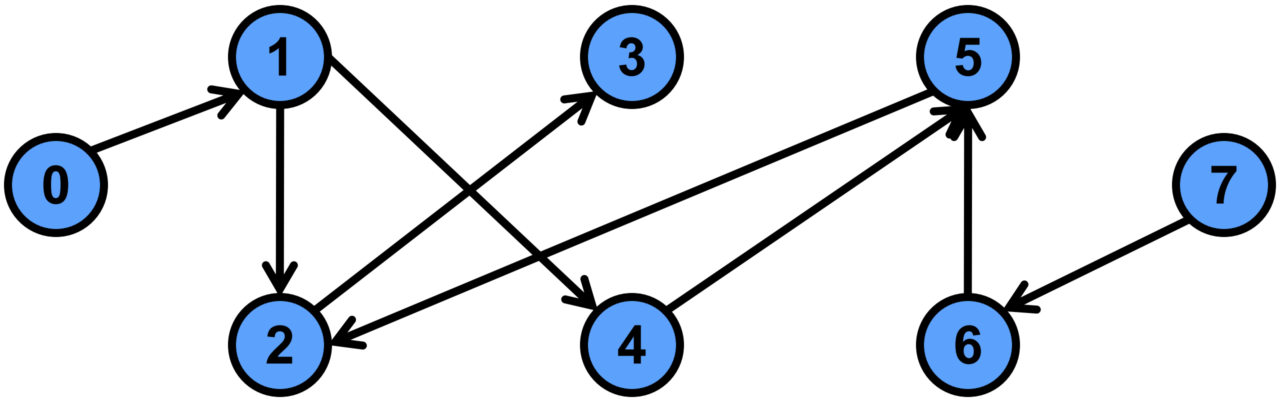
Use three Boolean variables (a,b,c) to encode each node?

Graph Reachability



Use three Boolean variables (a,b,c) to encode each node?

Graph Reachability

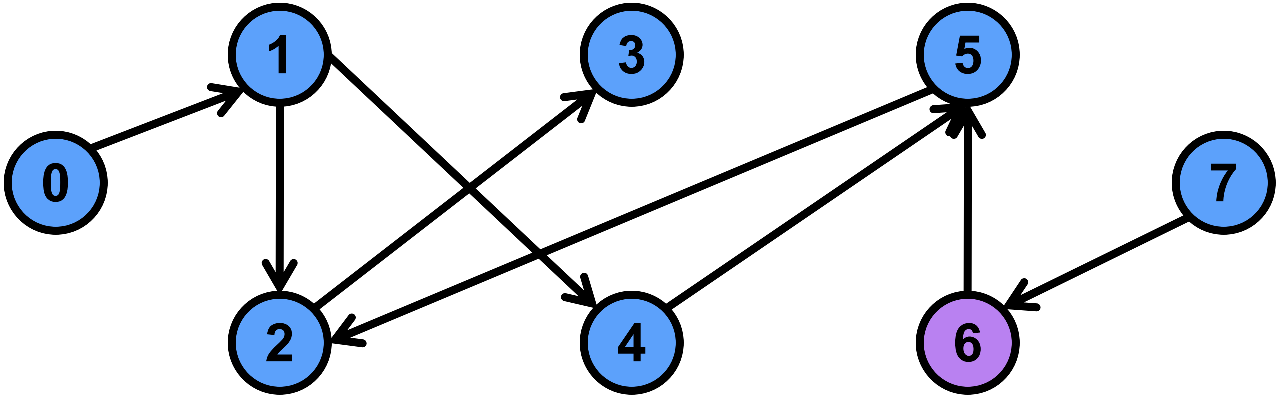


a ∧ b ∧ ¬ c = ?

Key Idea 1: Every Boolean formula represents a set of nodes!

The nodes whose encodings satisfy the formula.

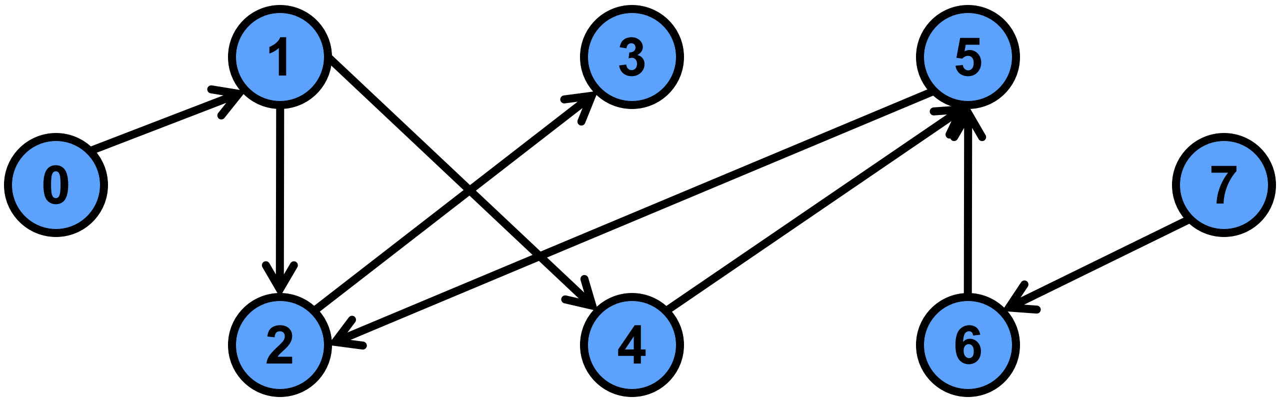
Graph Reachability



$a \wedge b \wedge \neg c = \{6\}$

Key Idea 1: Every Boolean formula represents a set of nodes!

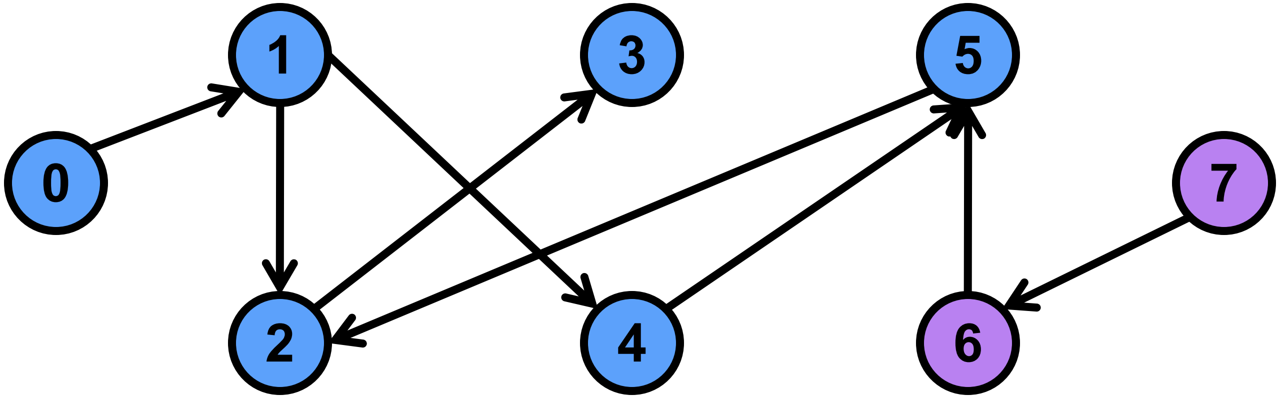
Graph Reachability



$a \wedge b = \ ?$

Key Idea 1: Every Boolean formula represents a set of nodes!

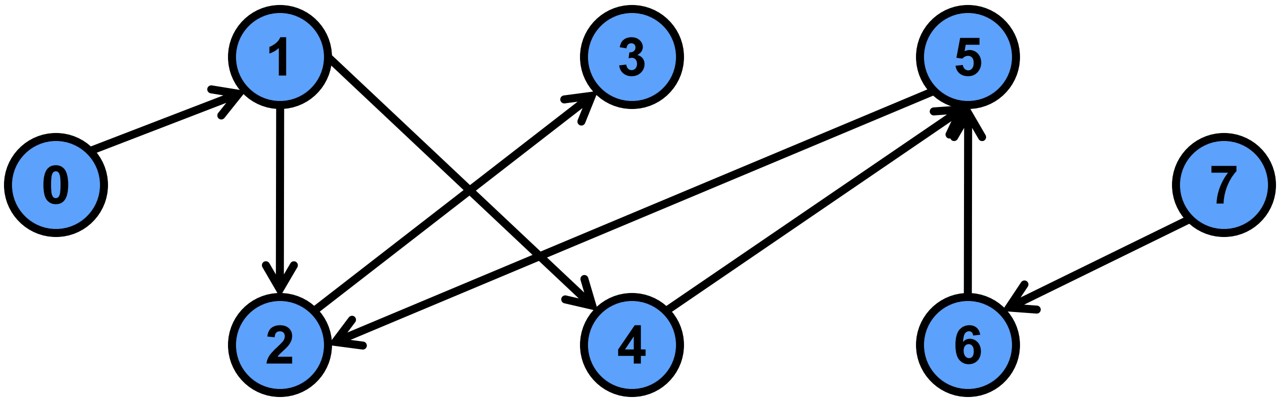
Graph Reachability



a ∧b = {6,7}

Key Idea 1: Every Boolean formula represents a set of nodes!

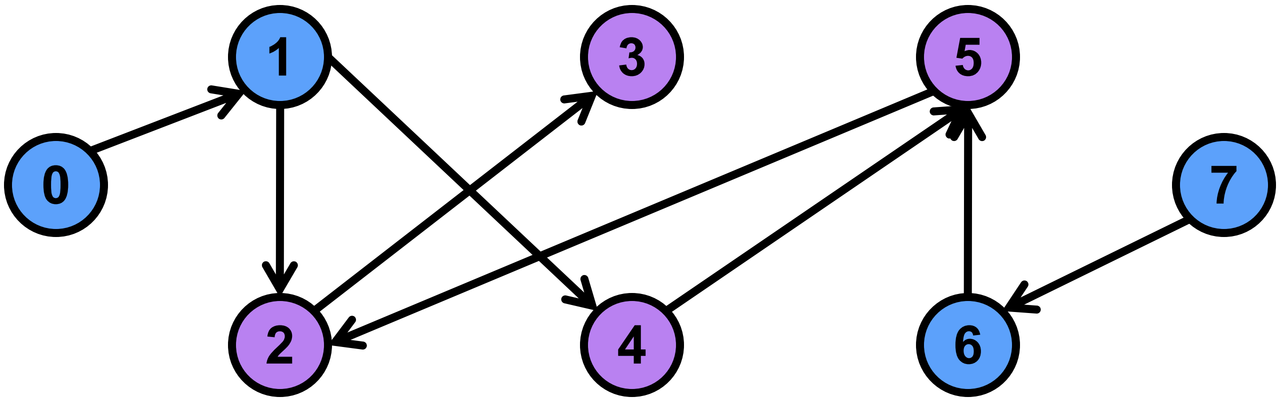
Graph Reachability



a xor b = ?

Key Idea 1: Every Boolean formula represents a set of nodes!

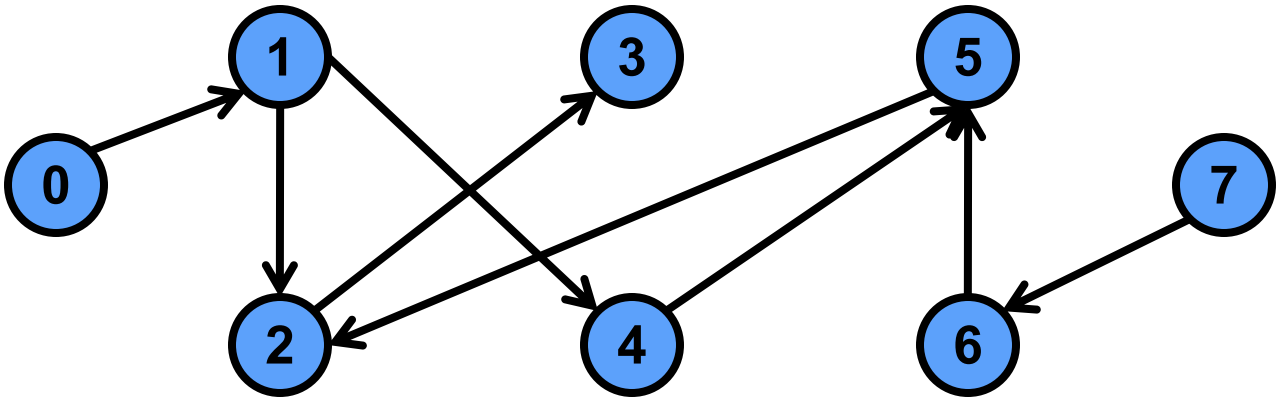
Graph Reachability



a xor b = {2,3,4,5}

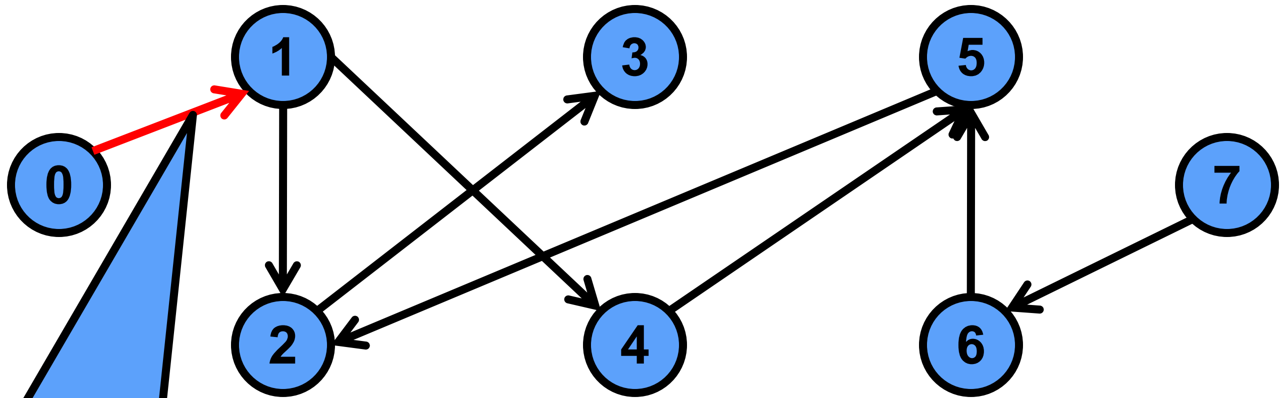
Key Idea 1: Every Boolean formula represents a set of nodes!

Graph Reachability



- Key Idea 2: Edges can also be represented by Boolean formulas
- An edge is just a pair of nodes
- Introduce three new variables¬ a', b', c'
- Formula *Φ* represents all pairs of nodes (n,n') that satisfy *Φ* when n is encoded using (a,b,c) and n' is encoded using (a',b',c')

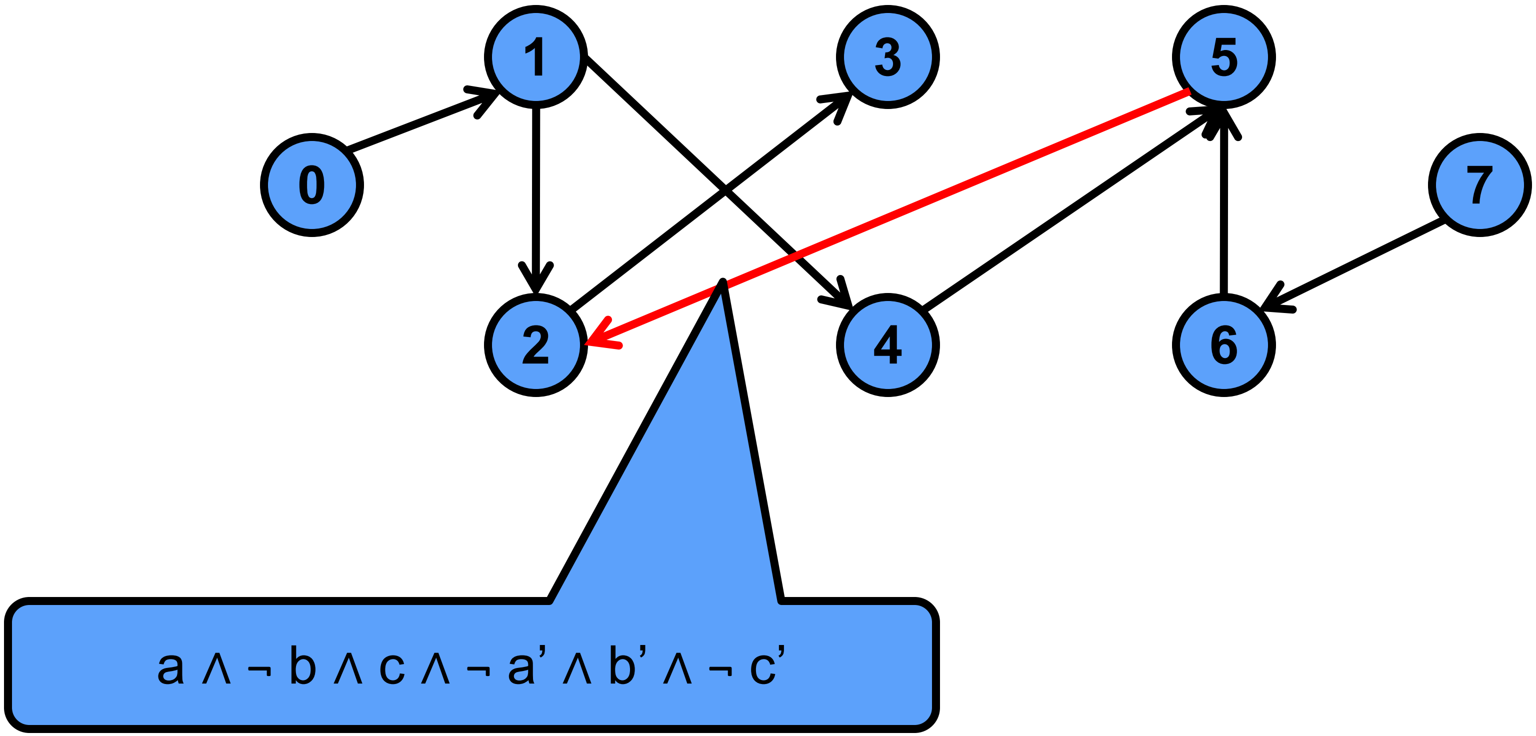
Graph Reachability



$\neg a \land \neg b \land \neg c \land \neg a' \land \neg b' \land c'$

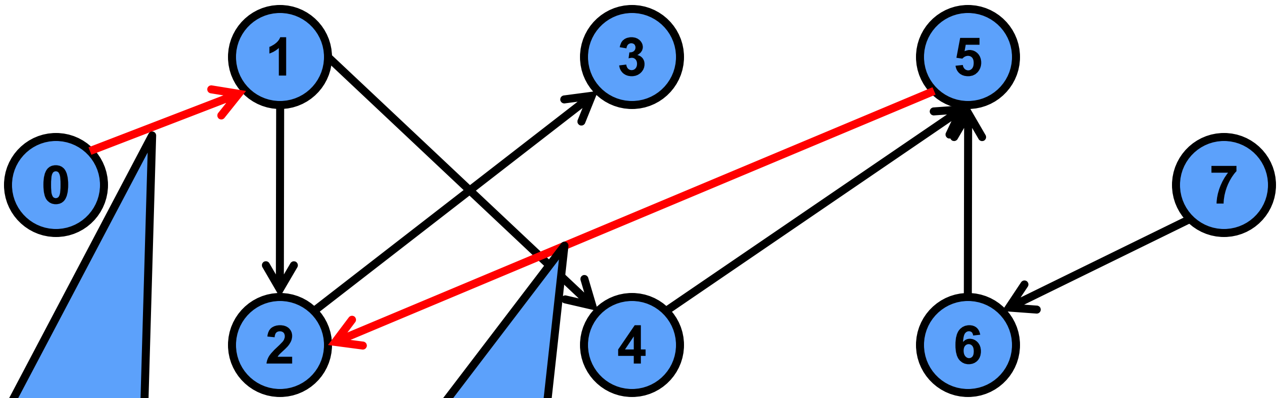
Key Idea 2: Edges can also be represented by Boolean formulas

Graph Reachability



Key Idea 2: Edges can also be represented by Boolean formulas

Graph Reachability



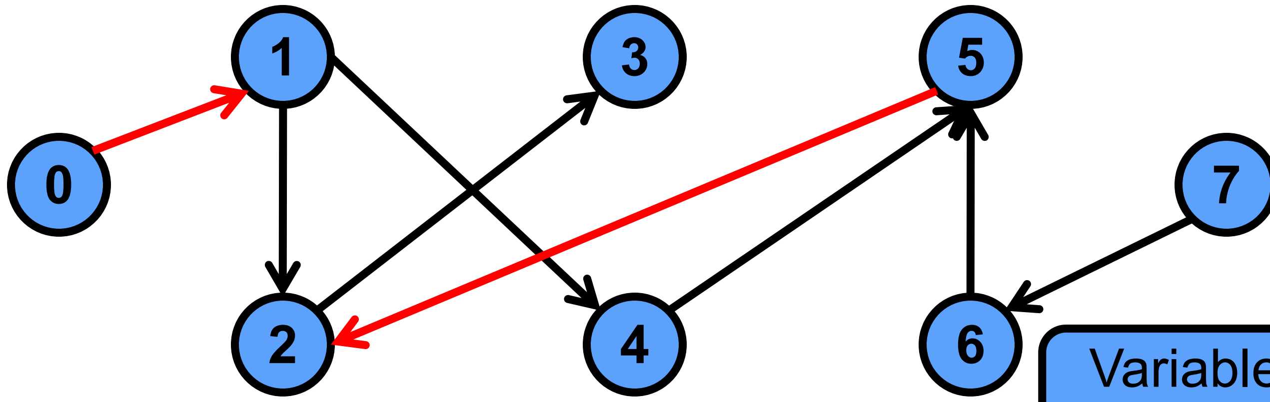
$a \wedge \neg b \wedge c \wedge \neg a' \wedge b' \wedge \neg c'$

$\vee$

$\neg a \wedge \neg b \wedge \neg c \wedge \neg a' \wedge \neg b' \wedge c'$

Key Idea 2: Edges can also be represented by Boolean formulas

Graph Reachability



Variable renaming ¬ replace a' with a

Image(S,R) =

$$(\exists\, a,b,c \,.\, (S \wedge R))\,[\, a \backslash a',\ b \backslash b',\ c \backslash c' \,]$$

Key Idea 3: Given the BDD for a set of nodes S, and the BDD for the set of all edges R, the BDD for all the nodes that are adjacent to S can be computed using the BDD operations

Graph Reachability Algorithm

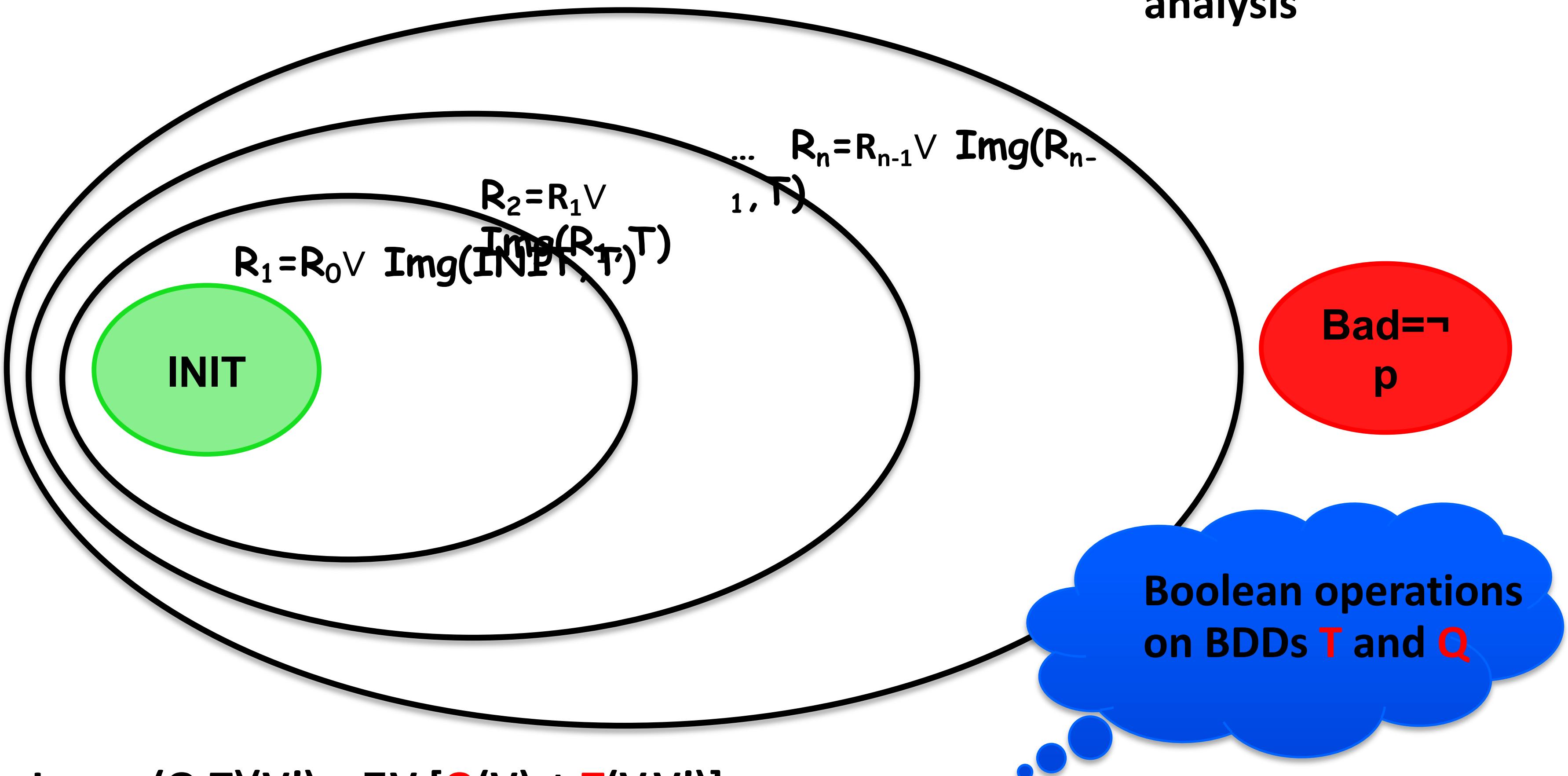
```
S = BDD for initial set of nodes;
R = BDD for all the edges of the graph;

while (true) {
   I = Image(S,R); // compute adjacent nodes to S
   if (And(Not(S),I) == False) // no new nodes found
      break;
   S = Or(S,I); // add newly discovered nodes to result
}

return S;
```

Symbolic Model Checking. Has been done for graphs with 10^{20} nodes.

Forward Reachability Analysis with BDDs

Does AG p hold?

All safety properties reduce to reachability analysis



$$Image(Q,T)(V') = \exists V\,[Q(V) \wedge T(V,V')]$$