

Constrained Horn Clauses (CHC)

Automated Program Verification (APV)
Fall 2019

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The Science, Art and Magic of Constrained Horn Clauses

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Numeric Algorithms for Scientific Computing

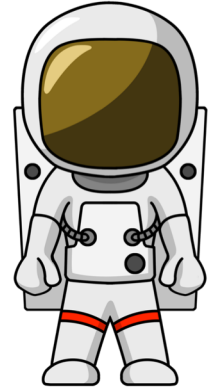


<https://notebooks.azure.com/arie-gurfinkel/projects/spacerexamples>

Software Model Checking of
Programs / Transitions Systems /
Push-down Systems

=

Satisfiability of Constrained
Horn Logic (CHC) fragment of
First Order Logic



Reduce Model Checking to
FOL Satisfiability



Constrained **Horn** Clauses (CHC)

Is it short for Horner?



Alfred Horn

Is it related to hornets?



Is it Santa Clause blowing a Horn?



Example CHC: Is this SAT?

$$\forall x \cdot x \leq 0 \implies P(x)$$

$$\forall x, x' \cdot P(x) \wedge x < 5 \wedge x' = x + 1 \implies P(x')$$

$$\forall x \cdot P(x) \wedge x \geq 10 \implies \textit{false}$$

This set of clauses is satisfiable

The model is an extension of the standard model of arithmetic with:

$$\begin{aligned} P(x) &\equiv \{x \mid x \leq 5\} \\ &\equiv \{5, 4, 3, 2, \dots\} \end{aligned}$$



Constrained Horn Clauses (CHC)

A Constrained Horn Clause (CHC) is a FOL formula of the form

$$\forall V \cdot (\varphi \wedge p_1[X_1] \wedge \cdots \wedge p_n[X_n]) \rightarrow h[X]$$

where

- φ - constraint in a background theory $\mathcal{T}$
- $\mathcal{T}$ - background theory
 - Linear Arithmetic, Arrays, Bit-Vectors, or combinations
- V - variables, and X_i are terms over V
- $p_1, \dots, p_n, h$ - n-ary predicates
- $p_i[X]$ - application of a predicate to first-order terms

CHC Satisfiability

Π - set of CHCs

M - $\mathcal{T}$ -**model** of a set of Π

- M satisfies $\mathcal{T}$
- M satisfies Π – through first-order interpretation of each predicate p_i

A set of clauses is **satisfiable** if and only if it has a model

- This is the usual FOL satisfiability

$\mathcal{T}$ -solution of a set of CHCs Π is a substitution σ from predicates p_i to $\mathcal{T}$ -formulas such that $\Pi\sigma$ is $\mathcal{T}$ -valid

In the context of program verification

Program $\models \varphi$	iff	$CHC_{Program} \rightarrow \varphi$
Inductive Invariant	=	Solution to CHC
Counter Example Trace	=	Resolution proof of CHC

Example CHC: Is this SAT?

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Validating the solution

Original CHC

$$\forall x \cdot x \leq 0 \implies P(x)$$

$$\forall x, x' \cdot P(x) \wedge x < 5 \wedge x' = x + 1 \implies P(x')$$

$$\forall x \cdot P(x) \wedge x \geq 10 \implies \text{false}$$

Validation of $P(x) = \{x \mid x \leq 5\}$

$$\vdash \forall x \cdot x \leq 0 \implies x \leq 5$$

$$\vdash \forall x, x' \cdot x \leq 5 \wedge x < 5 \wedge x' = x + 1 \implies x' \leq 5$$

$$\vdash \forall x \cdot x \leq 5 \wedge x \geq 10 \implies \text{false}$$



Example CHC: is this SAT?

$$\forall x \cdot x \leq 0 \implies Q(x)$$

$$\forall x, x' \cdot Q(x) \wedge x < 5 \wedge x' = x + 1 \implies Q(x')$$

$$\forall x \cdot Q(x) \wedge x \geq 2 \implies \textit{false}$$

This set of clauses is unsatisfiable

Justification is a refutation by resolution and instantiation



Example CHC: is this SAT?

$$\forall x \cdot x \leq 0 \implies Q(x)$$

$$\forall x, x' \cdot Q(x) \wedge x < 5 \wedge x' = x + 1 \implies Q(x')$$

$$\forall x \cdot Q(x) \wedge x \geq 2 \implies \text{false}$$

Refutation

$$\begin{array}{c} (x = 0) \frac{\forall x \cdot x \leq 0 \implies Q(x)}{Q(0)} \qquad \forall x \cdot Q(x) \wedge x < 5 \implies Q(x + 1) \\ \hline Q(1) \\ \forall x \cdot Q(x) \wedge x < 5 \implies Q(x + 1) \\ \hline Q(2) \\ \forall x \cdot Q(x) \wedge x \geq 2 \implies \text{false} \\ \hline \text{false} \end{array}$$



Horn Clauses for Program Verification

$e_{out}(x_0, w, e_o)$, which is an entry point into successor edges. with the edges are formulated as follows:

$$\begin{aligned} p_{init}(x_0, w, \perp) &\leftarrow x = x_0 && \text{where } x \text{ occurs in } w \\ p_{exit}(x_0, ret, \top) &\leftarrow \ell(x_0, w, \top) && \text{for each label } \ell, \text{ and } re \\ p(x, ret, \perp, \perp) &\leftarrow p_{exit}(x, ret, \perp) \\ p(x, ret, \perp, \top) &\leftarrow p_{exit}(x, ret, \top) \\ \ell_{out}(x_0, w', e_o) &\leftarrow \ell_{in}(x_0, w, e_i) \wedge \neg e_i \wedge \neg wlp(S, \neg(e_i = \end{aligned}$$

5. incorrect :- Z=W+1, W ≥ 0, W+1 < read(A, W, U), read(A, 2
6. p(I1, N, B) :- 1 ≤ I, I < N, D = I - 1, I1 = I + 1. V = U + 1. read(A, D, U), write(A
7. p(I, N, A) :- I = 1. N > 1.

De Angelis et al. Verifying Array Programs by Transforming Verification Conditions. VMCAI'14

Weakest Preconditions If we apply Boogie directly we obtain a translation from programs to Horn logic using a weakest liberal pre-condition calculus [26]:

$$\begin{aligned} \text{ToHorn}(\text{program}) &:= wlp(\text{Main}(), \top) \wedge \bigwedge_{\text{decl} \in \text{program}} \text{ToHorn}(\text{decl}) \\ \text{ToHorn}(\text{def } p(x) \{S\}) &:= wlp \left(\begin{array}{l} \text{havoc } x_0; \text{assume } x_0 = x; \\ \text{assume } p_{pre}(x); S, \end{array} p(x_0, ret) \right) \\ wlp(x := E, Q) &:= \text{let } x = E \text{ in } Q \\ wlp(\text{if } E \text{ then } S_1 \text{ else } S_2, Q) &:= wlp(((\text{assume } E; S_1) \square (\text{assume } \neg E; S_2)), Q) \\ wlp((S_1 \square S_2), Q) &:= wlp(S_1, Q) \wedge wlp(S_2, Q) \\ wlp(S_1; S_2, Q) &:= wlp(S_1, wlp(S_2, Q)) \\ wlp(\text{havoc } x, Q) &:= \forall x. Q \\ wlp(\text{assert } \varphi, Q) &:= \varphi \wedge Q \\ wlp(\text{assume } \varphi, Q) &:= \varphi \rightarrow Q \\ wlp(\text{while } E \text{ do } S, Q) &:= \text{inv}(w) \wedge \\ &\quad \forall w. \left(\begin{array}{l} ((\text{inv}(w) \wedge E) \rightarrow wlp(S, \text{inv}(w))) \\ \wedge ((\text{inv}(w) \wedge \neg E) \rightarrow Q) \end{array} \right) \end{aligned}$$

To translate a procedure call $\ell : y := q(E); \ell'$ within a procedure p , create the clauses:

$$\begin{aligned} p(w_0, w_4) &\leftarrow p(w_0, w_1), \text{call}(w_1, w_2), q(w_2, w_3), \text{return}(w_1, w_3, w_4) \\ q(w_2, w_2) &\leftarrow p(w_0, w_1), \text{call}(w_1, w_2) \\ \text{call}(w, w') &\leftarrow \pi = \ell, x' = E, \pi' = \ell_{q_{init}} \\ \text{return}(w, w', w'') &\leftarrow \pi' = \ell_{q_{exit}}, w'' = w[\text{ret}'/y, \ell'/\pi] \end{aligned}$$

Bjørner, Gurfinkel, McMillan, and Rybalchenko:
Horn Clause Solvers for Program Verification

Horn Clauses for Concurrent / Distributed / Parameterized Systems

For assertions $R_1, \dots, R_N$ over V and $E_1, \dots, E_N$ over V, V' ,

- CM1 : $init(V) \rightarrow R_i(V)$
 CM2 : $R_i(V) \wedge \rho_i(V, V') \rightarrow R_i(V')$
 CM3 : $(\bigvee_{i \in 1..N \setminus \{j\}} R_i(V) \wedge \rho_i(V, V')) \rightarrow E_j(V, V')$
 CM4 : $R_i(V) \wedge E_i(V, V') \wedge \rho_i^-(V, V') \rightarrow R_i(V')$
 CM5 : $R_1(V) \wedge \dots \wedge R_N(V) \wedge error(V) \rightarrow false$

multi-threaded program P is safe

Rybalchenko et al. Synthesizing Software Verifiers from Proof Rules. PLDI'12

- (initial) $init(g, x_1) \wedge \dots \wedge init(g, x_n) \rightarrow Inv(g, \ell_{init}, x_1, \dots, \ell_{init}, x_k)$
 (inductive) $Inv(g, \ell_1, x_1, \dots, \ell_i, x_i, \dots, \ell_k, x_k) \wedge s(g, x_i, g', x'_i) \rightarrow Inv(g', \ell_1, x_1, \dots, \ell'_i, x'_i, \dots, \ell_k, x_k)$
 (non-interference) $Inv(g, \ell_1, x_1, \dots, \ell_k, x_k) \wedge$
 $Inv(g, \ell^\dagger, x^\dagger, \ell_2, x_2, \dots, \ell_k, x_k) \wedge$
 $\vdots$
 $Inv(g, \ell_1, x_1, \dots, \ell_{k-1}, x_{k-1}, \ell^\dagger, x^\dagger) \wedge s(g, x^\dagger, g', \cdot) \rightarrow Inv(g', \ell_1, x_1, \dots, \ell_k, x_k)$
 (safe) $Inv(g, \ell_1, x_1, \dots, \ell_k, x_k) \wedge err(g, \ell_1, x_1, \dots, \ell_m, x_m) \rightarrow false$

Figure 6. Horn clause encoding for thread modularity at level k (where (ℓ_i, s, ℓ'_i) and $(\ell^\dagger, s, \cdot)$ refer to statement s on a thread from ℓ_i to ℓ'_i and, respectively, from $\ell^\dagger$ to some other location in the control flow graph)

Hoenicke et al. Thread Modularity at Many Levels. POPL'17

$$\left\{ R(g, p_{\sigma(1)}, l_{\sigma(1)}, \dots, p_{\sigma(k)}, l_{\sigma(k)}) \leftarrow dist(p_1, \dots, p_k) \wedge R(g, p_1, l_1, \dots, p_k, l_k) \right\}_{\sigma \in S_k} \quad (6)$$

$$R(g, p_1, l_1, \dots, p_k, l_k) \leftarrow dist(p_1, \dots, p_k) \wedge Init(g, l_1) \wedge \dots \wedge Init(g, l_k) \quad (7)$$

$$R(g', p_1, l'_1, \dots, p_k, l_k) \leftarrow dist(p_1, \dots, p_k) \wedge ((g, l_1) \xrightarrow{p_1} (g', l'_1)) \wedge R(g, p_1, l_1, \dots, p_k, l_k) \quad (8)$$

$$R(g', p_1, l_1, \dots, p_k, l_k) \leftarrow dist(p_0, p_1, \dots, p_k) \wedge ((g, l_0) \xrightarrow{p_0} (g', l'_0)) \wedge RConj(0, \dots, k) \quad (9)$$

$$false \leftarrow dist(p_1, \dots, p_r) \wedge \left(\bigwedge_{j=1, \dots, m} (p_j = p_j \wedge (g, l_j) \in E_j) \right) \wedge RConj(1, \dots, r) \quad (10)$$

Figure 4: Horn constraints encoding a homogeneous infinite system with the help of a k -indexed invariant. S_k is the symmetric group on $\{1, \dots, k\}$, i.e., the group of all permutations of k numbers; as an optimisation, any generating subset of S_k , for instance transpositions, can be used instead of S_k . In (10), we define $r = \max\{m, k\}$.

Hojjat et al. Horn Clauses for Communicating Timed Systems. HCVS'14

$$Init(i, j, \bar{v}) \wedge Init(j, i, \bar{v}) \wedge$$

$$Init(i, i, \bar{v}) \wedge Init(j, j, \bar{v}) \Rightarrow I_2(i, j, \bar{v})$$

$$I_2(i, j, \bar{v}) \wedge Tr(i, \bar{v}, \bar{v}') \Rightarrow I_2(i, j, \bar{v}') \quad (3)$$

$$I_2(i, j, \bar{v}) \wedge Tr(j, \bar{v}, \bar{v}') \Rightarrow I_2(i, j, \bar{v}') \quad (4)$$

$$I_2(i, j, \bar{v}) \wedge I_2(i, k, \bar{v}) \wedge I_2(j, k, \bar{v}) \wedge$$

$$Tr(k, \bar{v}, \bar{v}') \wedge k \neq i \wedge k \neq j \Rightarrow I_2(i, j, \bar{v}') \quad (5)$$

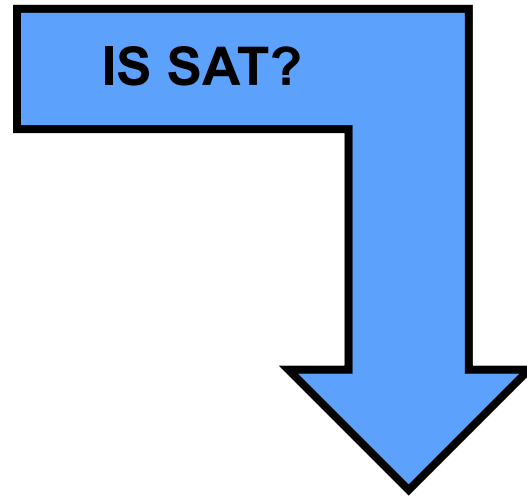
$$I_2(i, j, \bar{v}) \Rightarrow \neg Bad(i, j, \bar{v})$$

Figure 3: $VC_2(T)$ for two-quantifier invariants.

Gurfinkel et al. SMT-Based Verification of Parameterized Systems. FSE 2016

Program Verification with HORN(LIA)

```
z = x; i = 0;  
assume (y > 0);  
while (i < y) {  
    z = z + 1;  
    i = i + 1;  
}  
assert(z == x + y);
```



$z = x \ \& \ i = 0 \ \& \ y > 0$	$\rightarrow$	$\text{Inv}(x, y, z, i)$
$\text{Inv}(x, y, z, i) \ \& \ i < y \ \& \ z1=z+1 \ \& \ i1=i+1$	$\rightarrow$	$\text{Inv}(x, y, z1, i1)$
$\text{Inv}(x, y, z, i) \ \& \ i \geq y \ \& \ z \neq x+y$	$\rightarrow$	false

In SMT-LIB

```
(set-logic HORN)

;; Inv(x, y, z, i)
(declare-fun Inv ( Int Int Int Int) Bool)

(assert
  (forall ( (A Int) (B Int) (C Int) (D Int))
    (=> (and (> B 0) (= C A) (= D 0))
      (Inv A B C D)))
)
(assert
  (forall ( (A Int) (B Int) (C Int) (D Int) (C1 Int) (D1 Int) )
    (=>
      (and (Inv A B C D) (< D B) (= C1 (+ C 1)) (= D1 (+ D
1))))
    (Inv A B C1 D1)
  )
)
(assert
  (forall ( (A Int) (B Int) (C Int) (D Int))
    (=> (and (Inv A B C D) (>= D B) (not (= C (+ A B)))))
      false
    )
  )
)

(check-sat)
(get-model)
```

\$ z3 add-by-one.smt2

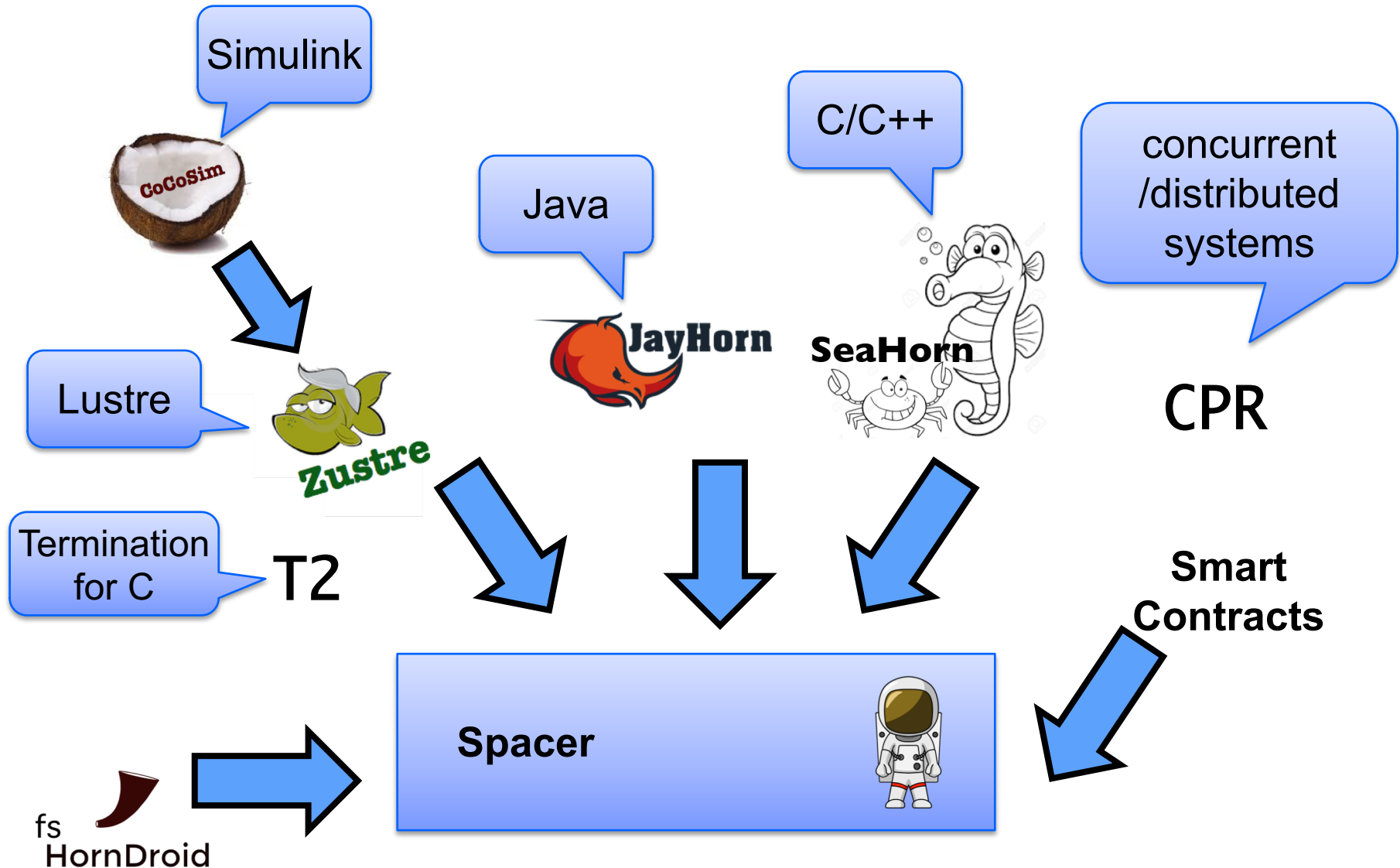
```
sat
(model
  (define-fun Inv ((x!0 Int) (x!1 Int) (x!2 Int) (x!3 Int)) Bool
    (and (<= (+ x!2 (* (- 1) x!0) (* (- 1) x!3)) 0)
      (<= (+ x!2 (* (- 1) x!0) (* (- 1) x!1)) 0)
      (<= (+ x!0 x!3 (* (- 1) x!2)) 0)))
  )
```

$\text{Inv}(x, y, z, i)$

$z = x + i$

$z \leq x + y$

Logic-based Algorithmic Verification



INTERACTIVE TUTORIAL



<https://notebooks.azure.com/arie-gurfinkel/projects/spacerexamples>

Procedures for Solving CHC(T)

Predicate abstraction by lifting Model Checking to HORN

- QARMC, Eldarica, ...

Maximal Inductive Subset from a finite Candidate space (Houdini)

- TACAS'18: hoice, FreqHorn

Machine Learning

- PLDI'18: sample, ML to guess predicates, DT to guess combinations

Abstract Interpretation (Poly, intervals, boxes, arrays...)

- Approximate least model by an abstract domain (SeaHorn, ...)

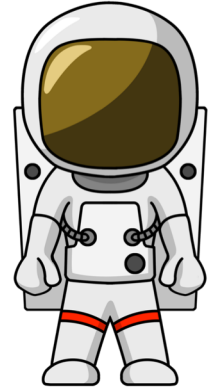
Interpolation-based Model Checking

- Duality, QARMC, ...

SMT-based Unbounded Model Checking (IC3/PDR)

- Spacer, Implicit Predicate Abstraction

Spacer: Solving SMT-constrained CHC



Spacer: SAT procedure for SMT-constrained Horn Clauses

- now the default CHC solver in Z3
 - <https://github.com/Z3Prover/z3>
 - dev branch at <https://github.com/agurfinkel/z3>

Supported SMT-Theories

- Linear Real and Integer Arithmetic
- Quantifier-free theory of arrays
- Universally quantified theory of arrays + arithmetic
- Best-effort support for many other SMT-theories
 - data-structures, bit-vectors, non-linear arithmetic

Support for Non-Linear CHC

- for procedure summaries in inter-procedural verification conditions
- for compositional reasoning: abstraction, assume-guarantee, thread modular, etc.

A little bit of complexity

Satisfiability of CHC over most interesting theories is undecidable

- e.g., CHC(Linear Real Arithmetic), CHC(Linear Integer Arithmetic)
- proof: many easy reductions, for example, counter automata

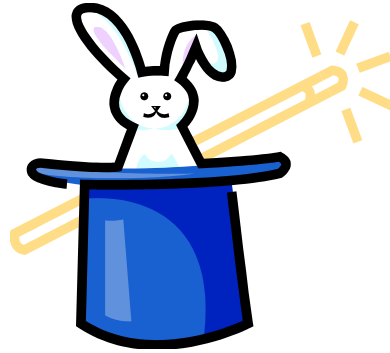
Satisfiability of Linear CHC over Propositional logic is decidable

- Finite state model checking of transition systems
- Complexity: linear in the size of the graph induced by the transition system

Satisfiability of Non-Linear CHC over Propositional logic is decidable

- Finite state model checking of pushdown systems
- Complexity: cubic in the size of the pushdown system

Decidability of some classes of CHC: Difference arithmetic (= timed automata)



SOLVING CONSTRAINED HORN CLAUSES

A Magician's Guide to Solving Undecidable Problems

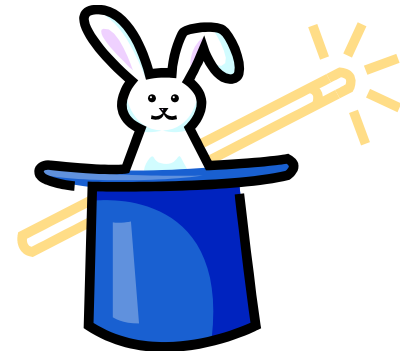
Develop a procedure **P** for a decidable problem

Show that **P** is a decision procedure for the problem

- e.g., model checking of finite-state systems

Choose one of

- Always terminate with some answer (over-approximation)
- Always make useful progress (under-approximation)



Extend procedure **P** to procedure **Q** that “solves” the undecidable problem

- Ensure that **Q** is still a decision procedure whenever **P** is
- Ensure that **Q** either always terminates or makes progress

Linear CHC Satisfiability

Satisfiability of a set of linear CHCs is reducible to satisfiability of THREE clauses of the form

$$\begin{array}{c} Init(X) \rightarrow P(X) \\ P(X) \wedge Tr(X, X') \rightarrow P(X') \\ P(X) \rightarrow \neg Bad(X) \end{array}$$

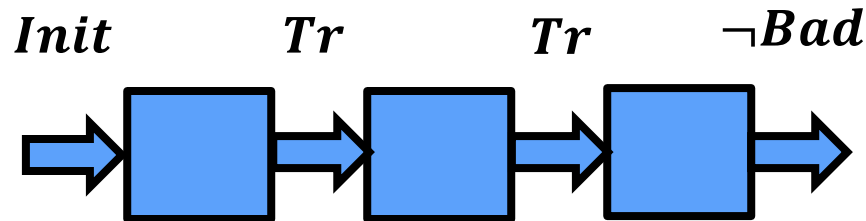
where, $X' = \{x' \mid x \in X\}$, P a fresh predicate, and $Init$, Bad , and Tr are constraints

Proof:

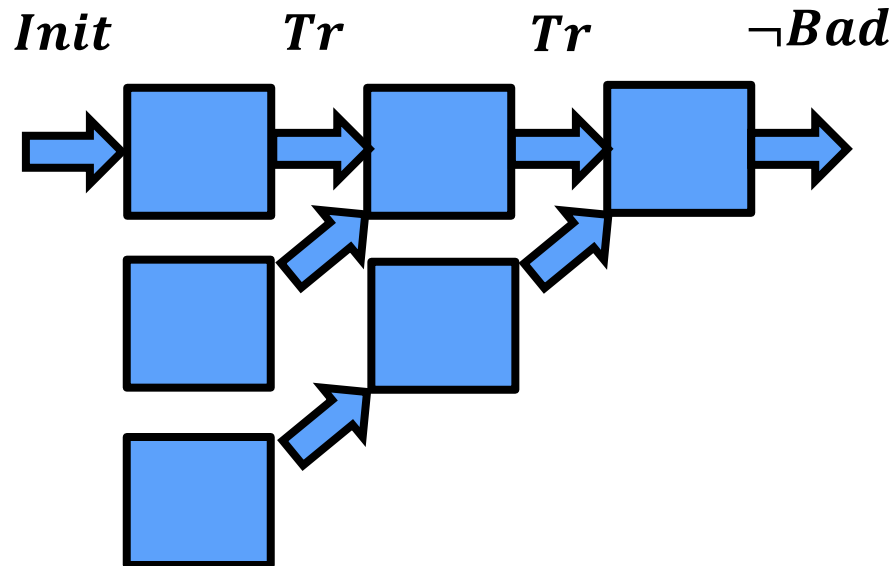
add extra arguments to distinguish between predicates

$$\frac{Q(y) \wedge \phi \rightarrow W(y, z)}{P(id='Q', y) \wedge \phi \rightarrow P(id='W', y, z)}$$

IC3, PDR and friends

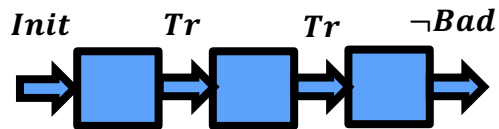


Finite State Machines
(HW model checking)
[Bradley, VMCAI 2011]



Push Down Machines
(SW model checking)
[Hoder&B, SAT 2012]

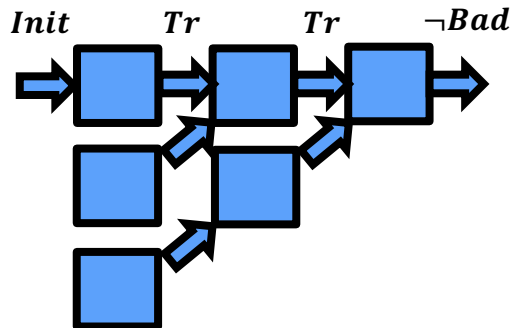
IC3, PDR and friends



Finite State Machines
(HW model checking)
[Bradley, VMCAI 2011]

Finite State

- Incremental SAT solving [Bradley, VMCAI 11]
- Fast prime implicants [Een& FMCAD 11]
- Basis for predicate abstraction [Cimatti& TACAS 14, Birgmeier& CAV 14]

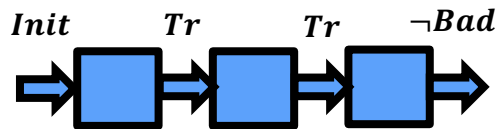


Push Down Machines
(SW model checking)
[Hoder, B, SAT 2012]

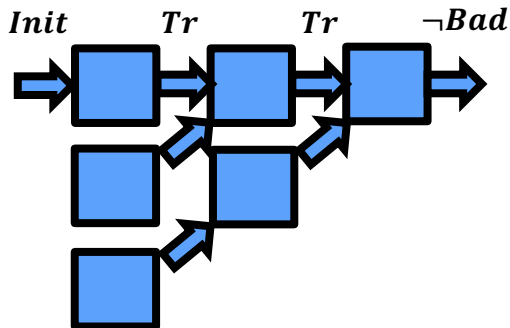
Infinite State

- Arithmetic + Farkas [H&B, SAT 12]
- Arithmetic + Model Based Projection [K&, CAV 14]
- Polyhedra + Convex Closure [B&G, VMCAI 15]
- Arithmetic + Arrays [K&, FMCAD 15]
- $\exists \forall$ - EPR fragment [K'&, CAV 15]
- $\exists \forall$ + Arithmetic/Arrays [G&, ATVA 18]

IC3, PDR and friends



Finite State Machines
(HW model checking)
[Bradley, VMCAI 2011]



Push Down Machines
(SW model checking)
[Hoder, B, SAT 2012]

Finite State

SAT

Infinite
State

SMT

Arithmetic
Arrays
Quantifiers

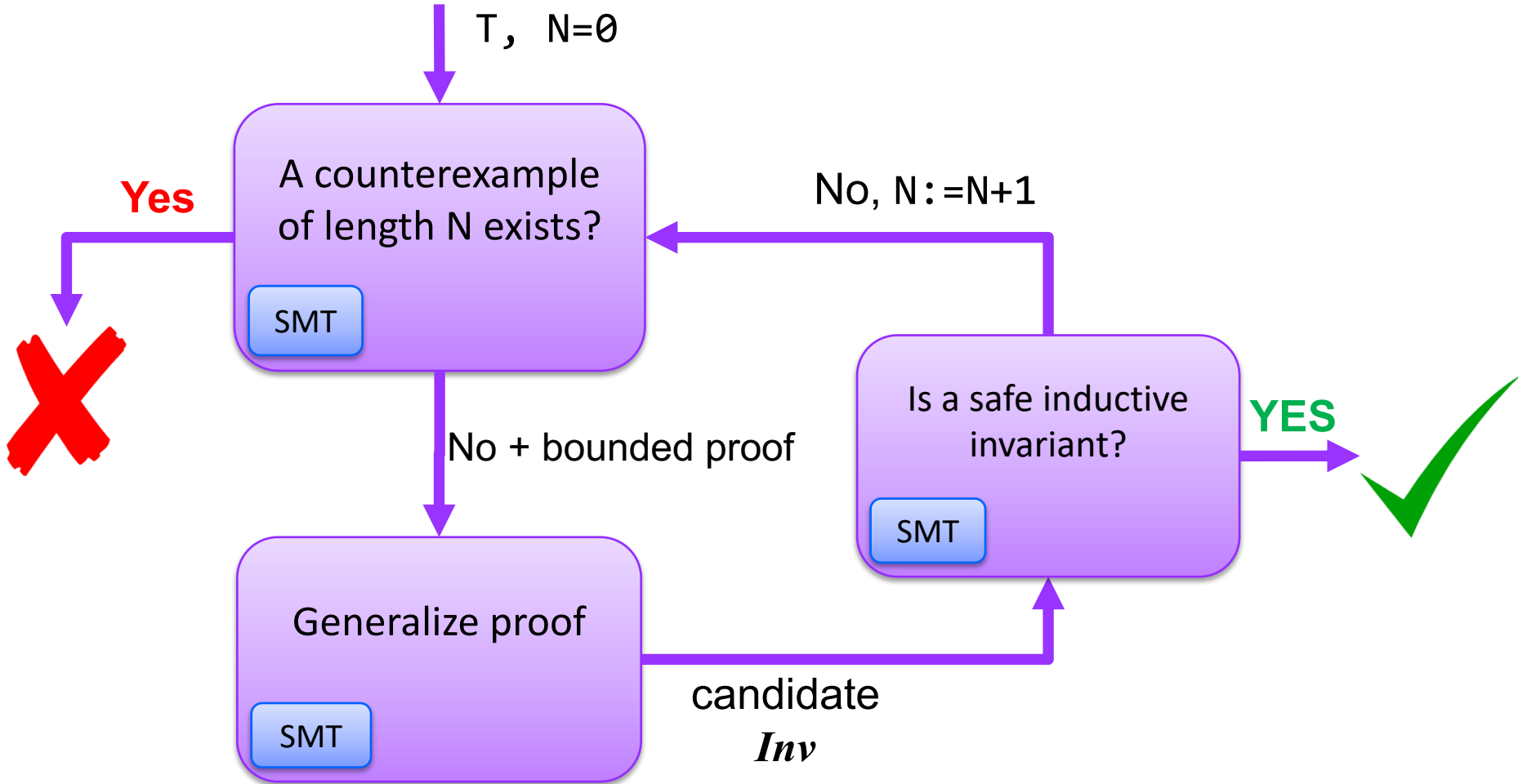
Search Strategies

[Bradley, VMCAI 11]
CTI – Counter Examples To Induction

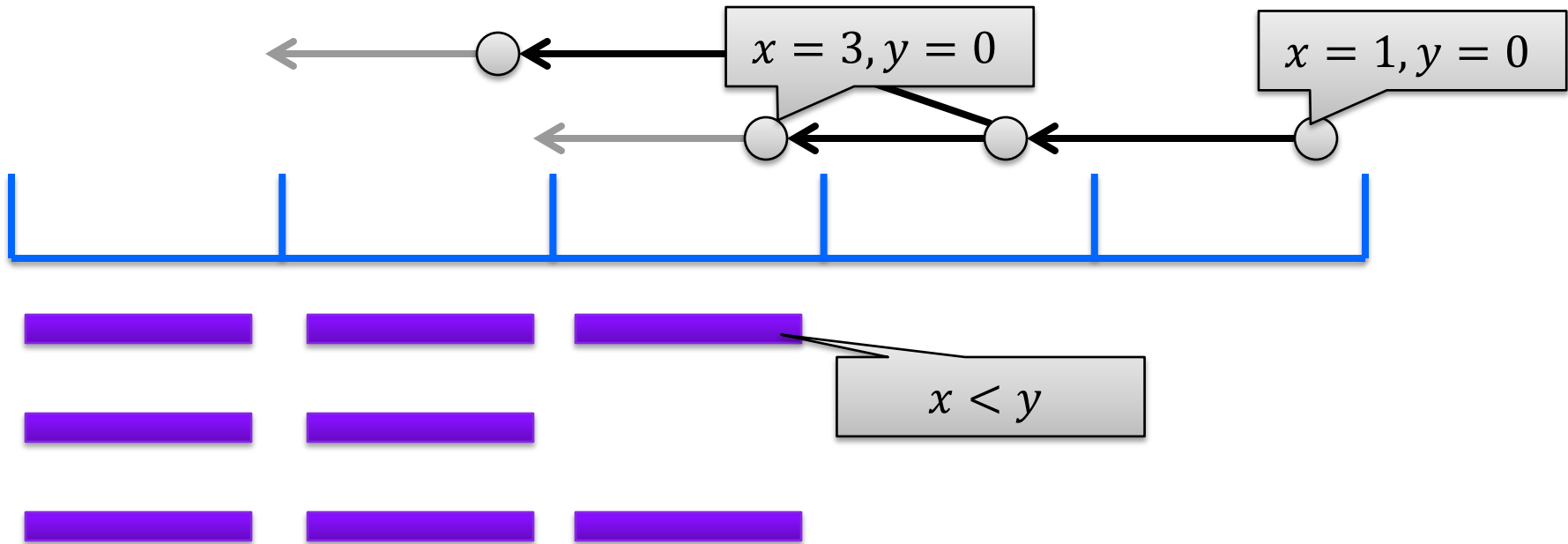
[G&Ivrii, FMCAD 15]
Under and over-approximations

[Vizel&G, CAV 14]
Use SAT for blocking
IC3 for pushing

Verification by Incremental Generalization



IC3/PDR In Pictures: MkSafe



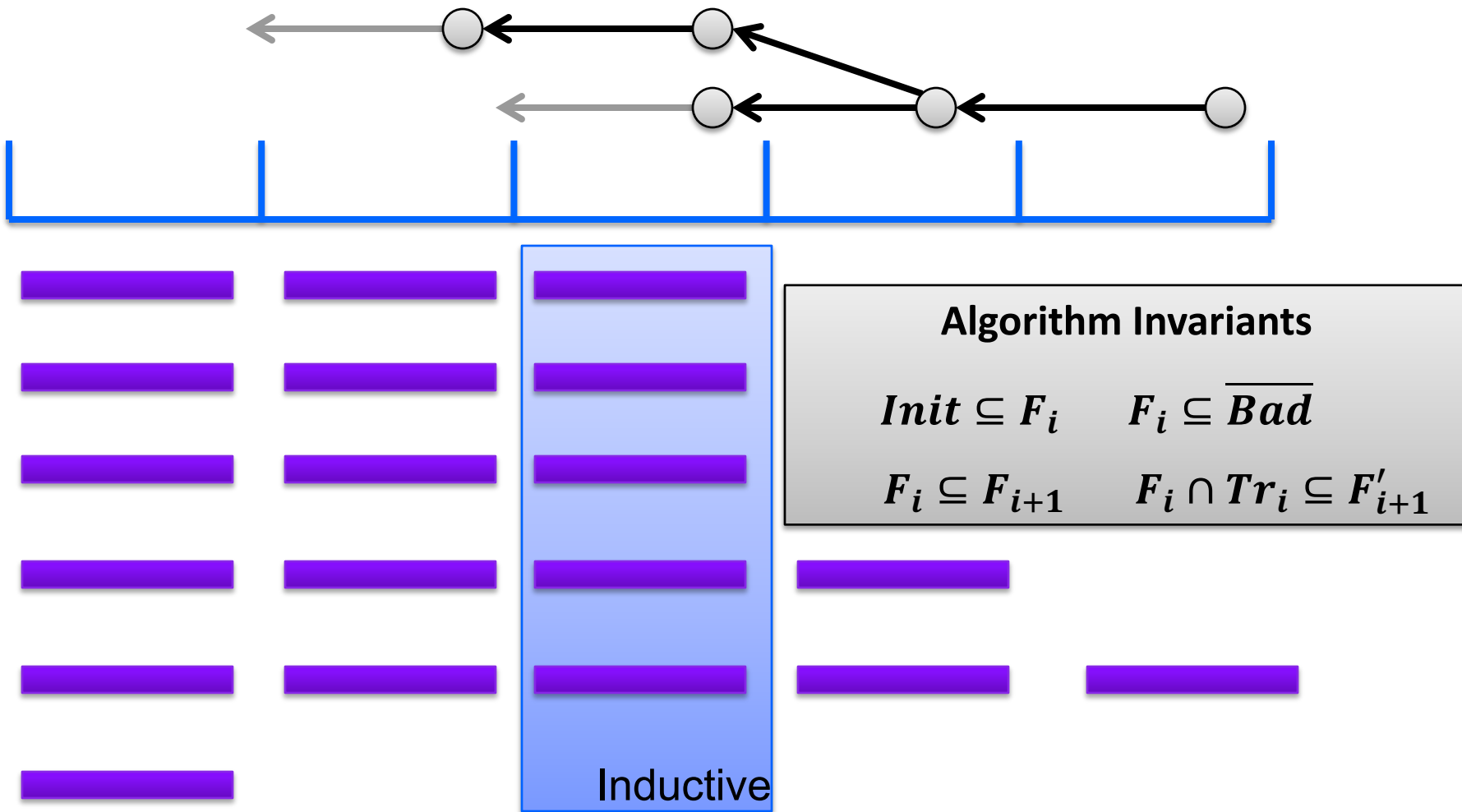
Predecessor

find M s.t. $M \models F_i \wedge Tr \wedge m'$

find m s.t. $(M \models m) \wedge (m \implies \exists V' \cdot Tr \wedge m')$

find ℓ s.t. $(F_i \wedge Tr \implies \ell') \wedge (\ell \implies \neg m)$

IC3/PDR in Pictures: Push



IC3/PDR: Solving Linear (Propositional) CHC

Unreachable and Reachable

- terminate the algorithm when a solution is found

Unfold

- increase search bound by 1

Candidate

- choose a bad state in the last frame

Decide

- extend a cex (backward) consistent with the current frame
- choose an assignment s s.t. $(s \wedge Fi \wedge Tr \wedge cex')$ is SAT

Conflict

- construct a lemma to explain why cex cannot be extended
- Find a clause L s.t. $L \Rightarrow \neg cex$, $Init \Rightarrow L$, and $F_i \wedge Tr \Rightarrow L'$

Induction

- propagate a lemma as far into the future as possible
- (optionally) strengthen by dropping literals

From Propositional PDR to Solving CHC

Theories with infinitely many models

- infinitely many satisfying assignments
- can't simply enumerate (when computing predecessor)
- can't block one assignment at a time (when blocking)

Non-Linear Horn Clauses

- multiple predecessors (when computing predecessors)

The problem is undecidable in general, but we want an algorithm that makes progress

- doesn't get stuck in a decidable sub-problem
- guaranteed to find a counterexample (if it exists)

IC3/PDR: Solving Linear (Propositional) CHC

Unreachable and Reachable

- terminate the algorithm when a solution is found

Unfold

- increase search bound by 1

Candidate

- choose a bad state in the last frame

Decide

- extend a cex (backward) consistent with the current frame
- choose an assignment s s.t. $(s \wedge Fi \wedge Tr \wedge cex')$ is SAT

Conflict

- construct a lemma to explain why cex cannot be extended
- Find a clause L s.t. $L \Rightarrow \neg cex$, $Init \Rightarrow L$, and $F_i \wedge Tr \Rightarrow L'$

Induction

- propagate a lemma as far into the future as possible
- (optionally) strengthen by dropping literals

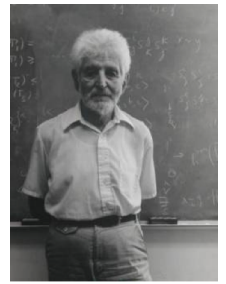
**Theory
dependent**

$$\left((F_i \wedge Tr) \vee Init' \right) \Rightarrow \varphi', \quad \varphi' \Rightarrow \neg cex'$$

Looking for φ'

CONFLICT (ARITHMETIC)

Craig Interpolation Theorem



Theorem (Craig 1957)

Let A and B be two First Order (FO) formulae such that $A \Rightarrow \neg B$, then there exists a FO formula I , denoted $ITP(A, B)$, such that

$$A \Rightarrow I \quad I \Rightarrow \neg B \quad \Sigma(I) \in \Sigma(A) \cap \Sigma(B)$$

A Craig interpolant $ITP(A, B)$ can be effectively constructed from a resolution proof of unsatisfiability of $A \wedge B$

In Model Checking, Craig Interpolation Theorem is used to safely over-approximate the set of (finitely) reachable states

Examples of Craig Interpolation for Theories

Boolean logic

$$A = (\neg b \wedge (\neg a \vee b \vee c) \wedge a)$$

$$B = (\neg a \vee \neg c)$$

$$ITP(A, B) = a \wedge c$$

Equality with Uninterpreted Functions (EUF)

$$A = (f(a) = b \wedge p(f(a)))$$

$$B = (b = c \wedge \neg p(c))$$

$$ITP(A, B) = p(b)$$

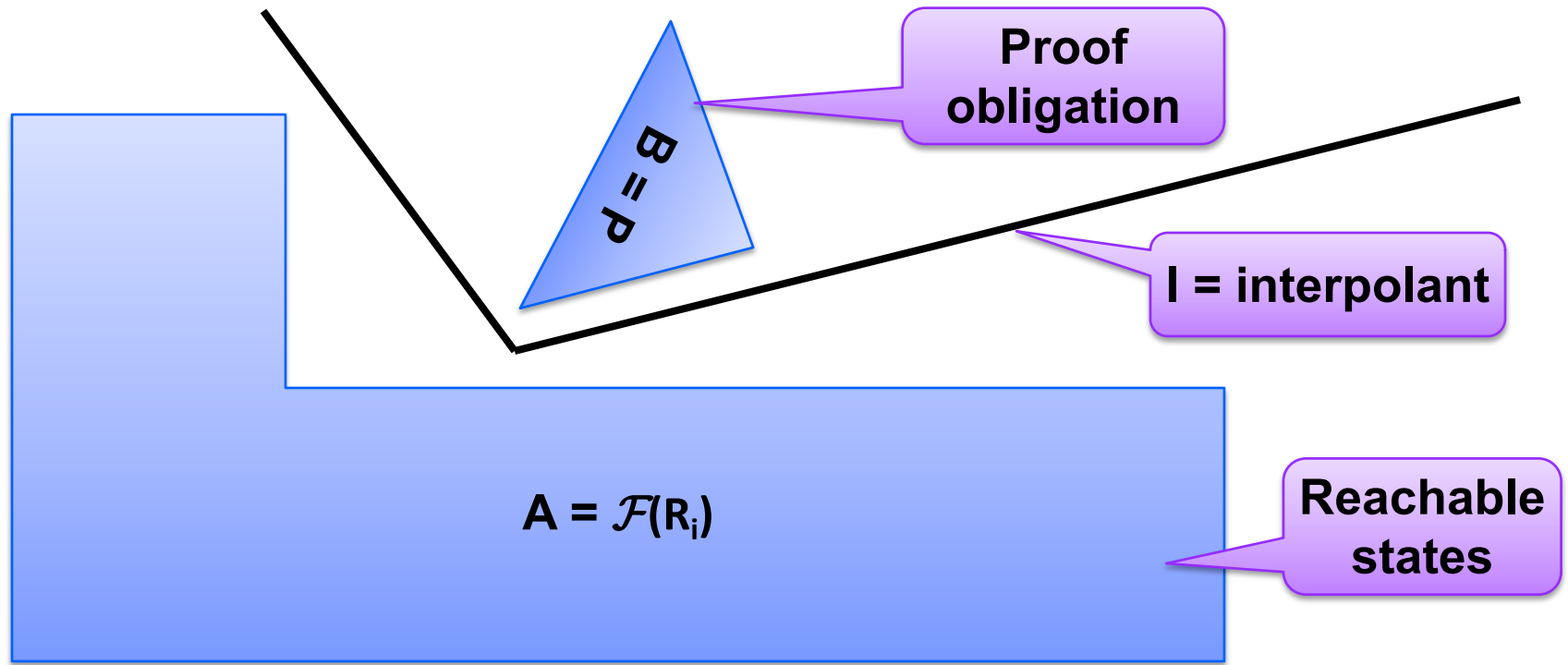
Linear Real Arithmetic (LRA)

$$A = (z + x + y > 10 \wedge z < 5)$$

$$B = (x < -5 \wedge y < -3)$$

$$ITP(A, B) = x + y > 5$$

Craig Interpolation for Linear Arithmetic



Useful properties of existing interpolation algorithms [CGS10] [HB12]

- $I \in \text{ITP}(A, B)$ then $\neg I \in \text{ITP}(B, A)$
- if A is syntactically convex (a monomial), then I is convex
- if B is syntactically convex, then I is co-convex (a clause)
- if A and B are syntactically convex, then I is a half-space

Arithmetic Conflict

Notation: $\mathcal{F}(A) = (A(X) \wedge Tr) \vee Init(X')$.

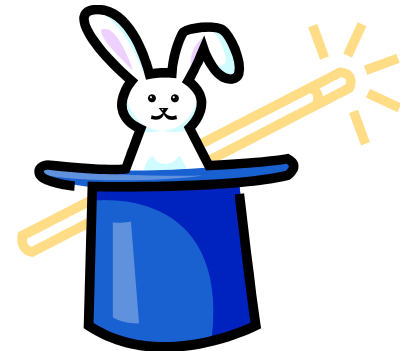
Conflict For $0 \leq i < N$, given a counterexample $\langle P, i+1 \rangle \in Q$ s.t.
 $\mathcal{F}(F_i) \wedge P'$ is unsatisfiable, add $P^\uparrow = \text{ITP}(\mathcal{F}(F_i), P')$ to F_j for $j \leq i+1$.

Counterexample is blocked using Craig Interpolation

- summarizes the reason why the counterexample cannot be extended

Generalization is not inductive

- weaker than IC3/PDR
- inductive generalization for arithmetic is still an open problem



Computing Interpolants for IC3/PDR

Much simpler than general interpolation problem for $A \wedge B$

- B is always a conjunction of literals
- A is dynamically split into DNF by the SMT solver
- DPLL(T) proofs do not introduce new literals

Interpolation algorithm is reduced to analyzing all theory lemmas in a DPLL(T) proof produced by the solver

- every theory-lemma that mixes B-pure literals with other literals is interpolated to produce a single literal in the final solution
- interpolation is restricted to clauses of the form $(\wedge B_i \Rightarrow \vee A_j)$

Interpolating (UNSAT) Cores

- improve interpolation algorithms and definitions to the specific case of PDR
- classical interpolation focuses on eliminating non-shared literals
- in PDR, the focus is on finding good generalizations

Farkas Lemma

Let $\Phi = t_1 \geq b_1 \wedge \dots \wedge t_n \geq b_n$, t_i are linear terms and b_i are constants

Φ is *unsatisfiable* iff $0 \geq 1$ is derivable from Φ by resolution

- $x + 2y > 10$,
- $-x > 5$,
- $-y > 3$
- $0 = (x + 2y - x - 2y) > (10 + 5 + 2 \cdot 3) > 21$

Proof uses *Farkas* coefficients $g_1, \dots, g_n$ such that

- $g_i > 0$
- $g_1 \cdot t_1 + \dots + g_n \cdot t_n = 0$
- $g_1 \cdot b_1 + \dots + g_n \cdot b_n > 1$

Frakas Lemma Example

Interpolants

$$z + x + y > 10 \quad \times 1$$

$$-z > -5 \quad \times 1$$

$$\left. \begin{array}{l} z + x + y > 10 \\ -z > -5 \end{array} \right\} x + y > 5$$

$$-x > 5 \quad \times 1$$

$$-y > 3 \quad \times 1$$

$$\left. \begin{array}{l} -x > 5 \\ -y > 3 \end{array} \right\} x + y < -8$$

$$0 > 13$$

Interpolation for Linear Real Arithmetic

Let $A \wedge B$ be UNSAT, where

- $A = t_1 \geq b_1 \wedge \dots \wedge t_i \geq b_i$, and
- $B = t_{i+1} \geq b_{i+1} \wedge \dots \wedge t_n \geq b_n$

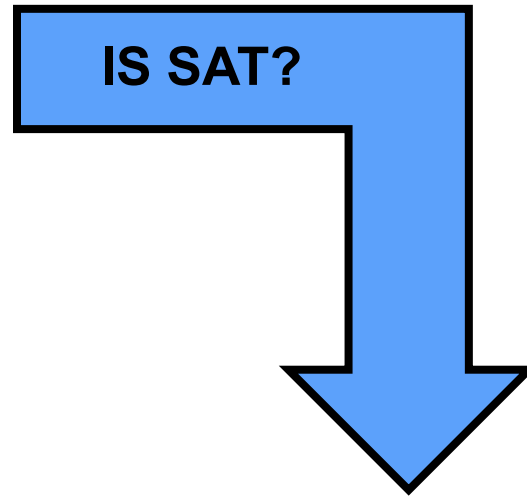
Let $g_1, \dots, g_n$ be the Farkas coefficients witnessing UNSAT

Then

- $g_1 \cdot (t_1 - b_1) + \dots + g_i \cdot (t_i - b_i) \geq 0$ is an interpolant between A and B
- $g_{i+1} \cdot (t_{i+1} - b_{i+1}) + \dots + g_n \cdot (t_n - b_n) \geq 0$ is an interpolant between B and A

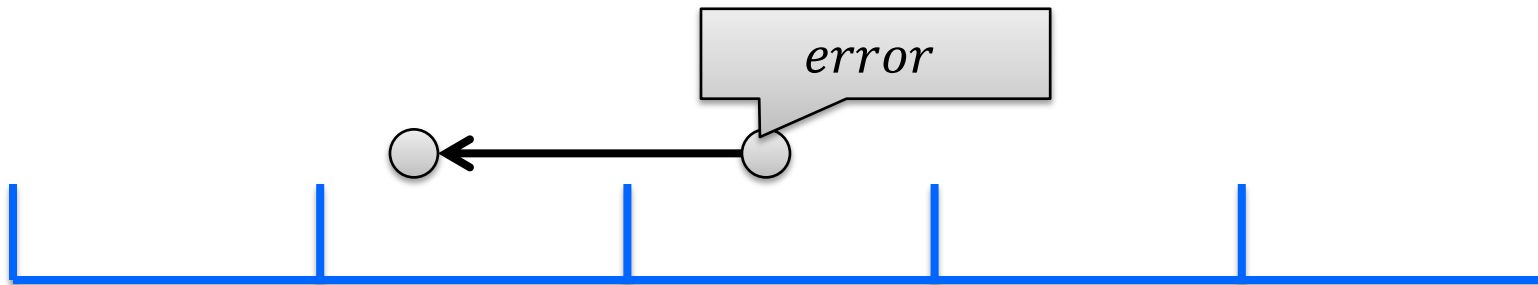
Program Verification with HORN(LIA)

```
z = x; i = 0;  
assume (y > 0);  
while (i < y) {  
    z = z + 1;  
    i = i + 1;  
}  
assert(z == x + y);
```



$z = x \ \& \ i = 0 \ \& \ y > 0$	$\rightarrow$	$\text{Inv}(x, y, z, i)$
$\text{Inv}(x, y, z, i) \ \& \ i < y \ \& \ z1=z+1 \ \& \ i1=i+1$	$\rightarrow$	$\text{Inv}(x, y, z1, i1)$
$\text{Inv}(x, y, z, i) \ \& \ i \geq y \ \& \ z \neq x+y$	$\rightarrow$	false

Lemma Generation Example



Transition Relation

$$x = x_0 \wedge z = z_0 + 1 \wedge i = i_0 + 1 \wedge y > i_0$$

Pob

$$i \geq y \wedge x + y > z$$

Farkas explanation for unsat

$$x_0 + y_0 \leq z_0, x \leq x_0, z_0 < z, i \leq i_0 + 1$$

$$x + i \leq z$$

$$i \geq y, x + y > z$$

$$x + i > z$$

false

Learn lemma:

$$x + i \leq z$$

Interpolation Problem in Spacer

Given an arbitrary LRA formula A and a conjunction of literals s such that $A \wedge s$ are UNSAT, compute an interpolant I such that

- $s \Rightarrow I$ $I \wedge A \Rightarrow \text{FALSE}$ I is over symbols common to s and A

Use an SMT solver to decide that $s \wedge A$ are UNSAT

- SMT solver uses LRA theory lemmas (called Farkas Theory Lemmas) of the form:
 $\neg ((s_1 \wedge \dots \wedge s_k) \wedge (a_1 \wedge \dots \wedge a_m))$
where s_i are literals from s and a_i are literals from A
- For each such lemma L_j , $((s_1 \wedge \dots \wedge s_k) \wedge (a_1 \wedge \dots \wedge a_m))$ is UNSAT
- Let t_j be an interpolant corresponding to L_j

Then, an interpolant between s and A is a clause of the form

$(\neg t_1 \vee \dots \vee \neg t_k)$ with one literal per each theory lemma

- in practice, interpolation is optimized by examining and restructuring SMT resolution proof, dealing with Boolean reasoning, and global optimization

Computing Interpolants in Spacer

Much simpler than general interpolation problem for $A \wedge B$

- B is always a conjunction of literals
- A is dynamically split into DNF by the SMT solver
- DPLL(T) proofs do not introduce new literals

Interpolation algorithm is reduced to analyzing all theory lemmas in a DPLL(T) proof produced by the solver

- every theory-lemma that mixes B-pure literals with other literals is interpolated to produce a single literal in the final solution
- interpolation is restricted to clauses of the form $(\wedge B_i \Rightarrow \vee A_j)$

Interpolating (UNSAT) Cores

- improve interpolation algorithms and definitions to the specific case of PDR
- classical interpolation focuses on eliminating non-shared literals
- in PDR, the focus is on finding good generalizations

$$\begin{aligned} s &\subseteq pre(cex) \\ &\equiv \\ s &\Rightarrow \exists X'. Tr(X, X') \wedge cex(X') \end{aligned}$$

Computing a predecessor s of a counterexample cex

DECIDE (ARITHMETIC)

Model Based Projection

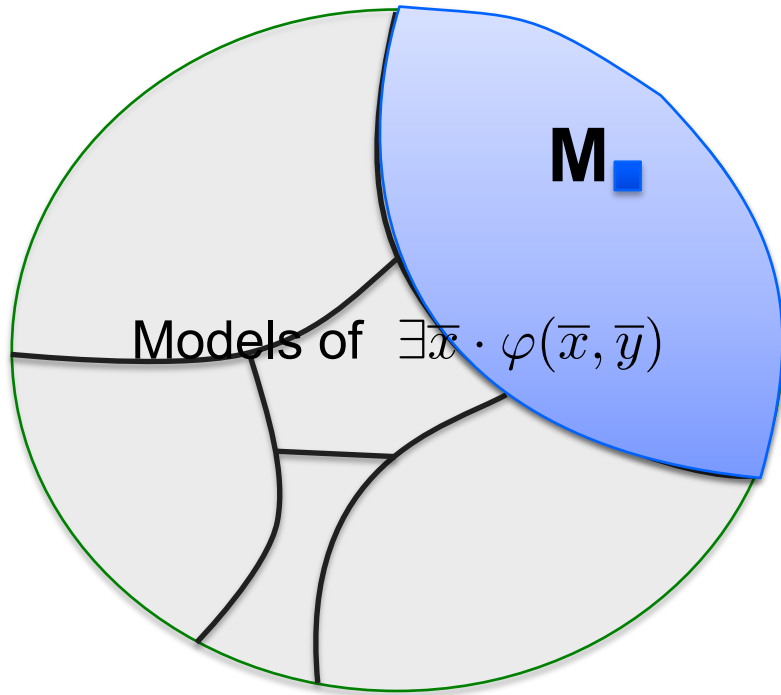
Definition: Let φ be a formula, X a set of variables, and M a model of φ . Then $\psi = MBP(X, M, \varphi)$ is a Model Based Projection of X, M, φ iff

1. ψ is a monomial
2. $Vars(\psi) \subseteq Vars(\varphi) \setminus X$
3. $M \models \psi$
4. $\psi \Rightarrow \exists X. \varphi$

Model Based Projection under-approximates existential quantifier elimination relative to a given model (i.e., satisfying assignment)

Model Based Projection

Expensive to find a quantifier-free $\psi(\bar{y}) \equiv \exists \bar{x} \cdot \varphi(\bar{x}, \bar{y})$



1. Find model M of $\varphi(x,y)$

2. Compute a partition containing M

Quantifier Elimination

Quantifier elimination procedure:

- **Input:** formula $\exists x \psi(x)$
- **Output:** equivalent φ without existential quantifier. x is eliminated.
- $\text{QELIM}(\exists x \psi(x)) = \varphi$ and $\exists x \psi(x) \Leftrightarrow \varphi$

Quantifier elimination in propositional logic

- $\text{QELIM}(\exists x \psi(x)) = \psi(\text{TRUE}) \vee \psi(\text{FALSE})$

Many theories support quantifier elimination (e.g., linear arithmetic)

- but not all. No quantifier elimination for EUF,
 - e.g., $(\exists x f(x) \neq g(x))$ cannot be expressed without the existential quantifier

Quantifier elimination is usually expensive

- e.g., propositional QELIM is exponential in the number of variables quantified

Loos-Weispfenning Quantifier Elimination for LRA

ϕ is LRA formula in Negation Normal Form

E is set of $x=t$ atoms, U set of $x < t$ atoms, and L set of $s < x$ atoms

There are no other occurrences of x in $\phi[x]$

$$\exists x. \varphi[x] \equiv \varphi[\infty] \vee \bigvee_{x=t \in E} \varphi[t] \vee \bigvee_{x < t \in U} \varphi[t - \epsilon]$$

where

$$(x < t')[t - \epsilon] \equiv t \leq t' \quad (s < x)[t - \epsilon] \equiv s < t \quad (x = e)[t - \epsilon] \equiv \text{false}$$

The case of lower bounds is dual

- using $-\infty$ and $t+\epsilon$

Fourier–Motzkin Quantifier Elimination for LRA

$$\begin{aligned} & \exists x \cdot \bigwedge_i s_i < x \wedge \bigwedge_j x < t_j \\ &= \bigwedge_i \bigwedge_j \text{resolve}(s_i < x, x < t_j, x) \\ &= \bigwedge_i \bigwedge_j s_i < t_j \end{aligned}$$

Quadratic increase in the formula size per each eliminated variable

Quantifier Elimination with Assumptions

$$\begin{aligned} & \left(\bigwedge_{j \neq 0} t_0 \leq t_j \right) \wedge \exists x \cdot \bigwedge_i s_i < x \wedge \bigwedge_j x < t_j \\ = & \left(\bigwedge_{j \neq 0} t_0 \leq t_j \right) \wedge \bigwedge_i \text{resolve}(s_i < x, x < t_0, x) \\ = & \left(\bigwedge_{j \neq 0} t_0 \leq t_j \right) \wedge \bigwedge_i s_i < t_0 \end{aligned}$$

Quantifier elimination is simplified by a choice of a minimal upper bound

- For each choice of minimal upper bound, no increase in term size
- Dually, can use largest lower bound

How to choose the assumptions?!

- MBP == use the order chosen by the model

MBP for Linear Rational Arithmetic

Compute a **single** disjunct from LW-QE that includes the model

- Use the Model to uniquely pick a substitution term for x

$$Mbp_x(M, x = s \wedge L) = L[x \leftarrow s]$$

$$Mbp_x(M, x \neq s \wedge L) = Mbp_x(M, s < x \wedge L) \text{ if } M(x) > M(s)$$

$$Mbp_x(M, x \neq s \wedge L) = Mbp_x(M, -s < -x \wedge L) \text{ if } M(x) < M(s)$$

$$Mbp_x(M, \bigwedge_i s_i < x \wedge \bigwedge_j x < t_j) = \bigwedge_i s_i < t_0 \wedge \bigwedge_j t_0 \leq t_j \text{ where } M(t_0) \leq M(t_i), \forall i$$

MBP techniques have been developed for

- Linear Rational Arithmetic, Linear Integer Arithmetic
- Theories of Arrays, and Recursive Data Types

Arithmetic Decide

Notation: $\mathcal{F}(A) = (A(X) \wedge Tr(X, X') \vee Init(X'))$.

Decide If $\langle P, i+1 \rangle \in Q$ and there is a model $m(X, X')$ s.t. $m \models \mathcal{F}(F_i) \wedge P'$,
add $\langle P_{\downarrow}, i \rangle$ to Q , where $P_{\downarrow} = MBP(X', m, \mathcal{F}(F_i) \wedge P')$.

Compute a predecessor using Model Based Projection

To ensure progress, Decide must be finite

- finitely many possible predecessors when all other arguments are fixed

Alternatively

- Completeness can follow from an interaction of **Decide** and **Conflict**
 - but requires more rules to propagate implicants backward (as in PDR) and forward (as in Spacer and Quip)

PolyPDR: Solving CHC(LRA)

Unreachable and Reachable

- terminate the algorithm when a solution is found

Unfold

- increase search bound by 1

Candidate

- choose a bad state in the last frame

Decide

- extend a cex (backward) consistent with the current frame
- find a model $\mathbf{M}$ of $\mathbf{s}$ s.t. $(F_i \wedge \text{Tr} \wedge \text{cex}')$, and let $\mathbf{s} = \text{MBP}(X', F_i \wedge \text{Tr} \wedge \text{cex}')$

Conflict

- construct a lemma to explain why cex cannot be extended
- Find an interpolant L s.t. $L \Rightarrow \neg \text{cex}$, $\text{Init} \Rightarrow L$, and $F_i \wedge \text{Tr} \Rightarrow L'$

Induction

- propagate a lemma as far into the future as possible

Non-Linear CHC Satisfiability

Satisfiability of a set of arbitrary (i.e., linear or non-linear) CHCs is reducible to satisfiability of THREE (3) clauses of the form

$$\mathit{Init}(X) \rightarrow P(X)$$

$$P(X) \wedge P(X^o) \wedge \mathit{Tr}(X, X^o, X') \rightarrow P(X')$$

$$P(X) \rightarrow \neg \mathit{Bad}(X)$$

where, $X' = \{x' \mid x \in X\}$, $X^o = \{x^o \mid x \in X\}$, P a fresh predicate, and Init , Bad , and Tr are constraints

Generalized GPDR

Input: A safety problem $\langle \text{Init}(X), \text{Tr}(X, X^o, X'), \text{Bad}(X) \rangle$.

Output: *Unreachable* or *Reachable*

Data: A cex queue Q , where a cex $\langle c_0, \dots, c_k \rangle \in Q$ is a tuple, each $c_j = \langle m, i \rangle$, m is a cube over state variables, and $i \in \mathbb{N}$. A level N .
A trace $F_0, F_1, \dots$.

Notation: $\mathcal{F}(A, B) = \text{Init}(X') \vee (A(X) \wedge B(X^o) \wedge \text{Tr})$, and $\mathcal{F}(A) = \mathcal{F}(A, A)$

Initially: $Q = \emptyset$, $N = 0$, $F_0 = \text{Init}$, $\forall i > 0 \cdot F_i = \emptyset$

Require: $\text{Init} \rightarrow \neg \text{Bad}$

repeat

Unreachable If there is an $i < N$ s.t. $F_i \subseteq F_{i+1}$ **return** *Unreachable*.

Reachable if exists $t \in Q$ s.t. for all $\langle c, i \rangle \in t$, $i = 0$, **return** *Reachable*.

Unfold If $F_N \rightarrow \neg \text{Bad}$, then set $N \leftarrow N + 1$ and $Q \leftarrow \emptyset$.

Candidate If for some m , $m \rightarrow F_N \wedge \text{Bad}$, then add $\langle \langle m, N \rangle \rangle$ to Q .

Decide If there is a $t \in Q$, with $c = \langle m, i + 1 \rangle \in t$, $m_1 \rightarrow m$, $l_0 \wedge m_0^o \wedge m'_1$ is satisfiable, and $l_0 \wedge m_0^o \wedge m'_1 \rightarrow F_i \wedge F_i^o \wedge \text{Tr} \wedge m'$ then add $\hat{t}$ to Q , where $\hat{t} = t$ with c replaced by two tuples $\langle l_0, i \rangle$, and $\langle m_0, i \rangle$.

Conflict If there is a $t \in Q$ with $c = \langle m, i + 1 \rangle \in t$, s.t. $\mathcal{F}(F_i) \wedge m'$ is unsatisfiable. Then, add $\varphi = \text{ITP}(\mathcal{F}(F_i), m')$ to F_j , for all $0 \leq j \leq i + 1$.

Leaf If there is $t \in Q$ with $c = \langle m, i \rangle \in t$, $0 < i < N$ and $\mathcal{F}(F_{i-1}) \wedge m'$ is unsatisfiable, then add $\hat{t}$ to Q , where $\hat{t}$ is t with c replaced by $\langle m, i + 1 \rangle$.

Induction For $0 \leq i < N$ and a clause $(\varphi \vee \psi) \in F_i$, if $\varphi \notin F_{i+1}$, $\mathcal{F}(\phi \wedge F_i) \rightarrow \phi'$, then add φ to F_j , for all $j \leq i + 1$.

until ∞ ;

counterexample
is a tree

two
predecessors

theory-aware
Conflict

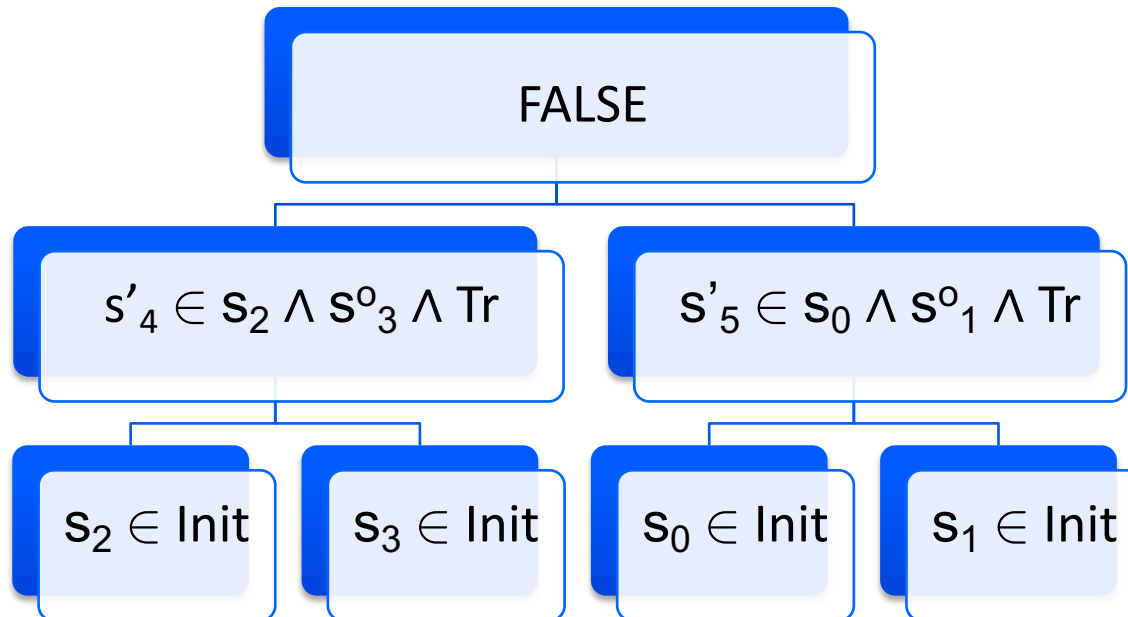
Counterexamples to non-linear CHC

A set S of CHC is unsatisfiable iff S can derive FALSE

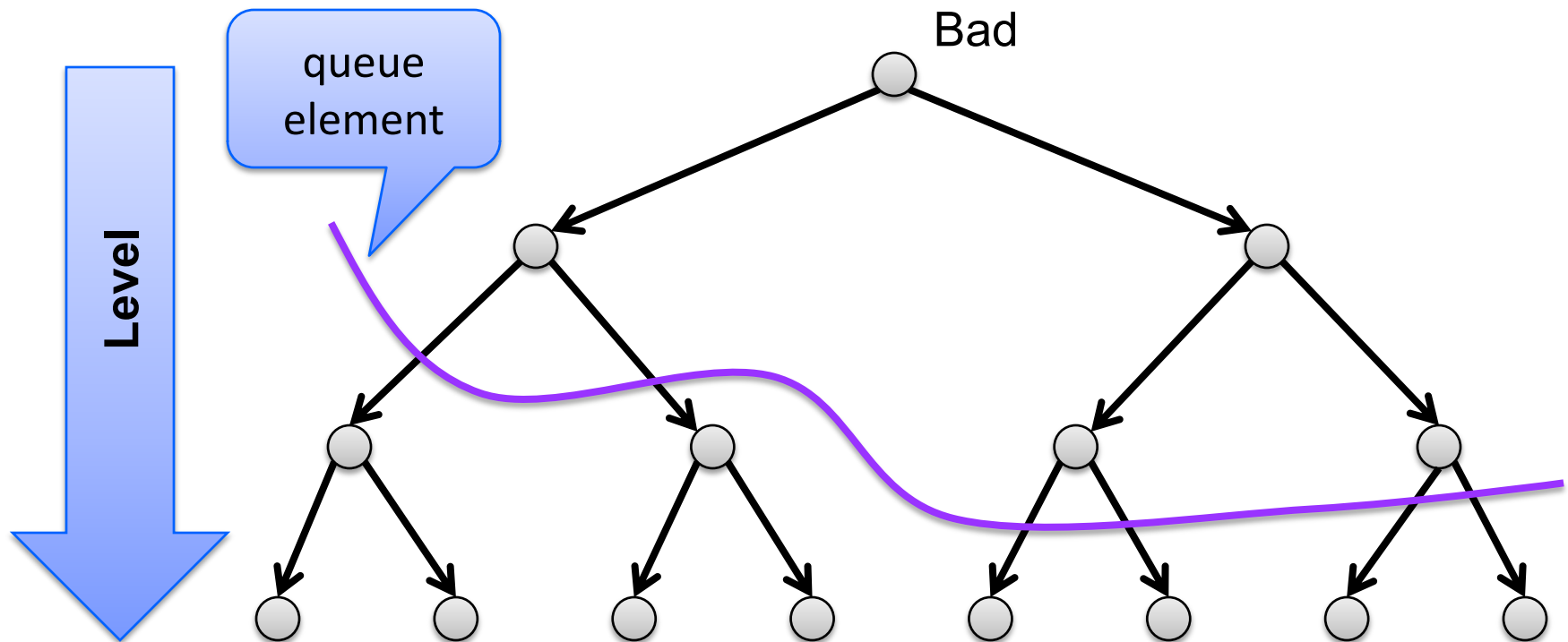
- we call such a derivation a counterexample

For linear CHC, the counterexample is a path

For non-linear CHC, the counterexample is a tree



GPDR Search Space



In Decide, one POB in the frontier is chosen and its two children are expanded

GPDR: Splitting predecessors

Consider a clause

$$P(x) \wedge P(y) \wedge x > y \wedge z = x + y \implies P(z)$$

How to compute a predecessor for a proof obligation $z > 0$

Predecessor over the constraint is:

$$\begin{aligned} & \exists z \cdot x > y \wedge z = x + y \wedge z > 0 \\ = & x > y \wedge x + y > 0 \end{aligned}$$

Need to create two separate proof obligation

- one for $P(x)$ and one for $P(y)$
- gpdr solution: split by substituting values from the model (incomplete)

GPDR: Deciding predecessors

Decide If there is a $t \in Q$, with $c = \langle m, i + 1 \rangle \in t$, $m_1 \rightarrow m$, $l_0 \wedge m_0^o \wedge m'_1$ is satisfiable, and $l_0 \wedge m_0^o \wedge m'_1 \rightarrow F_i \wedge F_i^o \wedge Tr \wedge m'$ then add $\hat{t}$ to Q , where $\hat{t} = t$ with c replaced by two tuples $\langle l_0, i \rangle$, and $\langle m_0, i \rangle$.

Compute two predecessors at each application of **GPDR/Decide**

Can explore both predecessors in parallel

- e.g., BFS or DFS exploration order

Number of predecessors is unbounded

- incomplete even for finite problem (i.e., non-recursive CHC)

No caching/summarization of previous decisions

- worst-case exponential for Boolean Push-Down Systems

Spacer

Same queue as
in IC3/PDR

Cache Reachable
states

Three variants of
Decide

Same **Conflict** as
in APDR/GPDR

Input: A safety problem $\langle \text{Init}(X), \text{Tr}(X, X^o, X'), \text{Bad}(X) \rangle$.

Output: *Unreachable* or *Reachable*

Data: A cex queue Q , where a cex $c \in Q$ is a pair $\langle m, i \rangle$, m is a cube over state variables, and $i \in \mathbb{N}$. A level N . A set of reachable states REACH . A trace $F_0, F_1, \dots$

Notation: $\mathcal{F}(A, B) = \text{Init}(X') \vee (A(X) \wedge B(X^o) \wedge \text{Tr})$, and $\mathcal{F}(A) = \mathcal{F}(A, A)$

Initially: $Q = \emptyset, N = 0, F_0 = \text{Init}, \forall i > 0 \cdot F_i = \emptyset, \text{REACH} = \text{Init}$

Require: $\text{Init} \rightarrow \neg \text{Bad}$

repeat

Unreachable If there is an $i < N$ s.t. $F_i \subseteq F_{i+1}$ **return** *Unreachable*.

Reachable If $\text{REACH} \wedge \text{Bad}$ is satisfiable, **return** *Reachable*.

Unfold If $F_N \rightarrow \neg \text{Bad}$, then set $N \leftarrow N + 1$ and $Q \leftarrow \emptyset$.

Candidate If for some $m, m \rightarrow F_N \wedge \text{Bad}$, then add $\langle m, N \rangle$ to Q .

Successor If there is $\langle m, i + 1 \rangle \in Q$ and a model $M \models \psi$, where $\psi = \mathcal{F}(\vee \text{REACH}) \wedge m'$. Then, add s to REACH , where $s' \in \text{MBP}(\{X, X^o\}, \psi)$.

DecideMust If there is $\langle m, i + 1 \rangle \in Q$, and a model $M \models \psi$, where $\psi = \mathcal{F}(F_i, \vee \text{REACH}) \wedge m'$. Then, add s to Q , where $s \in \text{MBP}(\{X^o, X'\}, \psi)$.

DecideMay If there is $\langle m, i + 1 \rangle \in Q$ and a model $M \models \psi$, where $\psi = \mathcal{F}(F_i) \wedge m'$. Then, add s to Q , where $s^o \in \text{MBP}(\{X, X'\}, \psi)$.

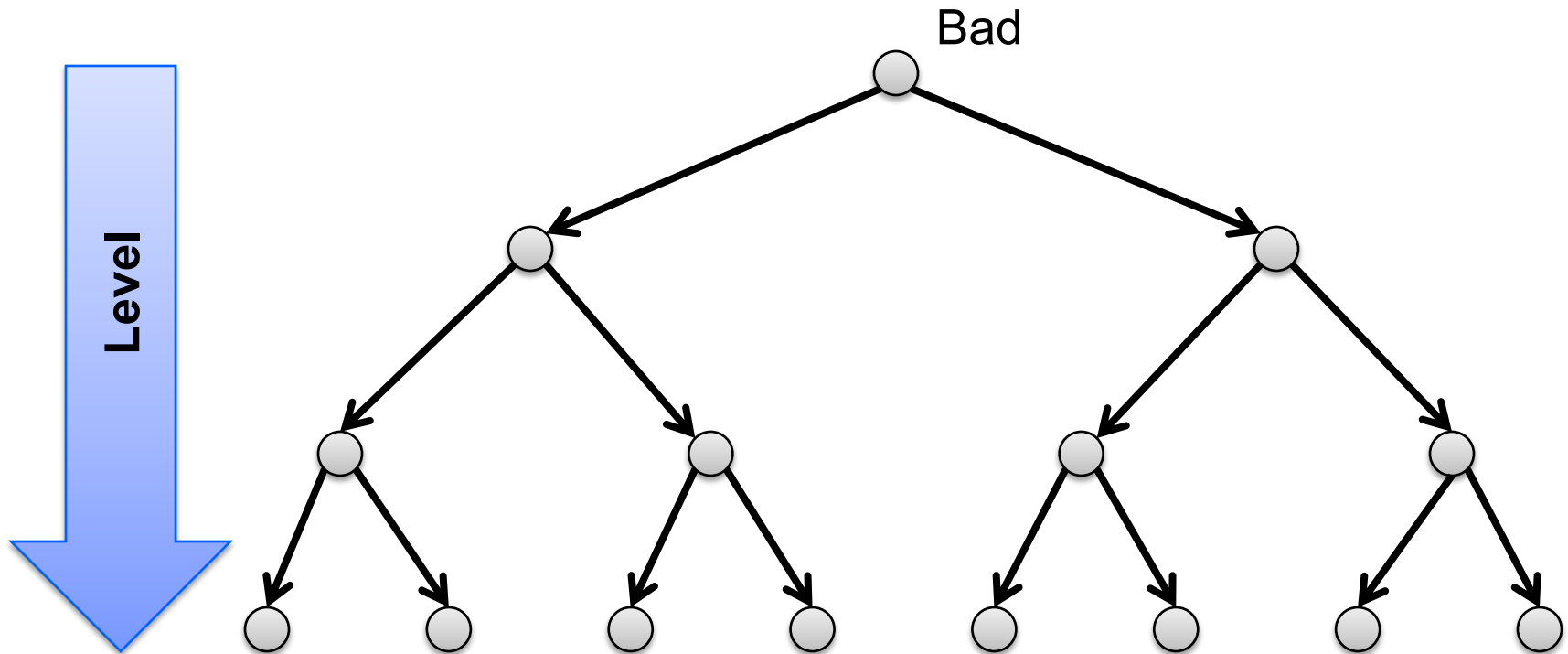
Conflict If there is an $\langle m, i + 1 \rangle \in Q$, s.t. $\mathcal{F}(F_i) \wedge m'$ is unsatisfiable. Then, add $\varphi = \text{ITP}(\mathcal{F}(F_i), m')$ to F_j , for all $0 \leq j \leq i + 1$.

Leaf If $\langle m, i \rangle \in Q, 0 < i < N$ and $\mathcal{F}(F_{i-1}) \wedge m'$ is unsatisfiable, then add $\langle m, i + 1 \rangle$ to Q .

Induction For $0 \leq i < N$ and a clause $(\varphi \vee \psi) \in F_i$, if $\varphi \notin F_{i+1}$, $\mathcal{F}(\phi \wedge F_i) \rightarrow \phi'$, then add φ to F_j , for all $j \leq i + 1$.

until ∞ ;

SPACER Search Space



In Decide, unfold the derivation tree in a fixed depth-first order

- use MBP to decide on counterexamples

Successor: Learn new facts (reachable states) on the way up

- use MBP to propagate facts bottom up

Successor Rule: Computing Reachable States

Successor If there is $\langle m, i + 1 \rangle \in Q$ and a model $M \models \psi$, where $\psi = \mathcal{F}(\text{REACH}) \wedge m'$. Then, add s to REACH, where $s' \in \text{MBP}(\{X, X^o\}, \psi)$.

Computing new reachable states by under-approximating forward image using MBP

- since MBP is finite, guarantee to exhaust all reachable states

Second use of MBP

- orthogonal to the use of MBP in Decide
- can allow REACH to contain auxiliary variables, but this might explode

For Boolean CHC, the number of reachable states is bounded

- complexity is polynomial in the number of states
- same as reachability in Push Down Systems

Decide Rule: Must and May refinement

DecideMust If there is $\langle m, i + 1 \rangle \in Q$, and a model $M \models \psi$, where $\psi = \mathcal{F}(F_i, \text{REACH}) \wedge m'$. Then, add s to Q , where $s \in \text{MBP}(\{X^o, X'\}, \psi)$.

DecideMay If there is $\langle m, i + 1 \rangle \in Q$ and a model $M \models \psi$, where $\psi = \mathcal{F}(F_i) \wedge m'$. Then, add s to Q , where $s^o \in \text{MBP}(\{X, X'\}, \psi)$.

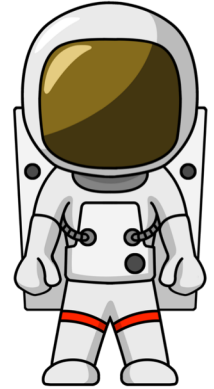
DecideMust

- use computed summary (REACH) to skip over a call site

DecideMay

- use over-approximation of a calling context to guess an approximation of the call-site
- the call-site either refutes the approximation (**Conflict**) or refines it with a witness (**Successor**)

Art, Science, and Magic



Verification of Safety Properties is FOL satisfiability

- Logic: Constrained Horn Clauses (CHC)
- “Decision” procedure: Spacer
- Now with (universal) quantifiers!

Art: finding the right encoding from the problem domain to logic

- the difference between easy to impossible
- encodings can “simulate” specialized algorithms

Science: Progress, termination (when decidable)

- while the underlying problem is undecidable, many fragment or sub-problems are decidable

Magic: actually solving useful problems

- interpolation, heuristics, generalizations, ...
- the list is endless

THE END