

Foundations: Syntax, Semantics, and Graphs

Testing, Quality Assurance, and Maintenance
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based on slides by Ruzica Pizkac, Claire Le Goues, Lin Tan, Marsha Chechik,
and others



Foundations

Syntax

- Syntax and BNF Grammar
- Abstract Syntax Trees (AST)

Semantics

- Natural Operational Semantics (a.k.a. big step)
- Structural Operational Semantics (a.k.a. small step)
- Judgements and derivations

Graphs

- Graph, cyclic, acyclic
- Nodes, edges, paths
- Trees, sub-graphs, sub-paths, ...
- Control Flow Graph (CFG)

SYNTAX

WHILE: A Simple Imperative Language

We will use a simple imperative language called WHILE

- *the language is also sometimes called IMP*

An example WHILE program:

```
{ p := 0; x := 1; n := 2 };  
while x ≤ n do {  
    x := x + 1;  
    p := p + m  
} ;  
print_state
```

‘;’ is a connective, not terminator as in C!

WHILE: Syntactic Entities

$n \in \mathbb{Z}$

– integers

$\text{true}, \text{false} \in \mathbb{B}$

– Booleans

$x, y \in L$

– locations (program variables)

} Terminal

$e \in Aexp$

– arithmetic expressions

$b \in Bexp$

– Boolean expressions

$c \in Stmt$

– statements

} Non-terminal

Terminals are atomic entities that are completely defined by their tokens

- integers, Booleans, and locations are terminals

Non-Terminals are composed of one or more terminals

- determined by rules of the **grammar**
- $Aexp$, $Bexp$, and $Stmt$ are non-terminals

WHILE: Syntax of Arithmetic Expressions

Arithmetic expressions (Aexp)

$e ::=$

n	for $n \in \mathbb{Z}$
$ -n$	for $n \in \mathbb{Z}$
$ x$	for $x \in L$
$ e_1 \text{ aop } e_2$	
$ '(' e ')'$	

BNF
grammar
rules

$\text{aop} ::= '*' \mid '/' \mid '-' \mid '+'$

Notes:

- Variables are not declared before use
- All variables have integer type
- Expressions have no side-effects

BNF: https://en.wikipedia.org/wiki/Backus%E2%80%93Naur_form

WHILE: Syntax of Boolean Expressions

Boolean expressions (Bexp)

$b ::=$
 'true'
 $| \text{'false'}$
 $| \text{'not' } b$
 $| e_1 \text{ rop } e_2$ for $e_1, e_2 \in \text{Aexp}$
 $| e_1 \text{ bop } b_2$ for $e_1, e_2 \in \text{Bexp}$
 $| \text{'(' } b \text{'})'$

$\text{rop} ::= \text{'<'} \mid \text{'<='} \mid \text{'='} \mid \text{'>='} \mid \text{'>'}$

$\text{bop} ::= \text{'and'} \mid \text{'or'}$

Syntax of Statements

Statements

$s ::=$ skip
| $x := e$
| if b then s [else s]
| while b do s
| '{' slist ' $\}$ '
| print_state
| assert b | assume b | havoc v_1, \dots, v_N

slist $::= s (';' s)^*$

prog $::=$ slist

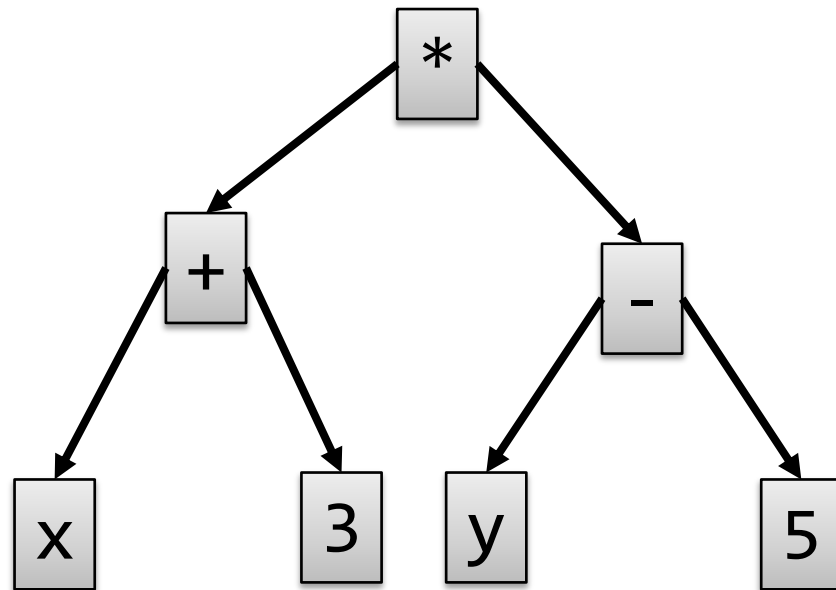
Notes:

- Semi-colon ';' is a statement composition, not statement terminator!!!
- Statements contain all the side-effects in the language
- Many usual features of a PL are missing: references, function calls, ...
 - the language is very very simple yet hard to analyze

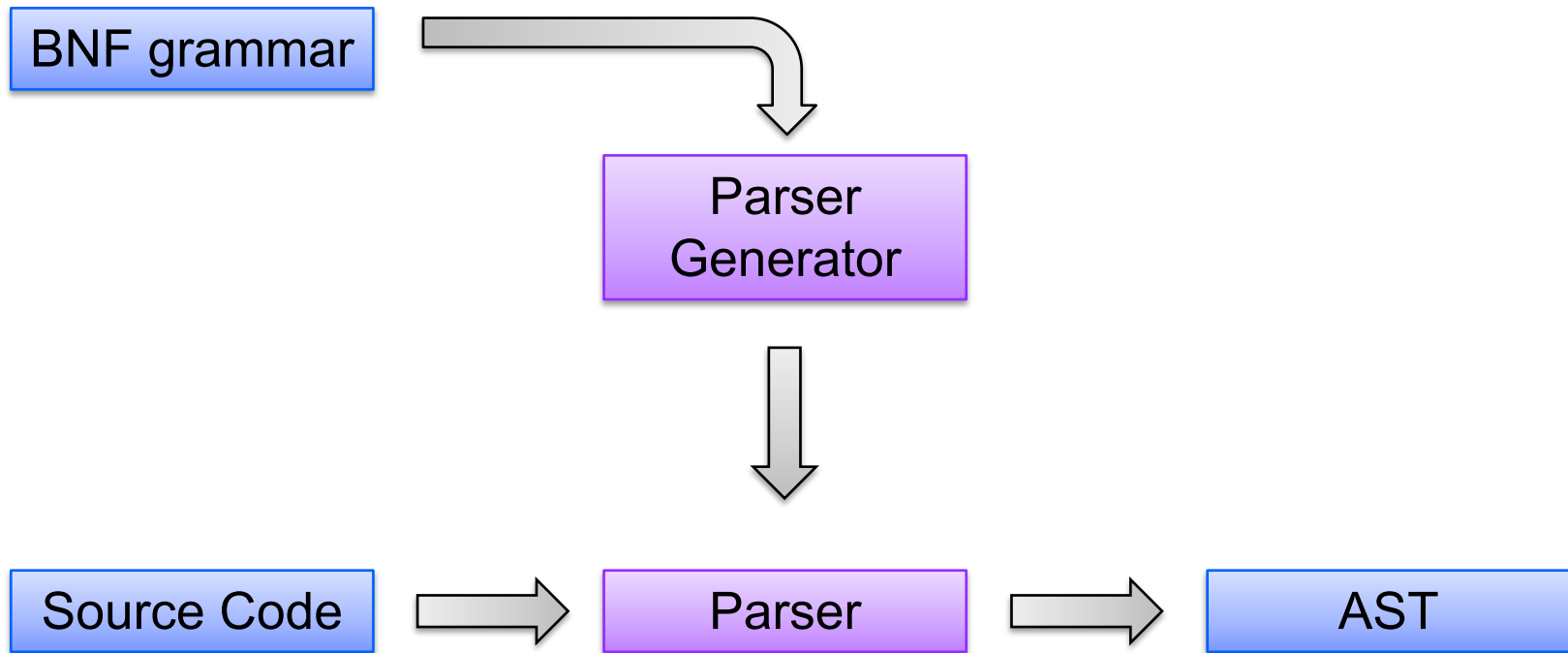
Abstract Syntax Tree (AST)

AST is an abstract tree representation of the source code

- each node represents a syntactic construct occurring in the code
 - statement, variable, operator, statement list
- called “**abstract**” because some details of concrete syntax are omitted
 - AST normalizes (provides common representation) of irrelevant differences in syntax (e.g., white space, comments, order of operations)
- example AST: $(x + 3) * (y - 5)$



Language Parsing in a Nutshell



Parser generator

- input: BNF grammar; output: parser (program)

Parser

- input: program source code; output: AST or error

WHILE AST in Python

One class per
syntactic entity

One field per child

Class hierarchy
corresponds to the
semantic one

```
Emacs-x86_64-10_9 Prelude - /tmp/ast.py
class Ast(object):
    """Base class of AST hierarchy"""
    pass

class Stmt (Ast):
    """A single statement"""
    pass

class AsgnStmt (Stmt):
    """An assignment statement"""
    def __init__(self, lhs, rhs):
        self.lhs = lhs
        self.rhs = rhs

class IfStmt (Stmt):
    """If-then-else statement"""
    def __init__(self, cond, then_stmt, else_stmt=None):
        self.cond = cond
        self.then_stmt = then_stmt
        self.else_stmt = else_stmt
```

Behavior Pattern: Visitor

Applicability

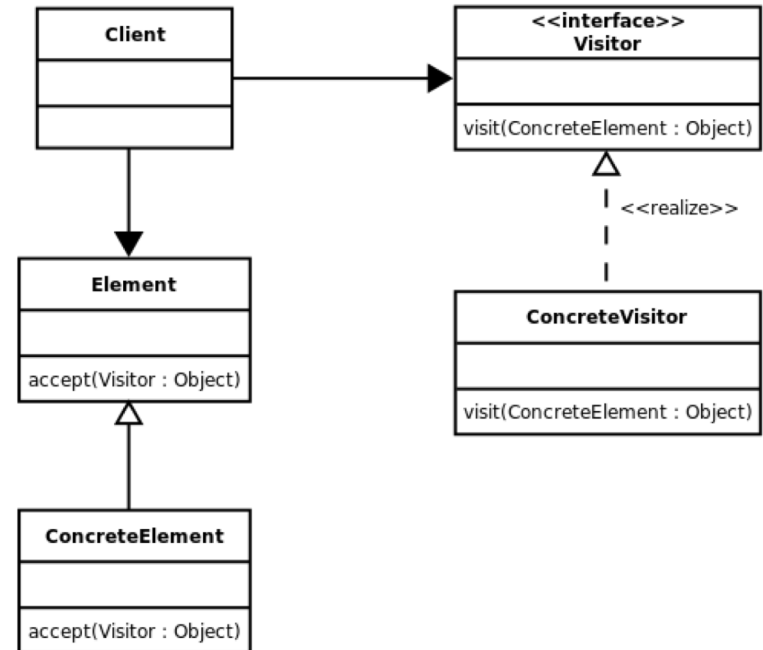
- Object hierarchy with many classes
- Operations depend on classes
- Set of classes is stable
- Want to define new operations

Consequences

- Simplifies adding new operations
- Groups related behavior in one class
- Extending class hierarchy is difficult
- Visitor can maintain state
- **Element** must expose interface

In Python

- Method name is used instead of polymorphism, e.g., `visit_stmt()`
- Visitor's `visit()` method dispatches calls based on reflection. No need for `accept()`



Example Visitor in Python

```
Emacs-x86_64-10.9 Prelude - /tmp/ast.py
class AstVisitor(object):
class AstVisitor(object):
    """Base class for AST visitor"""
    def __init__(self):
        pass

    def visit(self, node, *args, **kwargs):
        """Visit a node."""
        method = 'visit_' + node.__class__.__name__
        visitor = getattr(self, method)
        return visitor(node, *args, **kwargs)

    def visit_BoolConst(self, node, *args, **kwargs):
        visitor = getattr(self, 'visit_' + Const.__name__)
        return visitor(node, *args, **kwargs)
```

```
class PrintVisitor (AstVisitor):
    """A printing visitor"""
    def visit_IntVar (self, node, *args, **kwargs):
        self._write (node.name)

    def visit_Const (self, node, *args, **kwargs):
        self._write (node.val)

    def visit_Exp (self, node, *args, **kwargs):
        if node.is_unary ():
            self._write (node.op)
            self.visit (node.arg (0))
        else:
            self._open_brkt (**kwargs)
            self.visit (node.arg (0))
            for a in node.args [1:]:
                self._write (' ')
                self._write (node.op)
                self._write (' ')
                self.visit (a)
            self._close_brkt (**kwargs)
```

Exercise: Implement a state counting visitor

Write a visitor that counts the number of statements in a program

- (a) Implementation 1:
 - the visitor should be stateless and return the number of statements
- (b) Implementation 2:
 - uses an internal state (field) to keep track of the number of statements

Stateless Visitor

```
class StmtCounterStateless (wlang.ast.AstVisitor):
    def __init__ (self):
        super (StmtCounterStateless, self).__init__ ()

    def visit_StmtList (self, node, *args, **kwargs):
        if node.stmts is None:
            return 0
        res = 0
        for s in node.stmts:
            res = res + self.visit (s)
        return res

    def visit_IfStmt (self, node, *args, **kwargs):
        res = 1 + self.visit (node.then_stmt)
        if node.has_else ():
            res = res + self.visit (node.else_stmt)
        return res

    def visit_WhileStmt (self, node, *args, **kwargs):
        return 1 + self.visit (node.body)

    def visit_Stmt (self, node, *args, **kwargs):
        return 1
```

Statefull Visitor

```
class StmtCounterStatefull (wlang.ast.AstVisitor):
    def __init__ (self):
        super (StmtCounterStatefull, self).__init__ ()
        self._count = 0

    def get_num_stmts (self):
        return self._count

    def count (self, node, *args, **kwargs):
        self._count = 0
        self.visit (node, *args, **kwargs)

    def visit_StmtList (self, node, *args, **kwargs):
        if node.stmts is None:
            return
        for s in node.stmts:
            self.visit (s)

    def visit_Stmt (self, node, *args, **kwargs):
        self._count = self._count + 1
    def visit_IfStmt (self, node, *args, **kwargs):
        self.visit_Stmt (node)
        self.visit (node.then_stmt)
        if node.has_else ():
            self.visit (node.else_stmt)

    def visit_WhileStmt (self, node, *args, **kwargs):
        self.visit_Stmt (node)
        self.visit (node.body)
```

Tutorial Today @ 7pm in E7 5353

Today at 7pm in E7 5353

Topics

- Docker
- Python
- AST
- Visitors

From Programming to Modeling

Extend a programming language with 3 modeling features

Assertions

- `assert e` – aborts an execution when `e` is false, no-op otherwise

```
void assert (bool b) { if (!b) error(); }
```

Non-determinism

- `havoc x` – assigns variable `x` a non-deterministic value

```
void havoc(int &x) { int y; x = y; }
```

Assumptions

- `assume e` – blocks execution if `e` is false, noop otherwise

```
void assume (bool e) { while (!e) ; }
```

Safety Specifications as Assertions

A program is **correct** if all executions that satisfy all assumptions also satisfy all assertions

A program is **incorrect** if there exists an execution that satisfies all of the assumptions AND violates at least one an assertion

Assumptions express **pre-conditions** on which the program behavior relies

Assertions express desired **properties** that the program must maintain

Writing Specifications with Assert and Assume

```
int x, y;

void main (void)
{
    havoc (x);
    assume (x > 10);
    assume (x <= 100);

    y = x + 1;

    assert (y > x);
    assert (y < 200);

}
```


Order of Assumptions is IMPORTANT!!!

```
int x, y;

void main (void)
{
    havoc (x);

    y = x + 1;

    assume (x > 10);
    assume (x <= 100);

    assert (y > x);
    assert (y < 200);
}
```

```
int x, y;

void main (void)
{
    havoc (x);

    y = x + 1;

    assert (y > x);
    assert (y < 200);

    assume (x > 10);
    assume (x <= 100);
}
```

Non-determinism vs. Randomness

A *deterministic* function always returns the same result on the same input

- e.g., $F(5) = 10$

A *non-deterministic* function may return different values on the same input

- e.g., $G(5)$ in $[0, 10]$ “ $G(5)$ returns a non-deterministic value between 0 and 10”

A *random* function may choose a different value with a probability distribution

- e.g., $H(5) = (3 \text{ with prob. } 0.3, 4 \text{ with prob. } 0.2, \text{ and } 5 \text{ with prob. } 0.5)$

Non-deterministic choice cannot be implemented!

- used to model the worst possible adversary/environment

SEMANTICS

Reference for Semantics

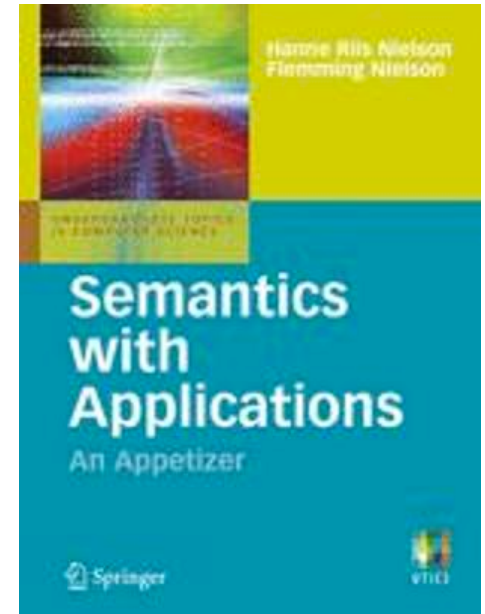
Nielson². Semantics with Applications: An Appetizer

Available FREE at SpringerLink

<https://link.springer.com/book/10.1007/978-1-84628-692-6>

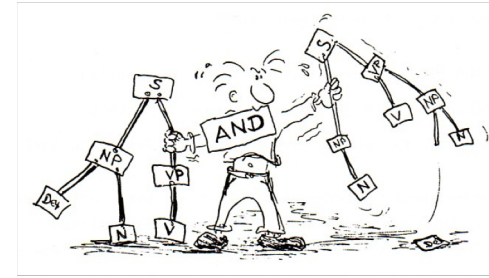
Chapter 1 and Chapter 2

Link on course web page



Syntax and Semantics

Syntax





- MW: the way in which linguistic elements (such as words) are put together to form constituents (such as phrases or clauses)
- Determines and restricts how things are written


Semantics

- MW: the study of meanings
- Determines how syntax is interpreted to give meaning

[[SEMANTICS]]
of a structure

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[[

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Meaning of WHILE Programs

Questions to answer:

- What is the “meaning” of a given WHILE expression/statement?
- How would we evaluate WHILE expressions and statements?
- How are the evaluator and the meaning related?
- How can we reason about the effect of a command?

Semantics of Programming Languages

Denotational Semantics

- Meaning of a program is defined as the mathematical object it computes (e.g., partial functions).
- example: Abstract Interpretation

Axiomatic Semantics

- Meaning of a program is defined in terms of its effect on the truth of logical assertions.
- example: Hoare Logic

Operational Semantics

- Meaning of a program is defined by formalizing the individual computation steps of the program.
- example: Natural (Big-Step) Semantics, Structural (Small-Step) Semantics

Semantics of WHILE

The meaning of WHILE expressions depends on the values of variables, i.e. the **current state**.

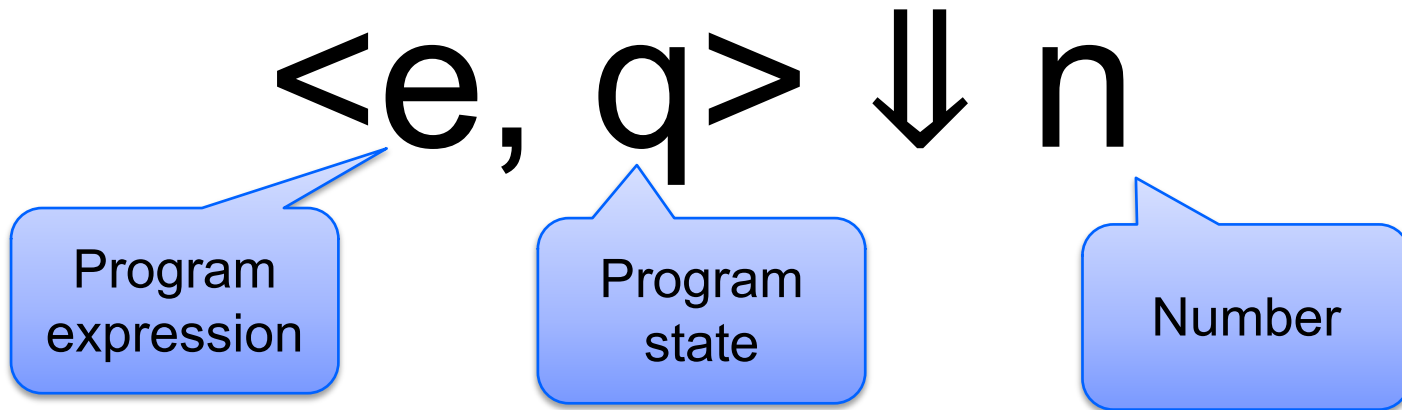
A **state** s is a function from L to Z

- assigns a value for every location/variable
- **notation**: $s(x)$ is the value of variable x in state s

The set of **all states** is $Q = L \rightarrow Z$

We use q to range over Q

Judgement: Natural Semantics



Expression e in state q has a value n

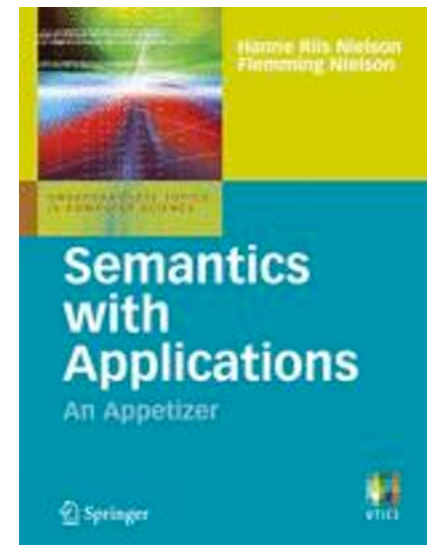
Natural Semantics in the Book

The book uses a slightly different notation

$$\langle e, q \rangle \Rightarrow n$$

versus in the slides

$$\langle e, q \rangle \Downarrow n$$



Judgments

We write $\langle e, q \rangle \Downarrow n$ to mean that expression e evaluates to n in state q .

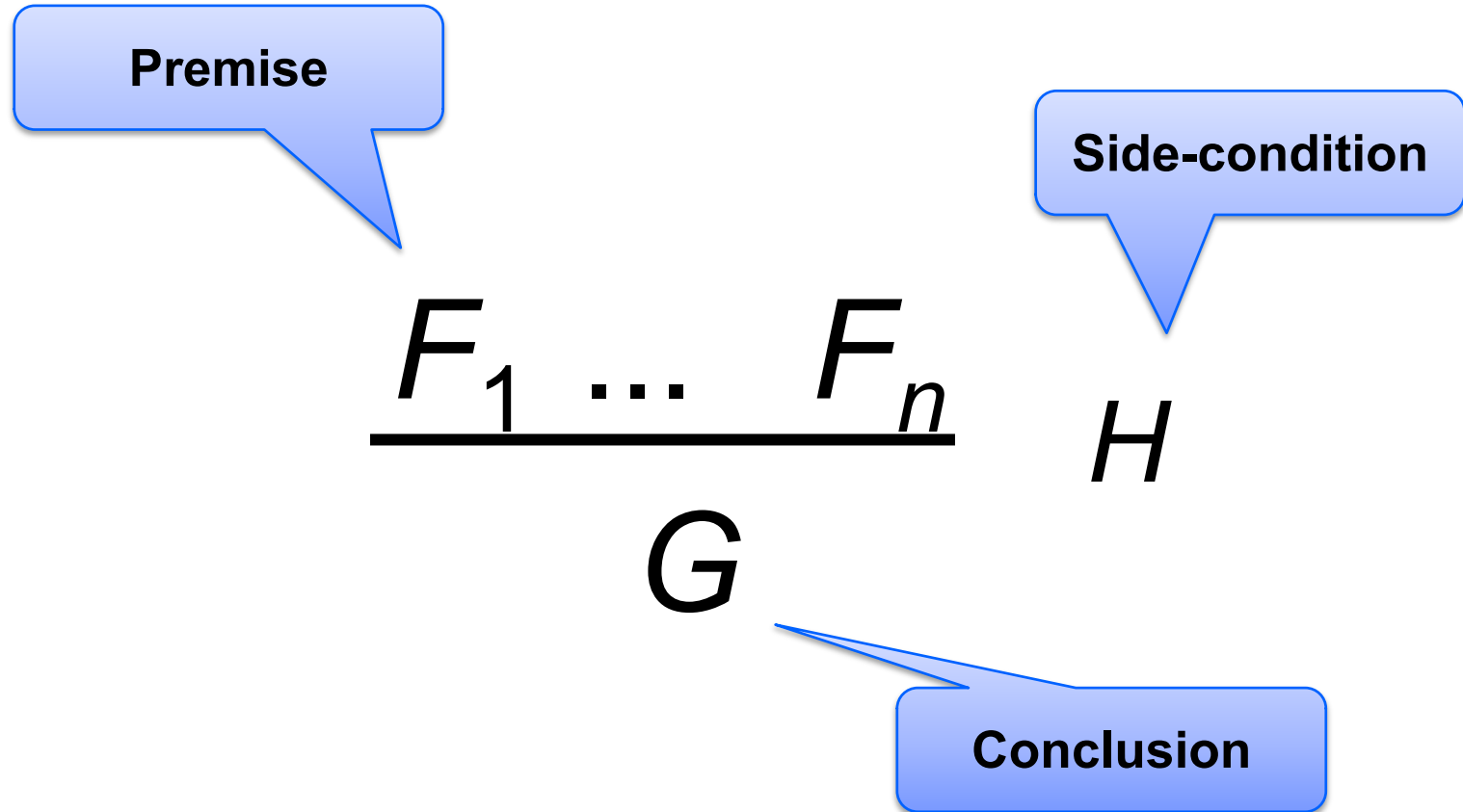
- The formula $\langle e, q \rangle \Downarrow n$ is called a **judgment**
(a judgement is a relation between an expression e , a state q and a number n)
- We can view \Downarrow as a function of two arguments e and q

This formulation is called **natural operational semantics**

- also known as *big-step* operational semantics
- the judgment relates the expression and its “meaning”

How to define $\langle e_1 + e_2, q \rangle \Downarrow \dots ?$

Notation: Inference Rule



Notation: Axiom

G

An axiom states that the conclusion G is true independently of any premises or side-conditions

Inference Rules

We express the evaluation rules as inference rules for our judgments.

The rules are also called **evaluation rules**.

An **inference rule**

$$\frac{F_1 \dots F_n}{G} \text{ where } H$$

defines a relation between judgments F_1, \dots, F_n and G .

- The judgments F_1, \dots, F_n are the **premises** of the rule;
- The judgment G is the **conclusion** of the rule;
- The formula H is called the **side condition** of the rule.

If $n=0$ the rule is called an **axiom**. In this case, the line separating premises and conclusion may be omitted.

Inference Rules for Aexp

In general, we have one rule per language construct:

$$\frac{}{\langle n, q \rangle \Downarrow n}$$

$$\frac{}{\langle x, q \rangle \Downarrow q(x)}$$

$$\frac{\langle e_1, q \rangle \Downarrow n_1 \quad \langle e_2, q \rangle \Downarrow n_2}{\langle e_1 + e_2, q \rangle \Downarrow (n_1 + n_2)}$$

$$\frac{\langle e_1, q \rangle \Downarrow n_1 \quad \langle e_2, q \rangle \Downarrow n_2}{\langle e_1 - e_2, q \rangle \Downarrow (n_1 - n_2)}$$

$$\frac{\langle e_1, q \rangle \Downarrow n_1 \quad \langle e_2, q \rangle \Downarrow n_2}{\langle e_1 * e_2, q \rangle \Downarrow (n_1 * n_2)}$$

This is called **structural operational semantics of expressions**.

- rules are defined based on the structure of the expressions.

Inference Rules for *Bexp*

$$\frac{}{\langle \text{true}, q \rangle \Downarrow \text{true}}$$

$$\frac{}{\langle \text{false}, q \rangle \Downarrow \text{false}}$$

$$\frac{\langle e_1, q \rangle \Downarrow n_1 \quad \langle e_2, q \rangle \Downarrow n_2}{\langle e_1 = e_2, q \rangle \Downarrow (n_1 = n_2)}$$

$$\frac{\langle e_1, q \rangle \Downarrow n_1 \quad \langle e_2, q \rangle \Downarrow n_2}{\langle e_1 \leq e_2, q \rangle \Downarrow (n_1 \leq n_2)}$$

$$\frac{\langle b_1, q \rangle \Downarrow t_1 \quad \langle e_2, q \rangle \Downarrow t_2}{\langle b_1 \wedge b_2, q \rangle \Downarrow (t_1 \wedge t_2)}$$

Derivation

A well-formed application of inference rules is called a *derivation*

Derivation infers new facts from existing ones

$$\frac{\frac{\langle 5, q \rangle \Downarrow 5 \quad \langle 7 * 2, q \rangle \Downarrow 14}{\langle 5 + (7 * 2), q \rangle \Downarrow 19}}{\frac{\langle 7, q \rangle \Downarrow 7 \quad \langle 2, q \rangle \Downarrow 2}{}}$$

Semantics of Statements

$$\langle s, q \rangle \Downarrow q'$$

Program
Statement

Input
Program
state

Output
Program
State

Statement s executed in state q results in state q'

Notation: state and state change

A *state* s is an assignment of values to locations (variables)

Notation

- empty state $[]$
- a state $[x := 10, y := 15, z := 5]$
- substitution $s[x := 10]$
 - a state like s , BUT, the value of x is 10

Evaluation of Statements

Evaluation of a statement produces a side-effect

- The *result* of evaluation of a statement is a **new state**

We write $\langle s, q \rangle \Downarrow q'$ to mean that evaluation of statement s in state q results in a new state q'

$$\frac{}{\langle \text{skip}, q \rangle \Downarrow q}$$

$$\frac{}{\langle \text{print_state}, q \rangle \Downarrow q}$$

$$\frac{\langle s_1, q \rangle \Downarrow q'' \quad \langle s_2, q'' \rangle \Downarrow q'}{\langle s_1 ; s_2, q \rangle \Downarrow q'}$$

$$\frac{\langle b, q \rangle \Downarrow \text{true} \quad \langle s_1, q \rangle \Downarrow q'}{\langle \text{if } b \text{ then } s_1 \text{ else } s_2, q \rangle \Downarrow q'}$$

$$\frac{\langle e, q \rangle \Downarrow n}{\langle x := e, q \rangle \Downarrow q[x:=n]}$$

$$\frac{\langle b, q \rangle \Downarrow \text{false} \quad \langle s_2, q \rangle \Downarrow q'}{\langle \text{if } b \text{ then } s_1 \text{ else } s_2, q \rangle \Downarrow q'}$$

Derivation and Execution

Derivation of statement facts corresponds to execution / interpretation

For example

- Show that $\langle p:=0; x:=1; n:=2, [] \rangle \Downarrow [p:=0, x:=1, n:=2]$

$$\begin{array}{c}
 \begin{array}{cc}
 \hline \langle 0, [] \rangle \Downarrow 0 & \hline \langle 1, [p:=0] \rangle \Downarrow 1 \\
 \hline \hline
 \end{array} \\
 \begin{array}{cc}
 \hline \langle p:=0, [] \rangle \Downarrow [p:=0] & \hline \langle x:=1, [p:=0] \rangle \Downarrow [p:=0, x:=1] \\
 \hline \hline
 \end{array} \\
 \hline
 \begin{array}{cc}
 \langle p:=0, x:=1, [] \rangle \Downarrow [p:=0, x:=1] & \langle n:=2, [p:=0, x:=1] \rangle \Downarrow [p:=0, x:=1, n:=2] \\
 \hline
 \end{array} \\
 \langle p:=0; x:=1; n:=2, [] \rangle \Downarrow [p:=0, x:=1, n:=2]
 \end{array}$$

Semantics of Loops

$$\frac{\langle b, q \rangle \Downarrow \text{false}}{\langle \text{while } b \text{ do } s, q \rangle \Downarrow q} \quad \frac{\langle b, q \rangle \Downarrow \text{true} \quad \langle s; \text{while } b \text{ do } s, q \rangle \Downarrow q'}{\langle \text{while } b \text{ do } s, q \rangle \Downarrow q'}$$

What about infinite execution?

- Can introduce a special state \top , called *top*, that represents divergence
- Infinite loop enters divergent state

$$\overline{\langle \text{while true do } s, q \rangle \Downarrow \top}$$

- Any statement in divergent state is treated like 'skip'

$$\overline{\langle s, \top \rangle \Downarrow \top}$$

Need structural (or *small step*) semantics to deal with reactive execution

- execution that does not terminate, but produces useful result

Properties of Semantics

A semantics is *deterministic* if every program statement has exactly one possible derivation in any state

- If $\langle s, q \rangle \Downarrow q_1$ and $\langle s, q \rangle \Downarrow q_2$ then $q_1 = q_2$

Two statements are *semantically equivalent* if for every input state they derive the same output state

- s_1 and s_2 are sem. equiv. if $\langle s_1, q \rangle \Downarrow q_1$ and $\langle s_2, q \rangle \Downarrow q_2$ imply $q_1 = q_2$
- e.g., $(\text{while } b \text{ do } s)$ and $\text{if } b \text{ then } (s ; \text{while } b \text{ do } s) \text{ else skip}$ are sem. equiv.

Structural induction: To prove a property P on a derivation tree

- **Base case**: prove P for all of the axioms
- **Inductive Hypothesis**: assume P holds before every rule
- **Induction**: prove that P holds at the end of every rule

Use structural induction to prove that our semantics are deterministic

Structural Operational Semantics (Small-Step)

The meaning of executing ONE statement of a program

$$\langle s, q \rangle \Rightarrow \langle t, q' \rangle$$

Program to
execute

Program
state

The REST of
the program

Output state

For the final statement, the output is only a state

$$\langle s, q \rangle \Rightarrow q'$$

Small-step semantics for WHILE

$$\overline{\langle \text{skip}, q \rangle \Rightarrow q}$$

$$\langle s_1, q \rangle \Rightarrow q'$$

$$\overline{\langle s_1 ; s_2, q \rangle \Rightarrow \langle s_2, q' \rangle}$$

$$\langle s_1, q \rangle \Rightarrow \langle s_3, q' \rangle$$

$$\overline{\langle s_1 ; s_2, q \rangle \Rightarrow \langle s_3 ; s_2, q' \rangle}$$

$$\langle b, q \rangle \Downarrow \text{true}$$

$$\overline{\langle \text{if } b \text{ then } s_1 \text{ else } s_2, q \rangle \Rightarrow \langle s_1, q \rangle}$$

$$\langle b, q \rangle \Downarrow \text{false}$$

$$\overline{\langle \text{if } b \text{ then } s_1 \text{ else } s_2, q \rangle \Rightarrow \langle s_2, q \rangle}$$

$$\overline{\langle \text{while } b \text{ do } s, q \rangle \Rightarrow \langle \text{if } b \text{ then } (s ; \text{while } b \text{ do } s) \text{ else skip}, q \rangle}$$

Properties of Small Step Semantics

Small step semantics can be viewed as a transition system $TS=(S, R)$

- S is a set of states; Each configuration $\langle s, q \rangle$ is a state.
- R is a transition relation on pair of states
 - (x, y) in R iff $(x \Rightarrow y)$ is a true judgement in small-step semantics

A path x_1, x_2, x_3, \dots in TS is called a *derivation sequence*

A derivation sequence in TS corresponds to a program execution

Properties of small-step semantics are established by induction on the length of the derivation

Small step semantics is deterministic if there is only one derivation for every configuration

Control-Flow Graph Semantics

$$\frac{}{\langle \text{skip}, q \rangle \Rightarrow q}$$

$$\frac{}{\langle x := e, q \rangle \Downarrow q}$$

$$\langle s_1, q \rangle \Rightarrow q$$

$$\frac{}{\langle s_1 ; s_2, q \rangle \Rightarrow \langle s_2, q \rangle}$$

$$\langle s_1, q \rangle \Rightarrow \langle s_3, q \rangle$$

$$\frac{}{\langle s_1 ; s_2, q \rangle \Rightarrow \langle s_3 ; s_2, q \rangle}$$

$$\frac{}{\langle \text{if } b \text{ then } s_1 \text{ else } s_2, q \rangle \Rightarrow \langle s_1, q \rangle}$$

$$\frac{}{\langle \text{if } b \text{ then } s_1 \text{ else } s_2, q \rangle \Rightarrow \langle s_2, q \rangle}$$

$$\frac{}{\langle \text{while } b \text{ do } s, q \rangle \Rightarrow \langle \text{if } b \text{ then } (s ; \text{while } b \text{ do } s) \text{ else skip}, q \rangle}$$

GRAPHS

Graphs

A **graph**, $G = (N, E)$, is an ordered pair consisting of

- a node set, N , and
- an edge set, $E = \{(n_i, n_j)\}$

If the pairs in E are ordered, then G is called a **directed graph** and is depicted with arrowheads on its edges

If not, the graph is called an **undirected graph**

Graphs are suggestive devices that help in the visualization of relations

- The set of edges in the graph are visual representations of the ordered pairs that compose relations

Graphs provide a mathematical basis for reasoning about programs

Paths

a **path**, P , through a directed graph $G = (N, E)$ is a sequence of edges, $((u_1, v_1), (u_2, v_2), \dots, (u_t, v_t))$ such that

- $v_{k-1} = u_k$ for all $1 < k \leq t$
- u_1 is called the **start** node and v_t is called the **end** node

The **length** of a path is the number of edges (or nodes-1 😊) in the path

Paths are also frequently represented by a sequence of nodes

- $(u_1, u_2, u_3, \dots, u_t)$

Cycles

A **cycle** in a graph G is a path whose start node and end node are the same

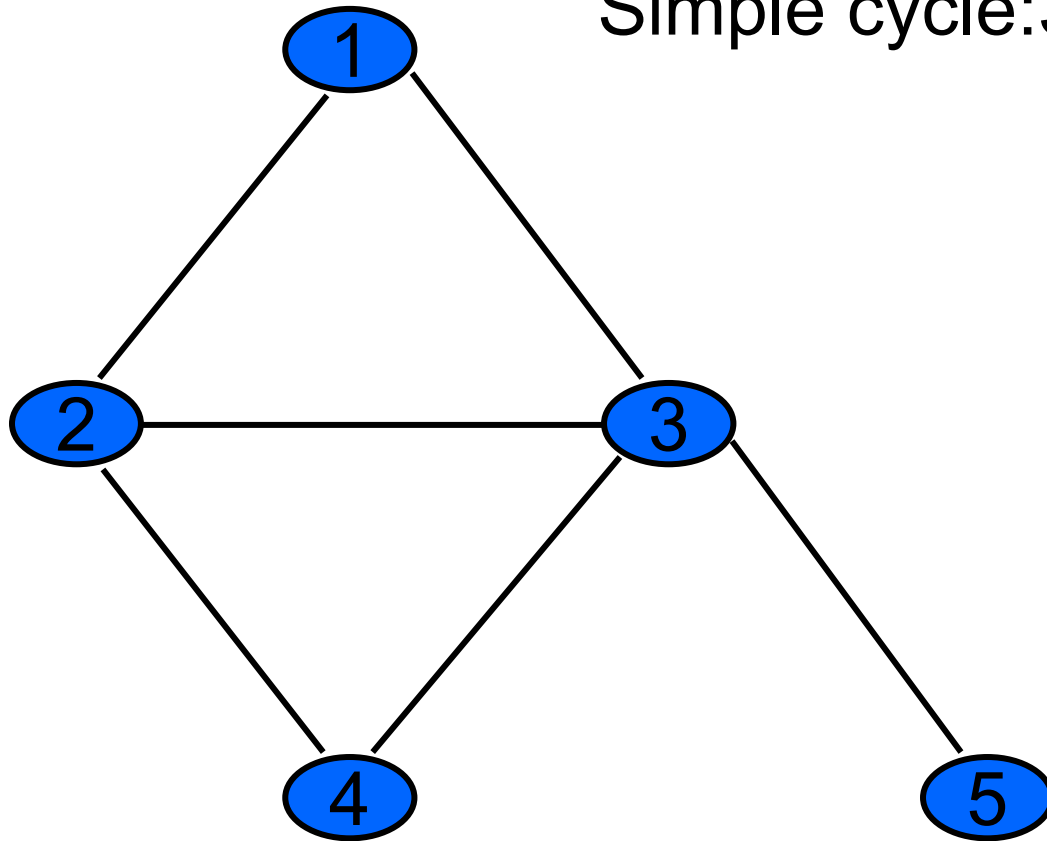
A **simple cycle** in a graph G is a cycle such that all of its nodes are different (except for the start and end nodes)

If a graph G has no path through it that is a cycle, then the graph is called **acyclic**

Example of Cycles

Cycle: 1,3,2,4,3,1

Simple cycle: 3,2,4,3



Trees

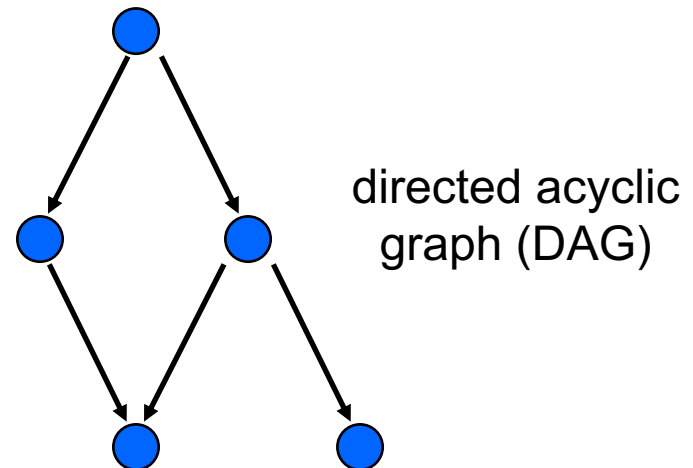
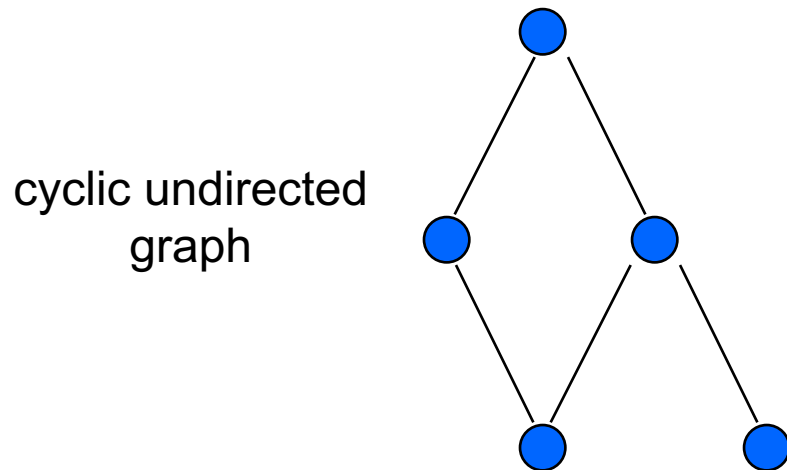
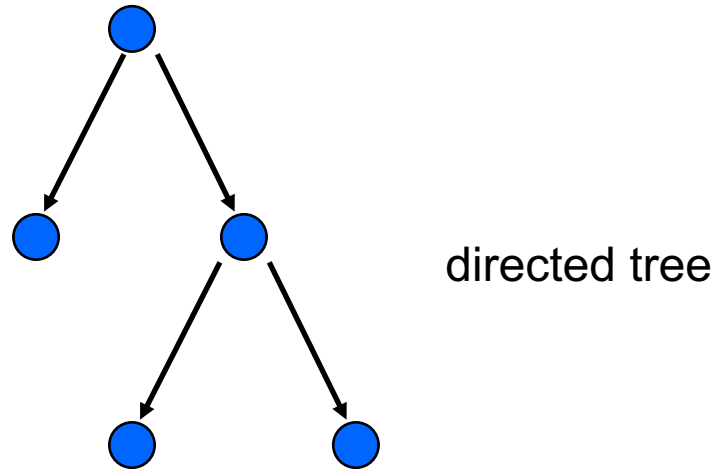
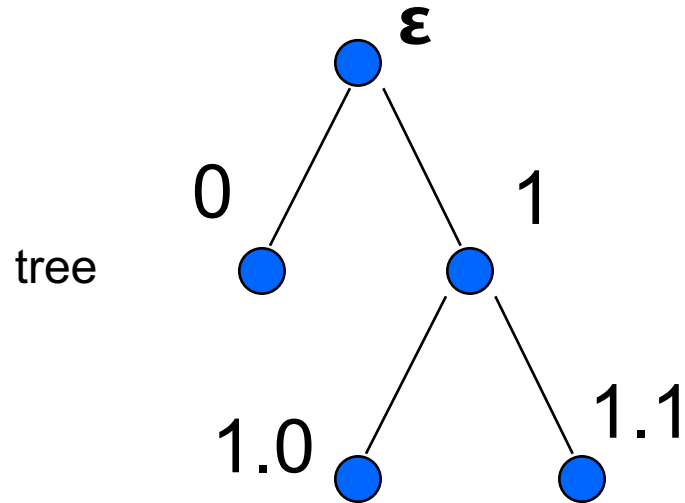
An acyclic, undirected graph is called a **tree**

If the undirected version of a directed graph is acyclic, then the graph is called a directed tree

If the undirected version of a directed graph has cycles, but the directed graph itself has no cycles, then the graph is called a **Directed Acyclic Graph (DAG)**

Every tree is isomorphic to a prefix-closed subset of N^* for some natural number N

Examples



GRAPHS AS MODELS OF COMPUTATION

Computation tree

A tree model of all the possible executions of a system

At each node represents a state of the system

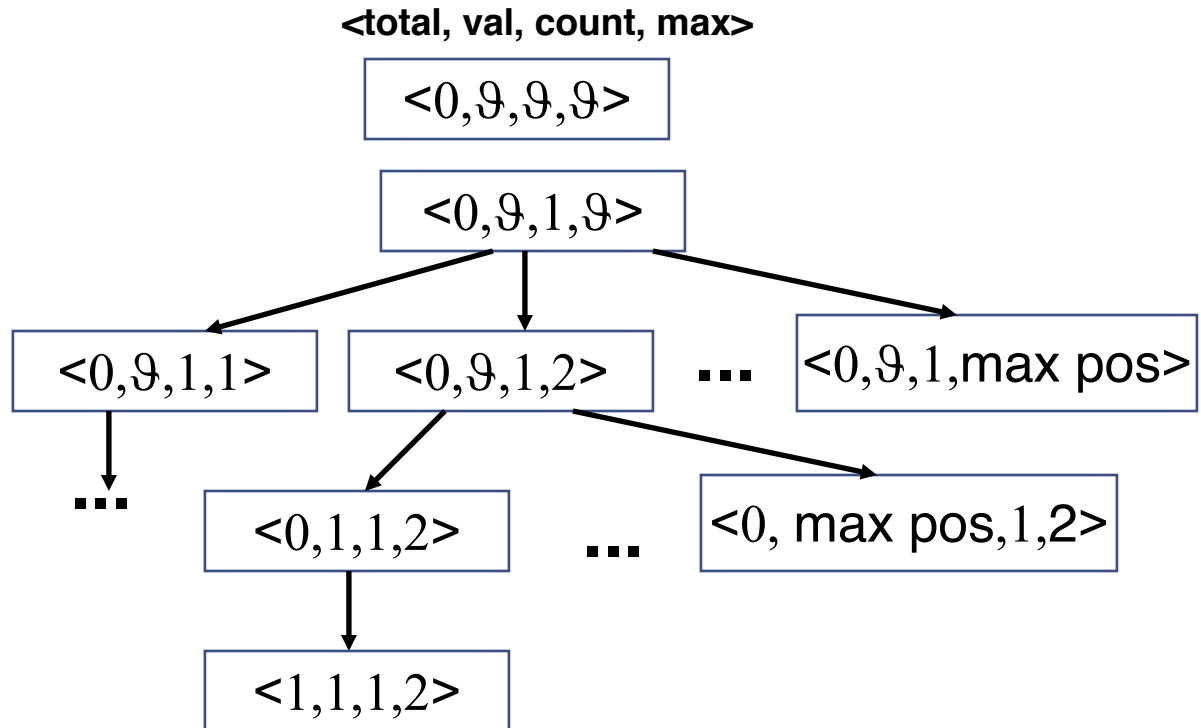
- valuation of all variables

Can have infinite number of paths

Can have infinite paths

Example Computation Tree

```
total := 0;  
count := 1;  
max := input();  
while (count <= max)  
do {  
  val := input();  
  total := total+val;  
  count := count+1};  
print (total)
```



Is this tree infinite?

Disadvantages of Computation Trees

Represent the space that we want to reason about

For anything interesting, they are too large to create or reason about

Other models of executable behavior are providing **abstractions** of the computation tree model

- Abstract values
- Abstract flow of control
- Specialize abstraction depending on focus of analysis

Control Flow Graph (CFG)

Represents the flow of execution in the program

$G = (N, E, S, T)$ where

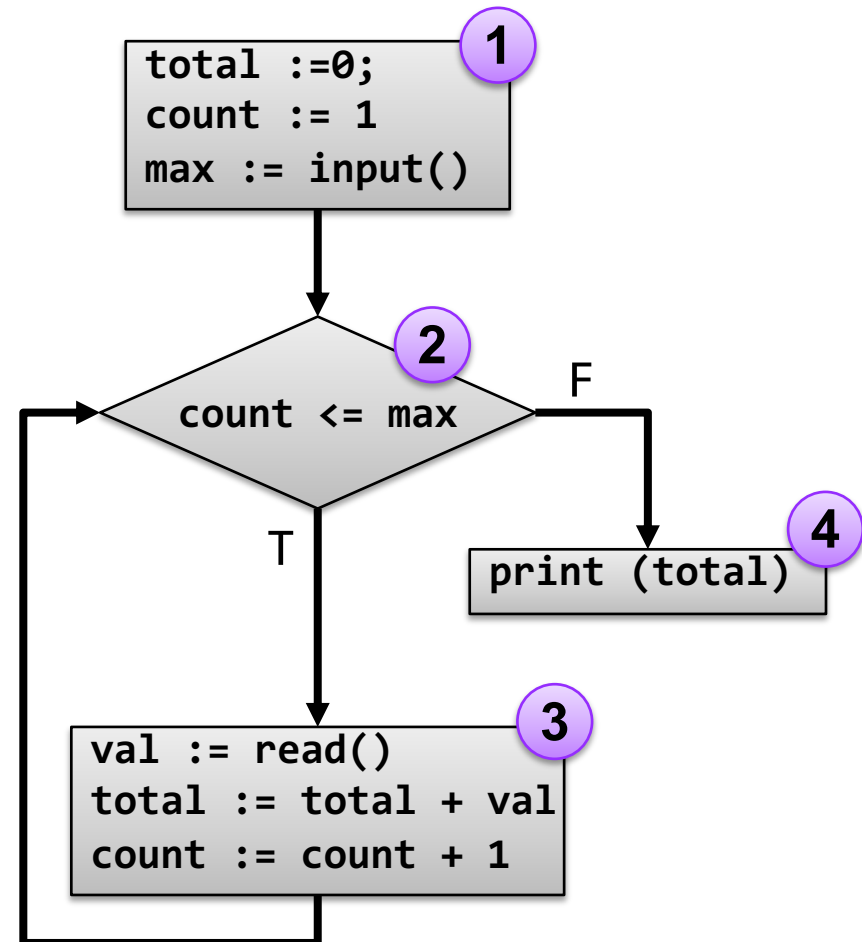
- the nodes N represent executable instructions (statement, statement fragments, or basic blocks);
- the edges E represent the **potential** transfer of control;
- S is a designated start node;
- T is a designated final node
- $E = \{ (n_i, n_j) \mid \text{syntactically, the execution of } n_j \text{ follows the execution of } n_i \}$

Nodes may correspond to single statements, parts of statements, or several statements (i.e., basic blocks)

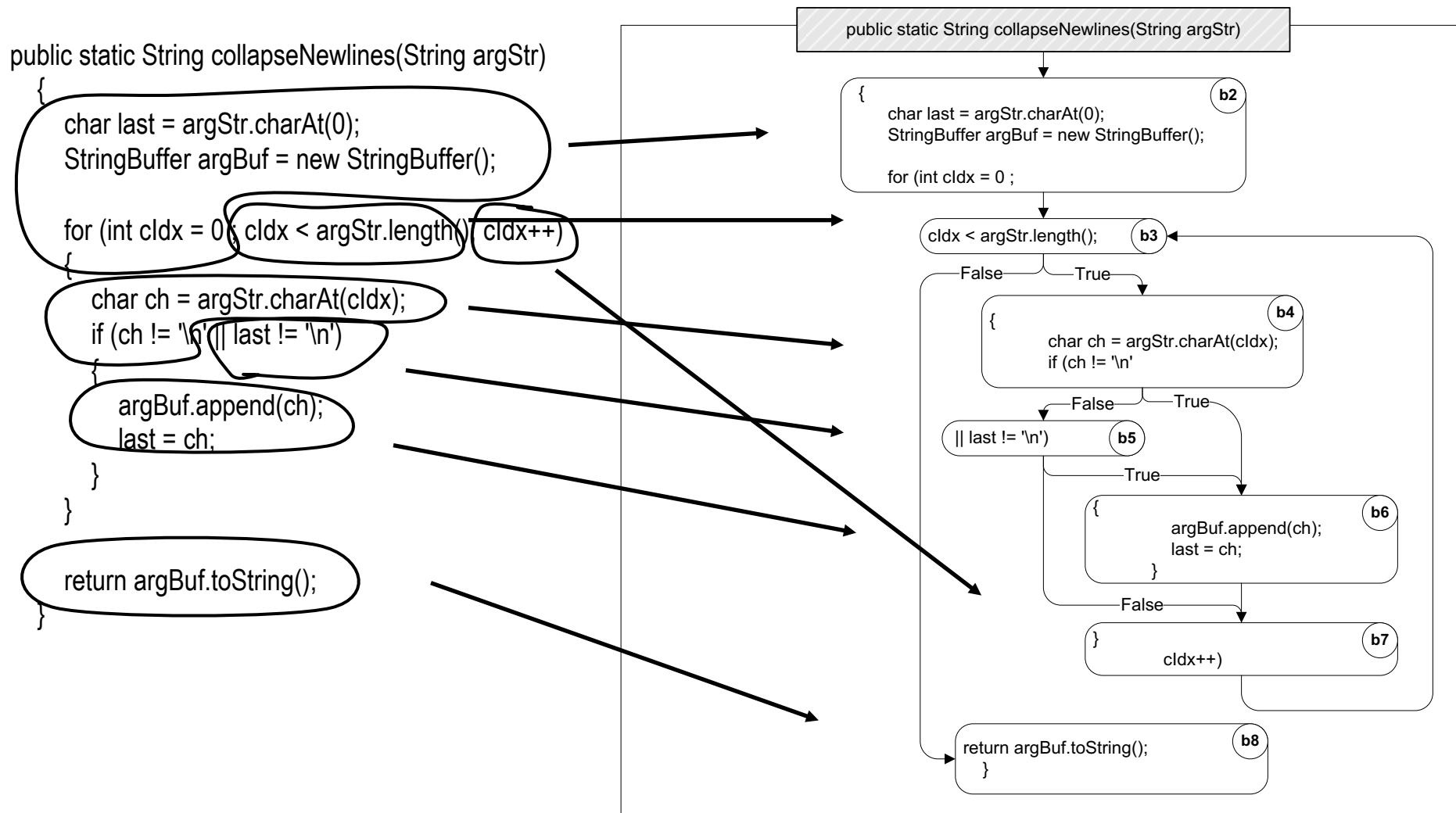
Execution of a node means that the instructions associated with a node are executed in order from the first instruction to the last

Example of a Control Flow Graph

```
total := 0;  
count := 1;  
max := input();  
while (count <= max)  
do {  
  val := input();  
  total := total+val;  
  count := count+1};  
print (total)
```



Deriving a Control Flow Graph



Control Flow Graph

A CFG is a graph of basic blocks

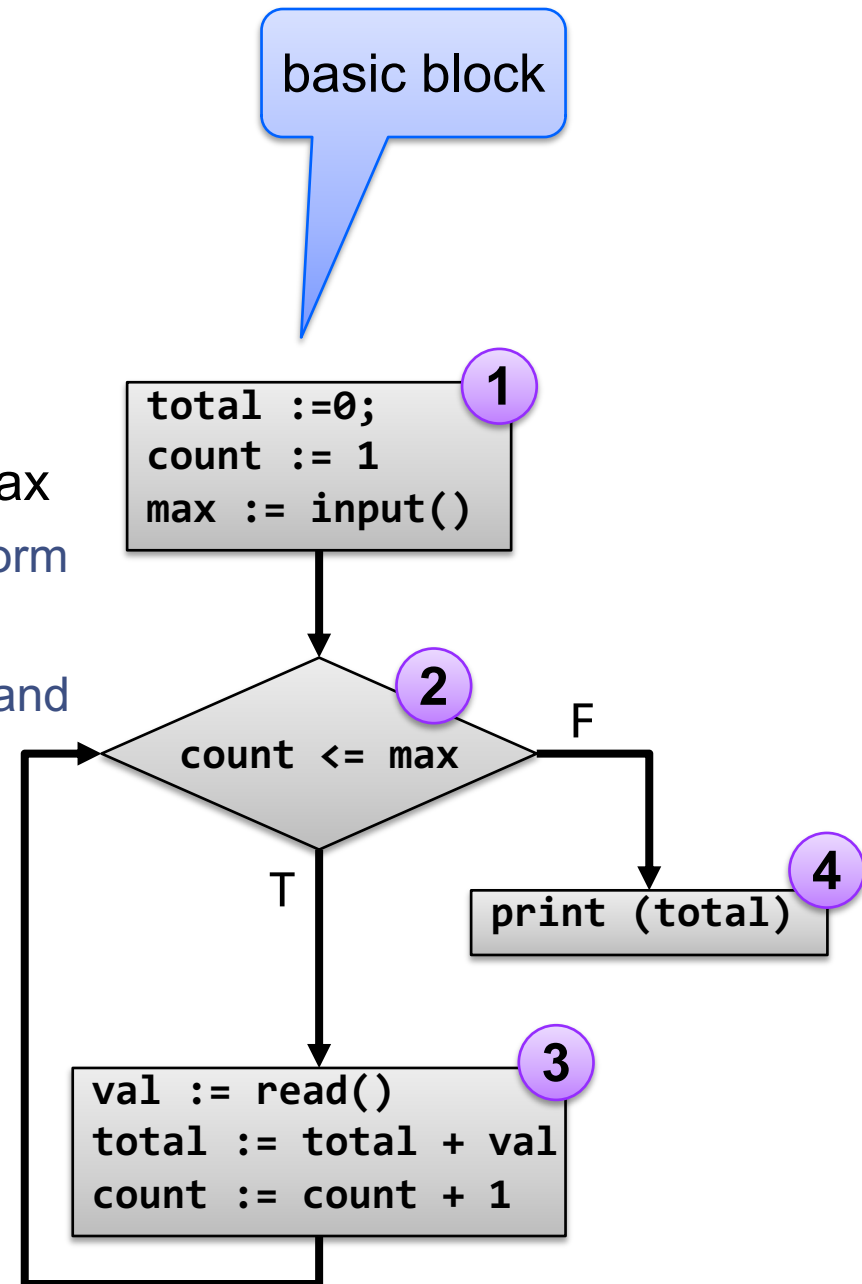
- edges represent different control flow

A CFG corresponds to a program syntax

- where statements are restricted to the form

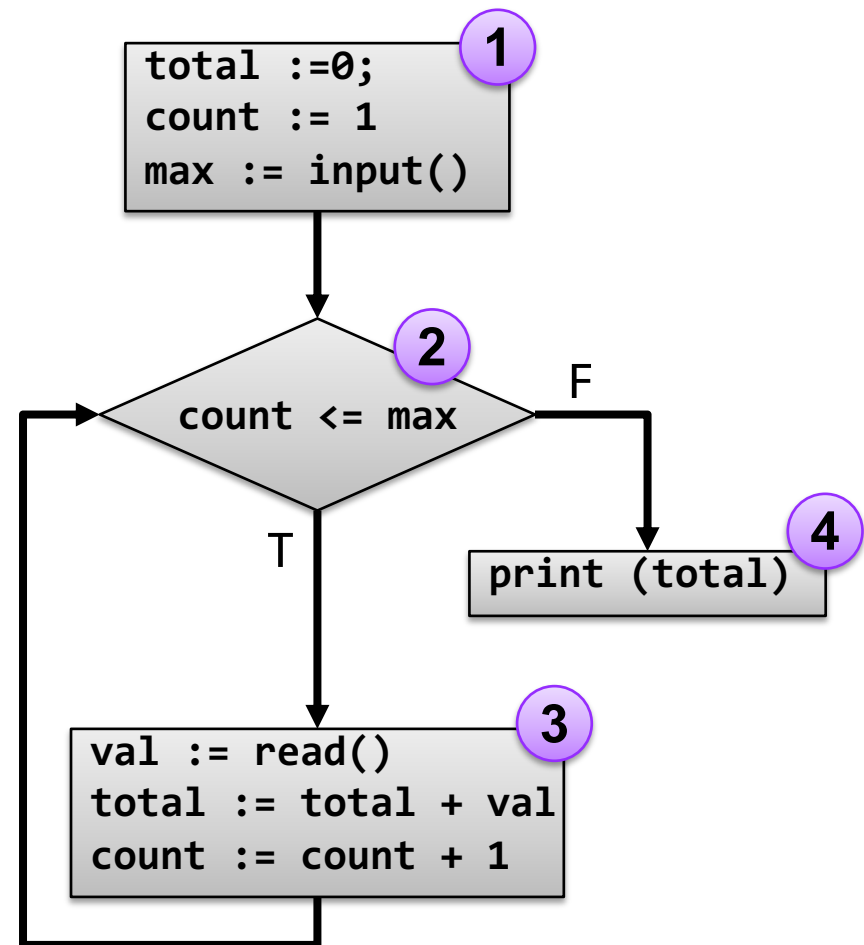
$L_i:S ; \text{goto } L_j$

and S is control-free (i.e., assignments and procedure calls)



Control Flow Graph

```
1: total:=0; count := 1;  
   max = input(); goto 2  
  
2: if count <= max  
   then goto 3 else goto 4  
  
3: val := read();  
   total := total + val;  
   count := count + 1; goto 2  
  
4: print(total)
```



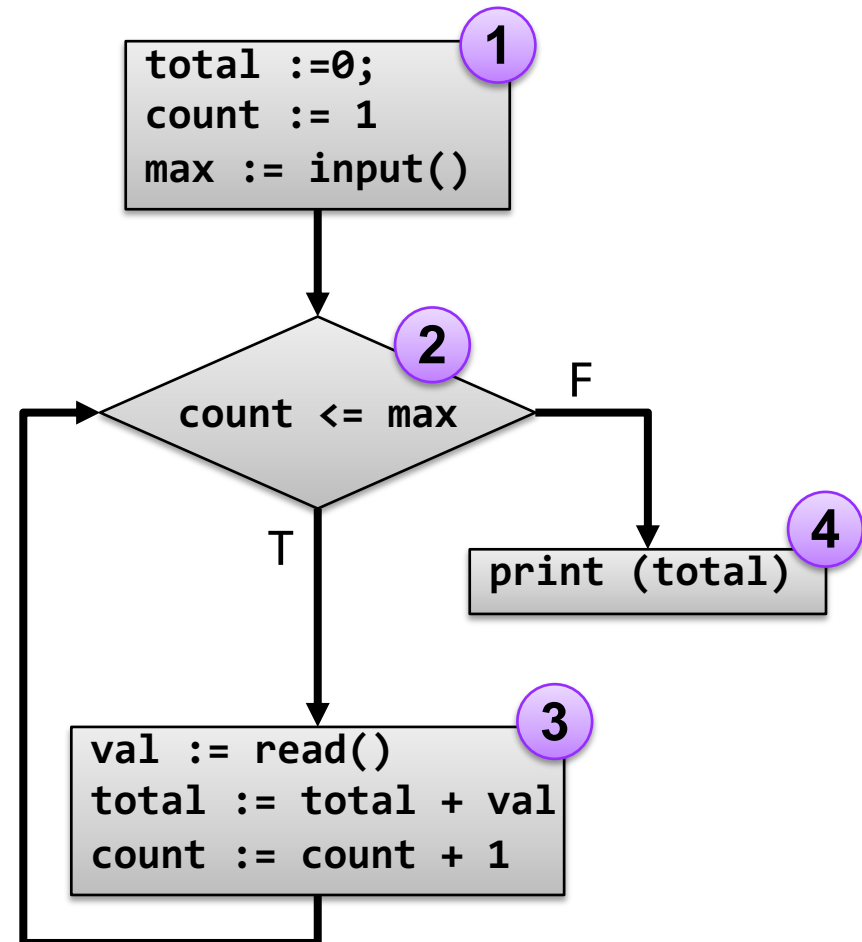
CFG: Sub-path and Complete Path

a **sub-path** through a CFG is a sequence of nodes $(n_i, n_{i+1}, \dots, n_t)$, $i \geq 1$ where for each n_k , $i \leq k < t$, (n_k, n_{k+1}) is an edge in the graph

- e.g., 2, 3, 2, 3, 2, 4

a **complete path** starts at the start node and ends at the final node

- e.g., 1, 2, 3, 2, 4



Infeasible Paths

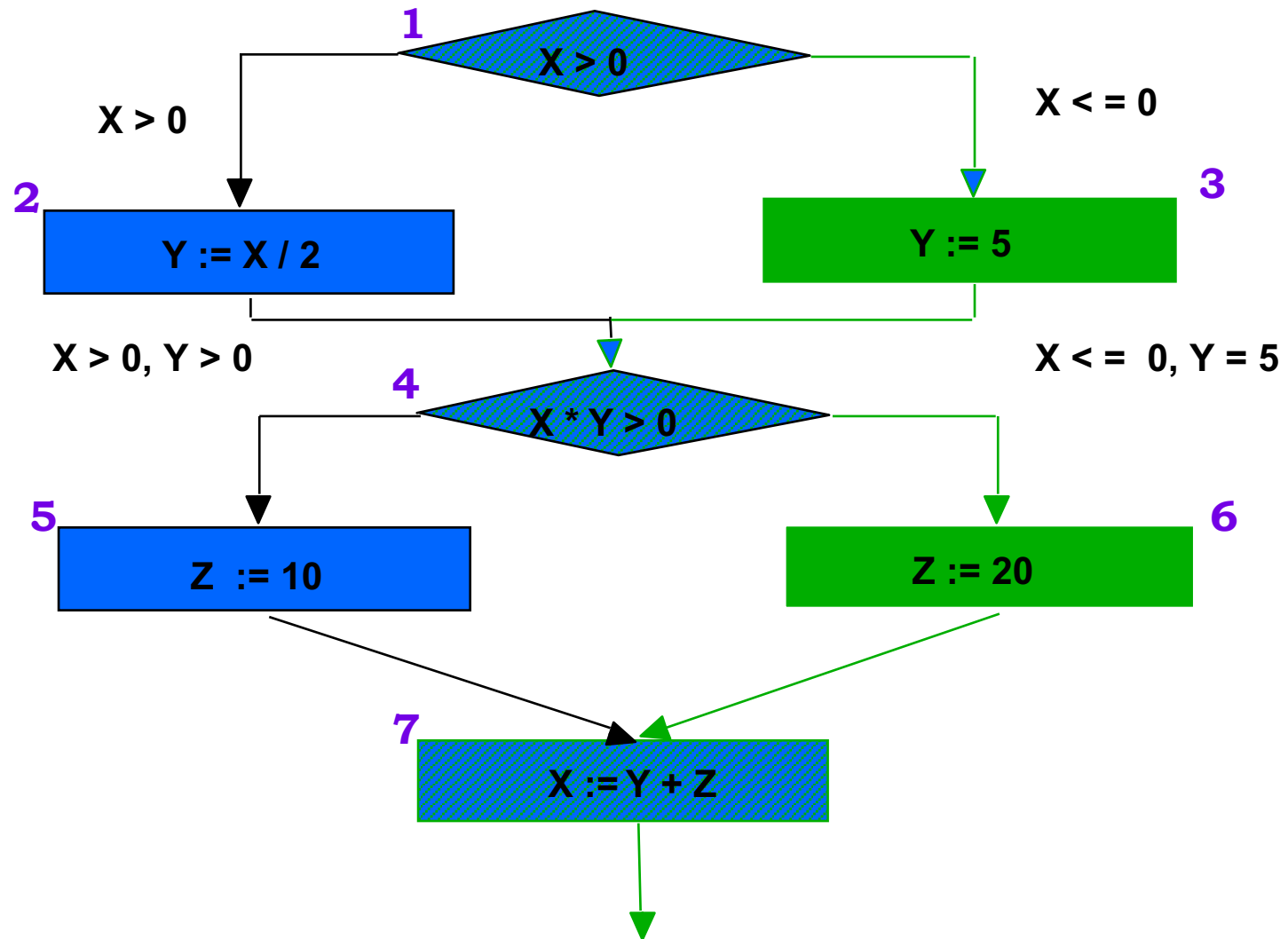
Every executable sequence in the represented component corresponds to a path in G

Not all paths correspond to executable sequences

- requires additional semantic information
- “infeasible paths” are not an indication of a fault

CFG usually **overestimates** the executable behavior

Example with an infeasible path



Example Paths

Feasible path: 1, 2, 4, 5, 7

Infeasible path: 1, 3, 4, 5, 7

Determining if a path is feasible or not requires additional semantic information

- In general, undecidable
- In practice, intractable
 - Some exceptions are studied in this course

Benefits of CFG

Probably the most commonly used representation

- Numerous variants

Basis for inter-component analysis

- Collections of CFGs

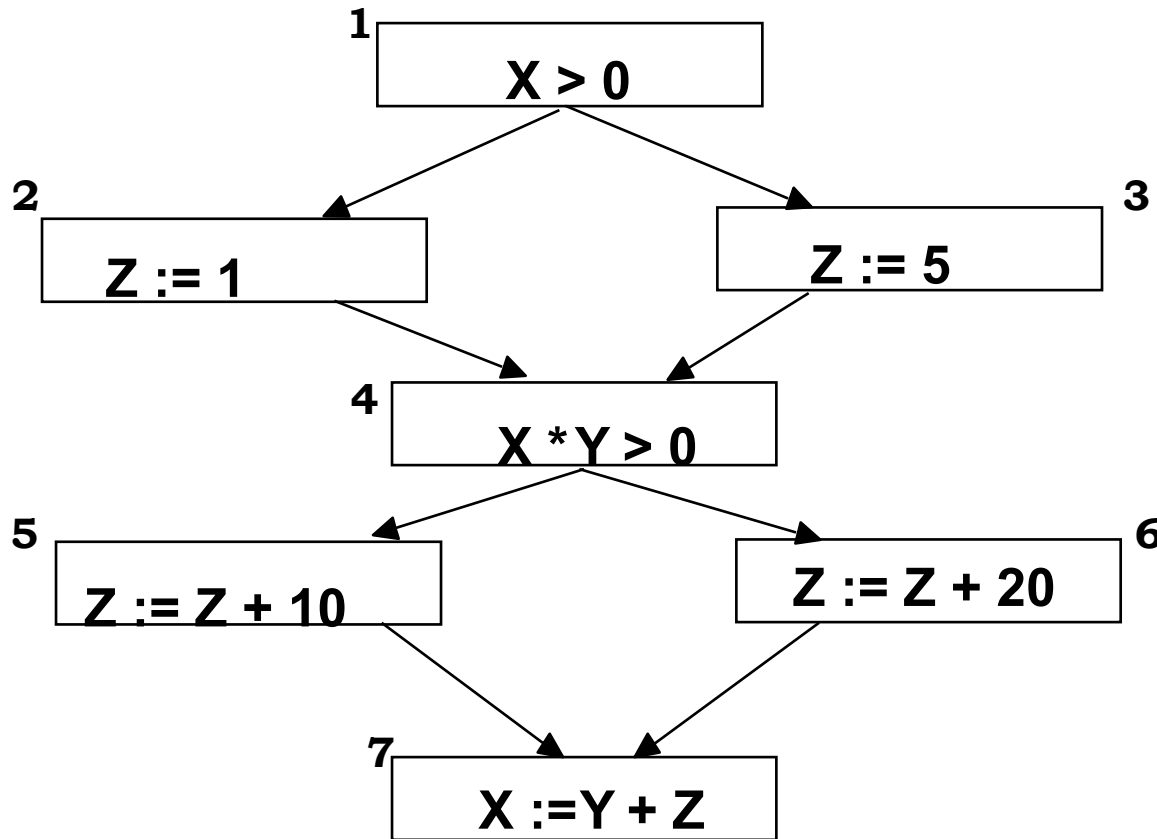
Basis for various transformations

- Compiler optimizations
- S/W analysis

Basis for automated analysis

- Graphical representations of interesting programs are too complex for direct human understanding

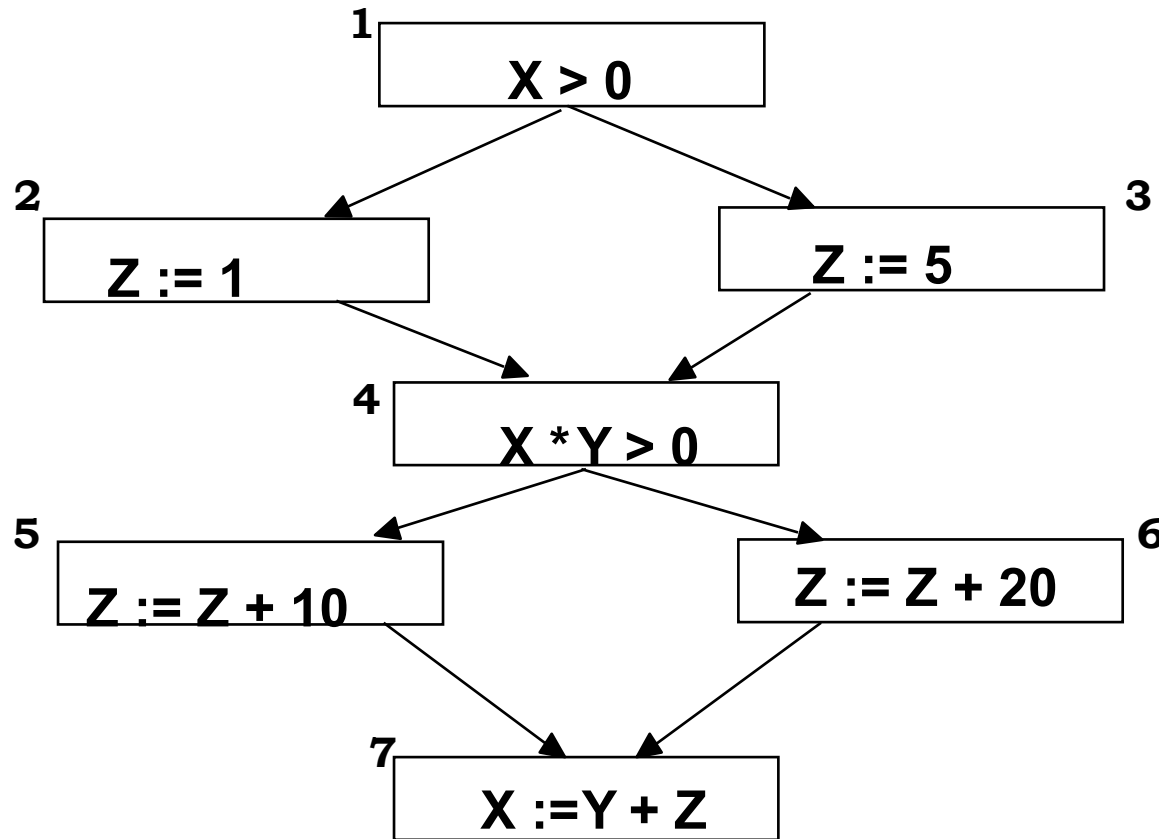
Paths



• Paths:

- 1, 2, 4, 5, 7
- 1, 2, 4, 6, 7
- 1, 3, 4, 5, 7
- 1, 3, 4, 6, 7

Paths can be identified by predicate outcomes



- outcomes

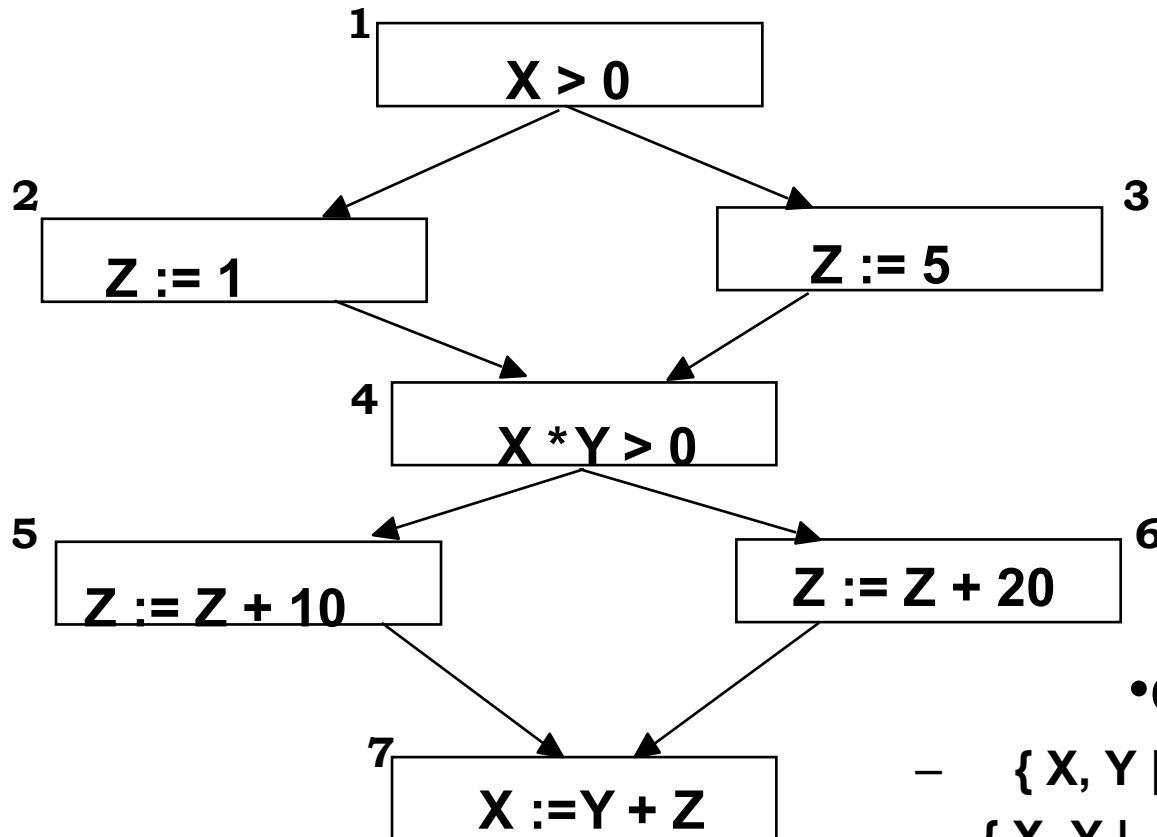
- t, t

- t, f

- f, t

- f, f

Paths can be identified by domains



•domains

- $\{ X, Y \mid X > 0 \text{ and } X * Y > 0 \}$
- $\{ X, Y \mid X > 0 \text{ and } X * Y \leq 0 \}$
- $\{ X, Y \mid X \leq 0 \text{ and } X * Y > 0 \}$
- $\{ X, Y \mid X \leq 0 \text{ and } X * Y \leq 0 \}$

CFG Abstraction Level?

Loop conditions? (yes)

Individual statements? (no)

Exception handling? (no)

What's best depends on type of analysis to be conducted

CFG Exercise (1)

Draw a control flow graph with 7 nodes.

```
int binary_search(int a[], int low, int high,
int target) { /* binary search for target in
the sorted a[low, high] */
1   while (low <= high) {
2       int middle = low + (high - low)/2;
3       if (target < a[middle])
4           high = middle - 1;
5       else if (target > a[middle])
6           low = middle + 1;
       else
7           return middle;
    }
8   return -1; /* return -1 if target is not
found in a[low, high] */
}
```

CFG Exercise (2)

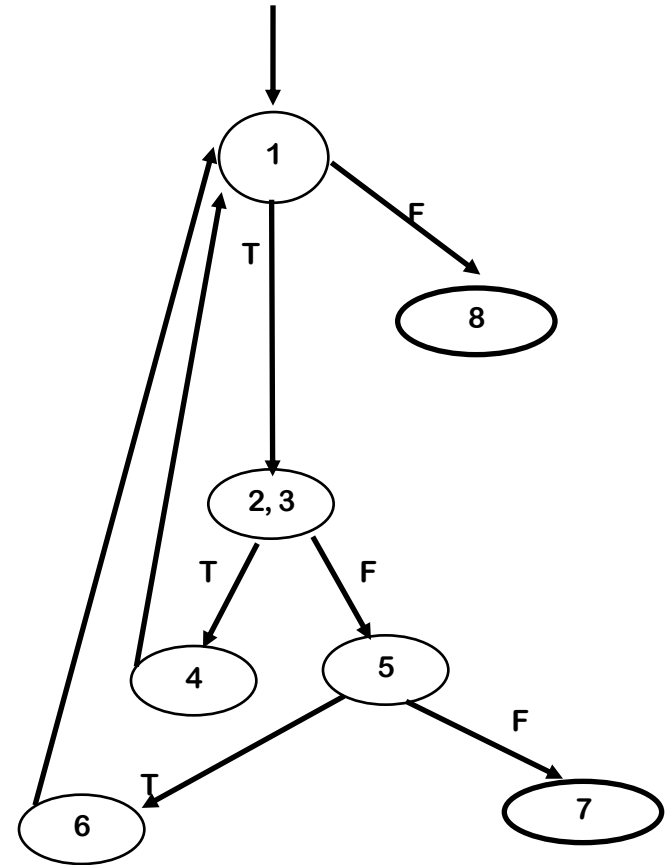
Draw a control flow graph with 8 nodes.

```
int binary_search(int a[], int low, int high,
int target) { /* binary search for target in
the sorted a[low, high] */
1   while (low <= high) {
2       int middle = low + (high - low)/2;
3       if (target < a[middle])
4           high = middle - 1;
5       else if (target > a[middle])
6           low = middle + 1;
       else
7           return middle;
    }
8   return -1; /* return -1 if target is not
found in a[low, high] */
}
```

CFG Exercise (1) Solution

Draw a control flow graph with **7** nodes.

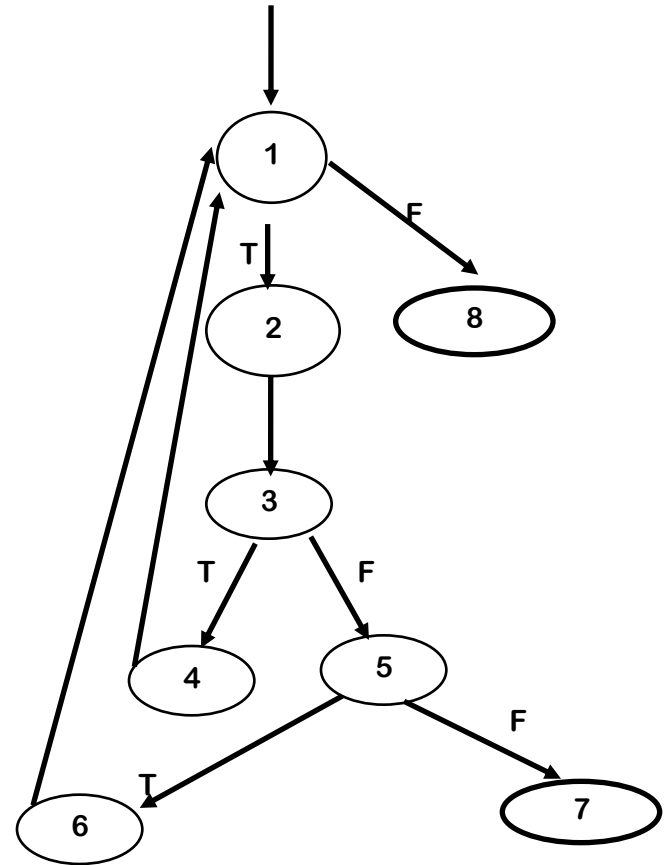
```
int binary_search(int a[], int low, int high,
int target) { /* binary search for target in
the sorted a[low, high] */
1  while (low <= high) {
2      int middle = low + (high - low)/2;
3      if (target < a[middle])
4          high = middle - 1;
5      else if (target > a[middle])
6          low = middle + 1;
7      else
8          return middle;
9  }
10 return -1; /* return -1 if target is not
found in a[low, high] */
}
```



CFG Exercise (2) Solution

Draw a control flow graph with **8** nodes.

```
int binary_search(int a[], int low, int high,
int target) { /* binary search for target in
the sorted a[low, high] */
1  while (low <= high) {
2      int middle = low + (high - low)/2;
3      if (target < a[middle])
4          high = middle - 1;
5      else if (target > a[middle])
6          low = middle + 1;
7      else
8          return middle;
9  }
10 return -1; /* return -1 if target is not
found in a[low, high] */
}
```



CFG Exercise (3)

/ Function: ReturnAverage Computes the average of all those numbers in the input array in the positive range [MIN, MAX]. The max size of the array is AS. But, the array size could be smaller than AS in which case the end of input is designated by -999. */*

```
1    public static double ReturnAverage(int value[], int AS, int MIN, int MAX) {
2        int i, ti, tv, sum;
3        double av;
4        i = 0; ti = 0; tv = 0; sum = 0;
5        while (ti < AS && value[i] != -999) {
6            ti++;
7            if (value[i] >= MIN && value[i] <= MAX) {
8                tv++;
9                sum = sum + value[i];
10           }
11           i++;
12       }
13       if (tv > 0) av = (double)sum/tv;
14       else av = (double) -999;
15       return (av);
16   }
```

CFG of ReturnAverage

