SAT Solving

Testing, Quality Assurance, and Maintenance
Winter 2020

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based on slides by Prof. Ruzica Piskac, Nikolaj Bjorner, and others

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Boolean Satisfiability (CNF-SAT)

Let V be a set of variables

A *literal* is either a variable v in V or its negation ~v

A *clause* is a disjunction of literals

- e.g., (v1 || ~v2 || v3)

A Boolean formula in *Conjunctive Normal Form* (CNF) is a conjunction of clauses

- e.g., (v1 || ~v2) && (v3 || v2)

An *assignment* *s* of Boolean values to variables *satisfies* a clause *c* if it evaluates at least one literal in *c* to true

An assignment *s* *satisfies* a formula *C* in CNF if it satisfies every clause in *C*

Boolean Satisfiability Problem (CNF-SAT):

- determine whether a given CNF C is satisfiable

Algorithms for SAT

SAT is NP-complete

DPLL (Davis-Putnam-Logemman-Loveland, ‘60)

- smart enumeration of all possible SAT assignments
- worst-case EXPTIME
- alternate between deciding and propagating variable assignments

CDCL (GRASP ‘96, Chaff ‘01)

- conflict-driven clause learning
- extends DPLL with
  - smart data structures, backjumping, clause learning, heuristics, restarts…
- scales to millions of variables
- N. Een and N. Sörensson, “An Extensible SAT-solver”, in SAT 2013.

Background Reading: SAT

http://cacm.acm.org/magazines/2009/8/34498-boolean-satisfiability-from-theoretical-h

Boolean Satisfiability: From ...

Find: currency Previous Next Options

REVIEW ARTICLES

Boolean Satisfiability: From Theoretical Hardness to Practical Success

By Sharad Malik, Lintao Zhang

Communications of the ACM, Vol. 52 No. 8, Pages 76-82
10.1145/1536616.1536637

Comments

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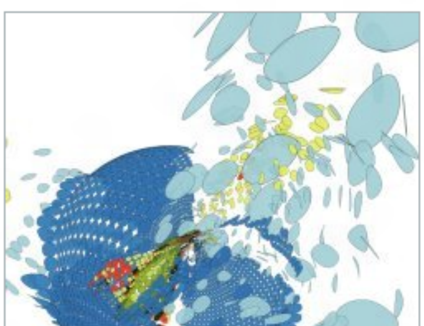

There are many practical situations where we need to satisfy several potentially conflicting constraints. Simple examples of this abound in daily life, for example, determining a schedule for a series of games that resolves the availability of players and venues, or finding a seating assignment at dinner consistent with various rules the host would like to impose. This also applies to applications in computing, for example, ensuring that a hardware/software system functions correctly with its overall behavior constrained by the behavior of its components and their composition, or finding a plan for a robot to reach a goal that is

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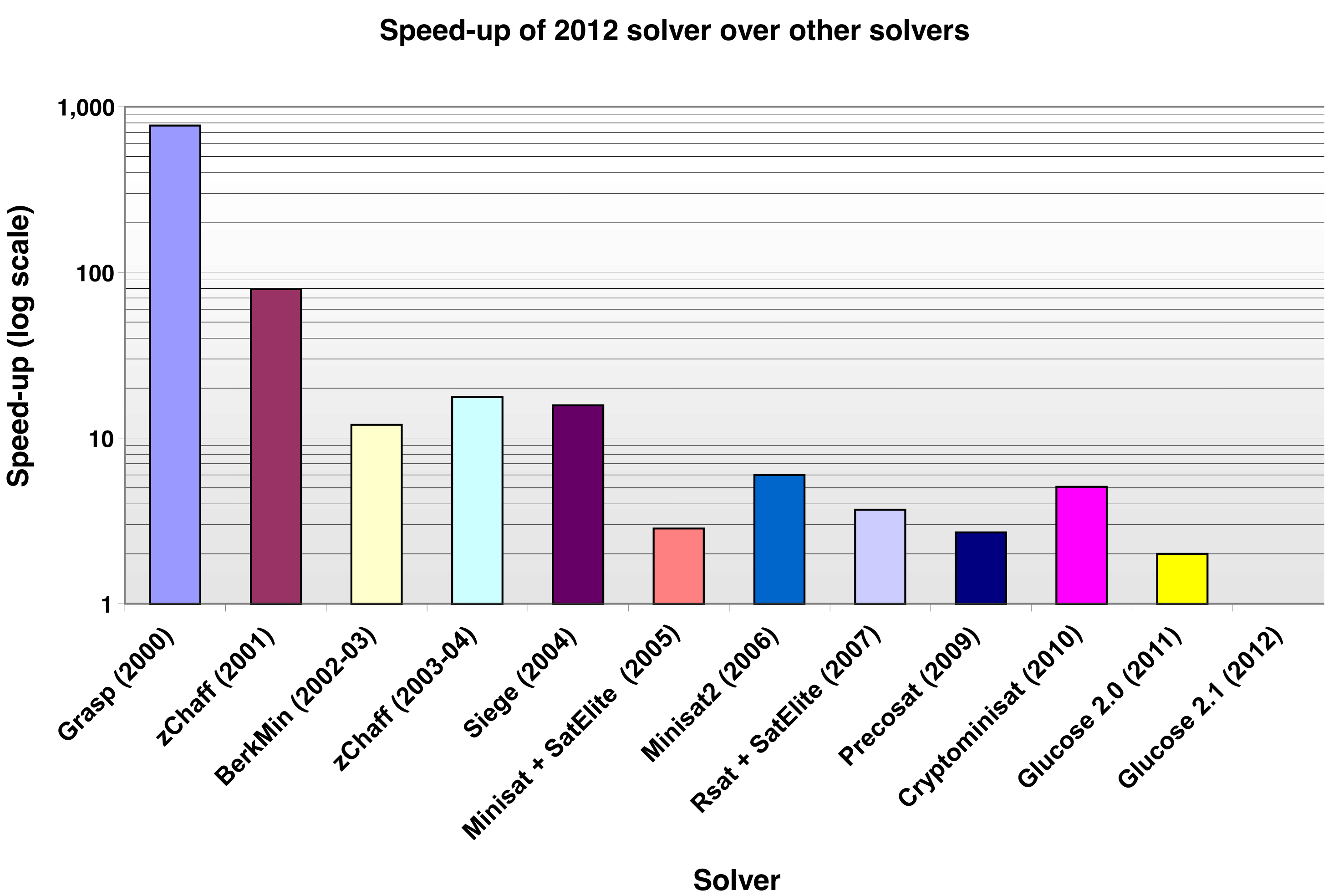
ARTICLE CONTENTS:

Some Experience with SAT Solving



from M. Vardi, https://www.cs.rice.edu/~vardi/papers/highlights15.pdf

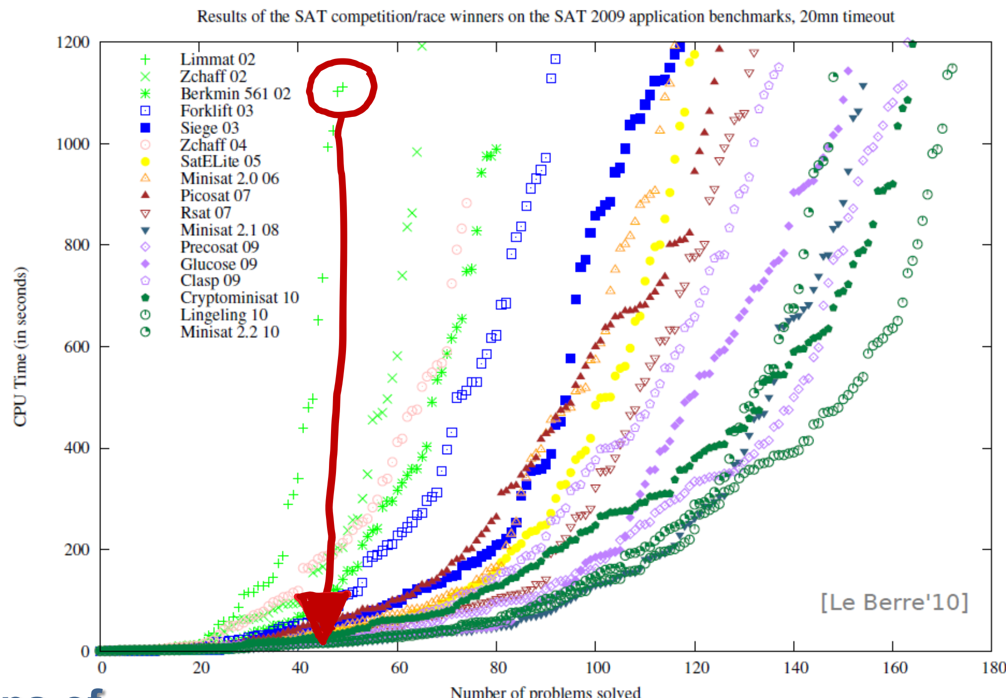
SAT - Milestones

Problems impossible 10 years ago are trivial today

year	Milestone
1960	Davis-Putnam procedure
1962	Davis-Logeman-Loveland
1984	Binary Decision Diagrams
1992	DIMACS SAT challenge
1994	SATO: clause indexing
1997	GRASP: conflict clause learning
1998	Search Restarts
2001	zChaff: 2-watch literal, VSIDS
2005	Preprocessing techniques
2007	Phase caching
2008	Cache optimized indexing
2009	In-processing, clause management
2010	Blocked clause elimination

Concept

2002 2010



Millions of variables from HW designs

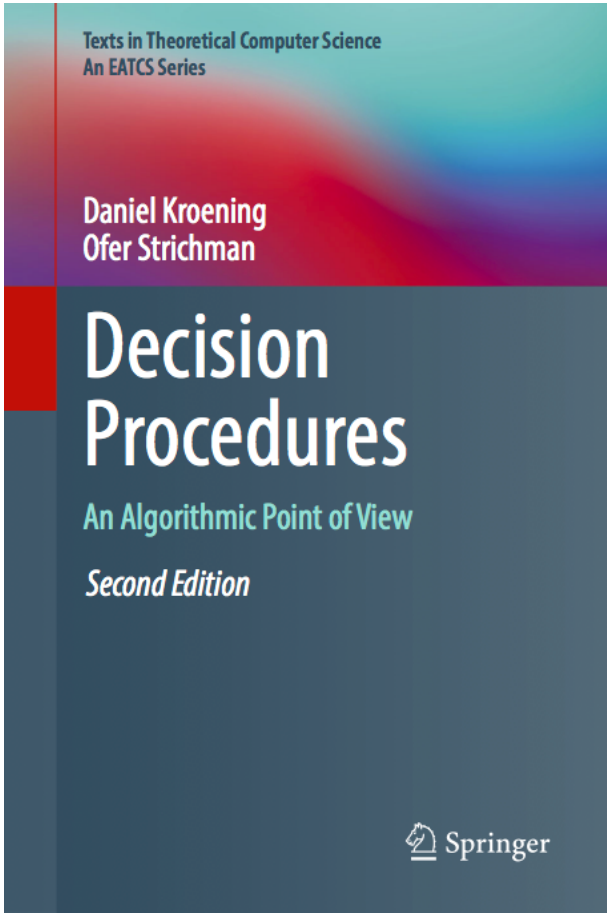
Courtesy Daniel le Berre

Davis Putnam Logemann Loveland

DPLL PROCEDURE

References

Chapter 2: Decision Procedures for Propositional Logic



Decision Procedure for Satisfiability

Algorithm that in some finite amount of computation decides if a given propositional logic (PL) formula F is satisfiable

- NP-complete problem

Modern decision procedures for PL formulae are called SAT solvers

Naïve approach

- Enumerate models (i.e., truth tables)
- Enumerate resolution proofs

Modern SAT solvers

- DPLL algorithm
  - Davis-Putnam-Logemann-Loveland
- Combines model- and proof-based search
- Operates on Conjunctive Normal Form (CNF)

Propositional Resolution

Pivot

$$\frac{C \vee p \qquad\qquad D \vee \neg p}{C \vee D}$$

Resolvent

Res({C, p}, {D, !p}) = {C, D}

Given two clauses (C, p) and (D, !p) that contain a literal p of different polarity, create a new clause by taking the union of literals in C and D

Resolution Lemma

Lemma:

Let F be a CNF formula. Let R be a resolvent of two clauses X and Y in F. Then, $F \cup \{R\}$ is equivalent to F

$$\frac{C \vee p \qquad\qquad D \vee \neg p}{C \vee D}$$

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Resolution Theorem

Let F be a set of clauses

$Res(F) = F \cup \{R \mid R \text{ is a resolvent of two clauses in } F\}$

$$Res^0(F) = F$$

$$Res^{n+1}(F) = Res(Res^n(F)), \text{ for } n \geq 0$$

$$Res^*(F) = \bigcup_{n \geq 0} Res^n(F)$$

Theorem: A CNF F is UNAT iff Res*(F) contains an empty clause

Resolution Theorem

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Theorem: A CNF F is UNAT iff Res*(F) contains an empty clause

Proof of the Resolution Theorem

(Soundness) By Resolution Lemma, F is equivalent to $Res^i(F)$ for any i. Let n be such that $Res^{n+1}(F)$ contains an empty clause, but $Res^n(F)$ does not. Then $Res^n(F)$ must contain to unit clauses L and ¬L. Hence, it is UNSAT.

(Completeness) By induction on the number of different atomic propositions in F.

Base case is trivial: F contains an empty $_{\text{clause}}$.

IH: Assume F has atomic propositions A1, … A_{n+1}

Let F_0 be the result of replacing A_{n+1} by 0

Let F_1 be the result of replacing A_{n+1} by 1

Apply IH to F_0 and F_1 . Restore replaced literals. Combine the two resolutions.

Proof System

$$P_1, \ldots, P_n \vdash C$$

An inference rule is a tuple (P_1, …, P_n, C)

- where, P_1, …, P_n, C are formulas
- P_i are called premises and C is called a conclusion
- intuitively, the rules says that the conclusion is true if the premises are

A proof system P is a collection of inference rules

A proof in a proof system P is a tree (or a DAG) such that

- nodes are labeled by formulas
- for each node *n*, (parents(n), n) is an inference rule in P

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Propositional Resolution

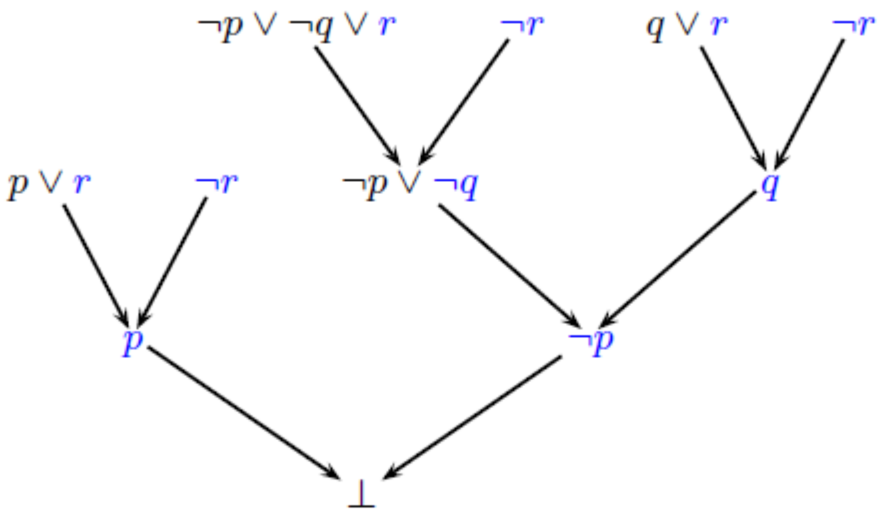
$$\frac{C \lor p \qquad\qquad D \lor \neg p}{C \lor D}$$

Propositional resolution is a sound inference rule

Proposition resolution system consists of a single propositional resolution rule

Example of a resolution proof

A refutation of $\neg p \vee \neg q \vee r$, $p \vee r$, $q \vee r$, $\neg r$:



Resolution Proof Example

Show by resolution that the following CNF is UNSAT

$$\neg b \land (\neg a \lor b \lor \neg c) \land a \land (\neg a \lor c)$$

$$\dfrac{\dfrac{\dfrac{\neg a \lor b \lor \neg c \qquad a}{b \lor \neg c} \qquad b}{\neg c} \qquad \dfrac{a \qquad \neg a \lor c}{c}}{\bot}$$

Entailment and Derivation

A set of formulas F entails a set of formulas G iff every model of F and is a model of G

$$F \models G$$

A formula G is derivable from a formula F by a proof system P if there exists a proof whose leaves are labeled by formulas in F and the root is labeled by G

$$F \vdash_P G$$

Soundness and Completeness

A proof system P is sound iff

$$(F \vdash_P G) \implies (F \models G)$$

A proof system P is complete iff

$$(F \models G) \implies (F \vdash_P G)$$

SAT solving by resolution (DP)

Assume that input formula F is in CNF

1. Pick two clauses C_1 and C_2 in F that can be resolved
2. If the resolvent C is an empty clause, return UNSAT
3. Otherwise, add C to F and go to step 1
4. If no new clauses can be resolved, return SAT

Termination: finitely many derived clauses

DPLL: David Putnam Logemann Loveland

Combines pure resolution-based search with case splitting on decisions

Proof search is restricted to unit resolution

- can be done very efficiently (polynomial time)

Case split restores completeness

DPLL can be described by the following two rules

- F is the input formula in CNF

$$\frac{F}{F, p \quad | \quad F, \neg p}\ \text{split} \qquad p\ and\ \neg p\ are\ not\ in\ F$$

$$\frac{F,\ C \vee \ell, \neg \ell}{F, C, \neg \ell}\ \text{unit}$$

Davis, Martin; Logemann, George; Loveland, Donald (1962). "A Machine Program for Theorem Proving". *C. ACM*. **5** (7): 394–397. doi:10.1145/368273.368557

The original DPLL procedure

Incrementally builds a satisfying truth assignment M for the input CNF formula F

M is grown by

- deducing the truth value of a literal from M and F, or
- guessing a truth value

If a wrong guess for a literal leads to an inconsistency, the procedure backtracks and tries the opposite value

DPLL: Illustration

M | F

Partial model

Set of clauses

DPLL: Illustration

Guessing (decide)

$$p \;\;|\;\; p \vee q, \neg q \vee r$$

$$\Downarrow$$

$$p, \neg q \;|\; p \vee q, \neg q \vee r$$

DPLL: Illustration

Deducing (unit propagate)

$$p \;\; | \;\; p \vee q, \neg p \vee s$$

$$\Downarrow$$

$$p, s \,|\, p \vee q, \neg p \vee s$$

DPLL: Illustration

Backtracking

$$p, \neg s, \; q \;\mid\; p \vee q, s \vee q, \neg p \vee \neg q$$

$$\downarrow$$

$$p, s \mid p \vee q, s \vee q, \neg p \vee \neg q$$

Pure Literals

A literal is pure if only occurs positively or negatively.

Example :

$\varphi = (\neg x_1 \vee x_2) \wedge (x_3 \vee \neg x_2) \wedge (x_4 \vee \neg x_5) \wedge (x_5 \vee \neg x_4)$

$\neg x_1$ and x_3 are pure literals

Pure literal rule :

Clauses containing pure literals can be removed from the formula (i.e. just satisfy those pure literals)

$$\varphi_{\neg x_1, x_3} = (x_4 \vee \neg x_5) \wedge (x_5 \vee \neg x_4)$$

Preserve satisfiability, not logical equivalency !

DPLL (as a procedure)

- Standard backtrack search
- DPLL(F) :
  - Apply unit propagation
  - If conflict identified, return UNSAT
  - Apply the pure literal rule
  - If F is satisfied (empty), return SAT
  - Select decision variable x
    - If DPLL($F \wedge x$)=SAT return SAT
    - return DPLL($F \wedge \neg x$)

The Original DPLL Procedure – Example

assign	$1 \vee 2,\ 2 \vee \neg 3 \vee 4,\ \neg 1 \vee \neg 2,\ \neg 1 \vee \neg 3 \vee \neg 4,\ \boxed{1}$
Deduce 1	
1	$1 \vee 2,\ 2 \vee \neg 3 \vee 4,\ \boxed{\neg 1 \vee \neg 2,}\ \neg 1 \vee \neg 3 \vee \neg 4,\ 1$
Deduce $\neg 2$	
1, 2	$1 \vee 2,\ 2 \vee \neg 3 \vee 4,\ \neg 1 \vee \neg 2,\ \neg 1 \vee \neg 3 \vee \neg 4,\ 1$
Guess 3	
1, 2, 3	$1 \vee 2,\ \boxed{2 \vee \neg 3 \vee 4,}\ \neg 1 \vee \neg 2,\ \neg 1 \vee \neg 3 \vee \neg 4,\ 1$
Deduce 4	
1, 2, 3, 4	$1 \vee 2,\ 2 \vee \neg 3 \vee 4,\ \neg 1 \vee \neg 2,\ \neg 1 \vee \neg 3 \vee \neg 4,\ 1$
Conflict	

The Original DPLL Procedure – Example

assign	$1 \vee 2,\ 2 \vee \neg 3 \vee 4,\ \neg 1 \vee \neg 2,\ \neg 1 \vee \neg 3 \vee \neg 4,\ \boxed{1}$
Deduce 1	
1	$1 \vee 2,\ 2 \vee \neg 3 \vee 4,\ \boxed{\neg 1 \vee \neg 2,}\ \neg 1 \vee \neg 3 \vee \neg 4,\ 1$
Deduce $\neg 2$	
1, 2	$1 \vee 2,\ 2 \vee \neg 3 \vee 4,\ \neg 1 \vee \neg 2,\ \neg 1 \vee \neg 3 \vee \neg 4,\ 1$
Guess 3	
1, 2, 3	$1 \vee 2,\ \boxed{2 \vee \neg 3 \vee 4,}\ \neg 1 \vee \neg 2,\ \neg 1 \vee \neg 3 \vee \neg 4,\ 1$
Deduce 4	
1, 2, 3, 4	$1 \vee 2,\ 2 \vee \neg 3 \vee 4,\ \neg 1 \vee \neg 2,\ \neg 1 \vee \neg 3 \vee \neg 4,\ 1$
Undo 3	

The Original DPLL Procedure – Example

assign	$1 \vee 2,\ 2 \vee \neg 3 \vee 4,\ \neg 1 \vee \neg 2,\ \neg 1 \vee \neg 3 \vee \neg 4,\ 1$
Deduce 1	
1	$1 \vee 2,\ 2 \vee \neg 3 \vee 4,\ \neg 1 \vee \neg 2,\ \neg 1 \vee \neg 3 \vee \neg 4,\ 1$
Deduce $\neg 2$	
1, 2	$1 \vee 2,\ 2 \vee \neg 3 \vee 4,\ \neg 1 \vee \neg 2,\ \neg 1 \vee \neg 3 \vee \neg 4,\ 1$
Guess $\neg 3$	
1, 2, 3	$1 \vee 2,\ 2 \vee \neg 3 \vee 4,\ \neg 1 \vee \neg 2,\ \neg 1 \vee \neg 3 \vee \neg 4,\ 1$
Model Found	

An Abstract Framework for DPLL

State

- **fail** or M ‖ F
- where
  - F is a CNF formula, a set of clauses, and
  - M is a sequence of annotated literals denoting a partial truth assignment

Initial State

- ∅ ‖ F, where F is to be checked for satisfiability

Expected final states:

- **fail** if F is unsatisfiable
- M ‖ G
  where
  - M is a model of G
  - G is logically equivalent to F

Transition Rules for DPLL

Extending the assignment:

UnitProp $M \parallel F, C \vee l \rightarrow M\, l \parallel F, C \vee l$ $\begin{cases} M \models \neg C \\ l \text{ is undefined in } M \end{cases}$

Decide $M \parallel F, C \rightarrow M\, l^d \parallel F, C$ $\begin{cases} l \text{ or } \neg l \text{ occur in } C \\ l \text{ is undefined in } M \end{cases}$

Notation: l^d is a decision literal

Transition Rules for DPLL

Repairing the assignment:

Fail $M \parallel F, C \to \text{fail}$ $\begin{cases} M \models \neg C \\ M \text{ does not contain decision literals} \end{cases}$

Backtrack $M\ l^d\ N \parallel F, C \to M\ \neg l \parallel F, C$ $\begin{cases} M\ l^d\ N \models \neg C \\ l \text{ is the last decision literal} \end{cases}$

Transition Rules DPLL – Example

$\varnothing \parallel 1 \vee 2,\ 2 \vee \neg 3 \vee 4,\ \neg 1 \vee \neg 2,\ \neg 1 \vee \neg 3 \vee \neg 4,\ 1$

UnitProp 1

$1 \parallel 1 \vee 2,\ 2 \vee \neg 3 \vee 4,\ \neg 1 \vee \neg 2,\ \neg 1 \vee \neg 3 \vee \neg 4,\ 1$

UnitProp $\neg 2$

$1, 2 \parallel 1 \vee 2,\ 2 \vee \neg 3 \vee 4,\ \neg 1 \vee \neg 2,\ \neg 1 \vee \neg 3 \vee \neg 4,\ 1$

Decide 3

$1, 2, 3^d \parallel 1 \vee 2,\ 2 \vee \neg 3 \vee 4,\ \neg 1 \vee \neg 2,\ \neg 1 \vee \neg 3 \vee \neg 4,\ 1$

UnitProp 4

$1, 2, 3^d, 4 \parallel 1 \vee 2,\ 2 \vee \neg 3 \vee 4,\ \neg 1 \vee \neg 2,\ \neg 1 \vee \neg 3 \vee \neg 4,\ 1$

Backtrack 3

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Transition Rules DPLL – Example

$$\emptyset \parallel 1 \vee 2,\ 2 \vee \neg 3 \vee 4,\ \neg 1 \vee \neg 2,\ \neg 1 \vee \neg 3 \vee \neg 4,\ 1$$

UnitProp 1

$$1 \parallel 1 \vee 2,\ 2 \vee \neg 3 \vee 4,\ \neg 1 \vee \neg 2,\ \neg 1 \vee \neg 3 \vee \neg 4,\ 1$$

UnitProp $\neg 2$

$$1, 2 \parallel 1 \vee 2,\ 2 \vee \neg 3 \vee 4,\ \neg 1 \vee \neg 2,\ \neg 1 \vee \neg 3 \vee \neg 4,\ 1$$

Decide 3

$$1, 2, 3^d \parallel 1 \vee 2,\ 2 \vee \neg 3 \vee 4,\ \neg 1 \vee \neg 2,\ \neg 1 \vee \neg 3 \vee \neg 4,\ 1$$

UnitProp 4

Backtrack 3

$$1, 2, 3 \parallel 1 \vee 2,\ 2 \vee \neg 3 \vee 4,\ \neg 1 \vee \neg 2,\ \neg 1 \vee \neg 3 \vee \neg 4,\ 1$$

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Transition Rules for DPLL (on one slide)

UnitProp $M \parallel F, C \vee l \rightarrow M\, l \parallel F, C \vee l$ $\begin{cases} M \models \neg C \\ l \text{ is undefined in } M \end{cases}$

Decide $M \parallel F, C \rightarrow M\, l^d \parallel F, C$ $\begin{cases} l \text{ or } \neg l \text{ occur in } C \\ l \text{ is undefined in } M \end{cases}$

Fail $M \parallel F, C \rightarrow \text{fail}$ $\begin{cases} M \models \neg C \\ M \text{ does not contain decision literals} \end{cases}$

Backtrack $M\, l^d\, N \parallel F, C \rightarrow M\, \neg l \parallel F, C$ $\begin{cases} M\, l^d\, N \models \neg C \\ l \text{ is the last decision literal} \end{cases}$

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The DPLL System – Correctness

Some terminology

- Irreducible state: state to which no transition rule applies.
- Execution: sequence of transitions allowed by the rules and starting with states of the form $\emptyset \parallel F$.
- Exhausted execution: execution ending in an irreducible state

Proposition (Strong Termination) Every execution in DPLL is finite

Proposition (Soundness) For every exhausted execution starting with $\emptyset \parallel F$ and ending in $M \parallel F$, $M \models F$

Proposition (Completeness) If F is unsatisfiable, every exhausted execution starting with $\emptyset \parallel F$ ends with fail

Maintained in more general rules + theories

Modern DPLL: CDCL

Conflict **D**riven **C**lause **L**earning

- two watched literals – efficient index to find clauses that can be used in unit resolution
- periodically restart backtrack search
- activity-based decision heuristic to choose decision variable
- **conflict resolution via clausal learning**

We will briefly look at clausal learning

More details on CDCL are available in

- Chapter 2 of Decision Procedures book
- http://gauss.ececs.uc.edu/SAT/articles/FAIA185-0131.pdf

Conflict Directed Clause Learning

Lemma learning

$\neg t, p, q, s \mid t \vee \neg p \vee q, \neg q \vee s, \neg p \vee \neg s$

$\Downarrow$

$\neg t, p, q, s \mid t \vee \neg p \vee q, \neg q \vee s, \neg p \vee \neg s \mid \neg p \vee \neg s$

$\Downarrow$

$\neg t, p, q, s \mid t \vee \neg p \vee q, \neg q \vee s, \neg p \vee \neg s \mid \neg p \vee \neg q$

$\Downarrow$

$\neg t, p, q, s \mid t \vee \neg p \vee q, \neg q \vee s, \neg p \vee \neg s \mid \neg p \vee t$

Learned Clause by Resolution

A new clause is learned by resolving the conflict clause with clauses deduced from the last decision

$$\neg t, p, q, s \mid t \vee \neg p \vee q, \neg q \vee s, \neg p \vee \neg s$$

$$\dfrac{\dfrac{\neg p \vee \neg s \qquad \neg q \vee s}{\neg p \vee \neg q} \qquad t \vee \neg p \vee q}{t \vee \neg p}$$

Trivial Resolution: at every resolution step, at least one clause is an input clause

Modern CDCL: Abstract Rules

Initialize	$\epsilon \mid F$	F *is a set of clauses*
Decide	$M \mid F \implies M, \ell \mid F$	ℓ *is unassigned*
Propagate	$M \mid F, C \vee \ell \implies M, \ell^{C \vee \ell} \mid F, C \vee \ell$	C *is false under* M
Sat	$M \mid F \implies M$	F *true under* M
Conflict	$M \mid F, C \implies M \mid F, C \mid C$	C *is false under* M
Learn	$M \mid F \mid C \implies M \mid F, C \mid C$	
Unsat	$M \mid F \mid \emptyset \implies Unsat$	
Backjump	$MM' \mid F \mid C \vee \ell \implies M\ell^{C \vee \ell} \mid F$	$\bar{C} \subseteq M, \neg\ell \in M'$
Resolve	$M \mid F \mid C' \vee \neg\ell \implies M \mid F \mid C' \vee C$	$\ell^{C \vee \ell} \in M$
Forget	$M \mid F, C \implies M \mid F$	C is a learned clause
Restart	$M \mid F \implies \epsilon \mid F$	

Model

Proof

Conflict Resolution

[Nieuwenhuis, Oliveras, Tinelli J.ACM 06] customized

SAT Algorithm

```
while(true) {
  if (!Decide())
     return SAT;
  while (!BCP())
    if (!ResolveConflict())
       return UNSAT
```

Choose next variable and value. Return FALSE if all variables are assigned

Apply unit resolution while it is applicable. Return FALSE if reached a conflict

Backtrack until no conflict. Return FALSE if impossible.

Conjuctive Normal Form

$$\varphi \leftrightarrow \psi \Rightarrow_{\mathrm{CNF}} \varphi \rightarrow \psi \wedge \psi \rightarrow \varphi$$

$$\varphi \rightarrow \psi \Rightarrow_{\mathrm{CNF}} \neg\varphi \vee \psi$$

$$\neg(\varphi \vee \psi) \Rightarrow_{\mathrm{CNF}} \neg\varphi \wedge \neg\psi$$

$$\neg(\varphi \wedge \psi) \Rightarrow_{\mathrm{CNF}} \neg\varphi \vee \neg\psi$$

$$\neg\neg\varphi \Rightarrow_{\mathrm{CNF}} \varphi$$

$$(\varphi \wedge \psi) \vee \xi \Rightarrow_{\mathrm{CNF}} (\varphi \vee \xi) \wedge (\psi \vee \xi)$$

Every propositional formula can be put in CNF

PROBLEM: (potential) exponential blowup of the resulting formula

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Tseitin Transformation – Main Idea

Introduce a fresh variable e_i for every subformula G_i of F

- intuitively, e_i represents the truth value of G_i

Assert that every e_i and G_i pair are equivalent

- $e_i \leftrightarrow G_i$
- and express the assertion as CNF

Conjoin all such assertions in the end

Formula to CNF Conversion

`mk_fresh_var()` returns a fresh variable not used anywhere before

```
def cnf (ϕ):
  p, F = cnf_rec (ϕ)
  return p ∧ F

def cnf_rec (ϕ):
  if is_atomic (ϕ): return (ϕ, True)
  elif ϕ == ψ ∧ ξ:
    q, F1 = cnf_rec (ψ)
    r, F2 = cnf_rec (ξ)

    p = mk_fresh_var ()
    # C is CNF for p↔(q∧r)
    C = (¬p∨q)∧(¬p∨r)∧(p∨¬q∨¬r)
    return (p, F1∧F2∧C)
  elif ϕ == ψ∨ξ:
    …
```

Exercise: Complete cases for $\phi == \psi \vee \xi$, $\phi == \neg\psi$, $\phi == \psi \leftrightarrow \xi$

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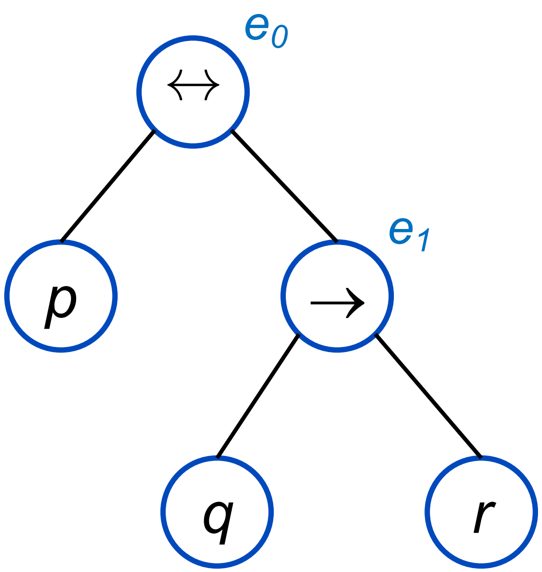
Tseitin Transformation: Example

$G : p \leftrightarrow (q \rightarrow r)$

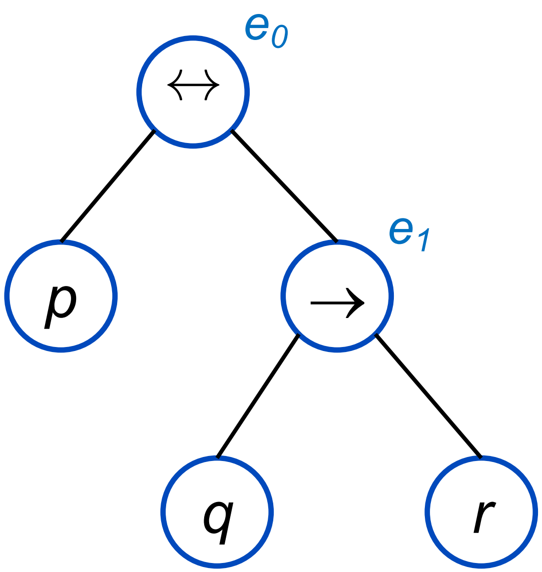


$G : e_0 \wedge (e_0 \leftrightarrow (p \leftrightarrow e_1)) \wedge (e_1 \leftrightarrow (q \rightarrow r))$

$$
\begin{aligned}
& e_1 \leftrightarrow (q \rightarrow r) \\
= \quad & (e_1 \rightarrow (q \rightarrow r)) \wedge ((q \rightarrow r) \rightarrow e_1) \\
= \quad & (\neg e_1 \vee \neg q \vee r) \wedge ((\neg q \vee r) \rightarrow e_1) \\
= \quad & (\neg e_1 \vee \neg q \vee r) \wedge (\neg q \rightarrow e_1) \wedge (r \rightarrow e_1) \\
= \quad & (\neg e_1 \vee \neg q \vee r) \wedge (q \vee e_1) \wedge (\neg r \vee e_1)
\end{aligned}
$$

Tseitin Transformation: Example

$G : p \leftrightarrow (q \rightarrow r)$

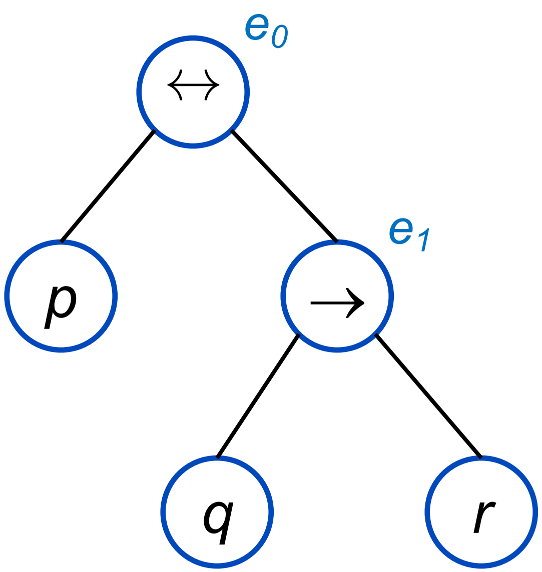


$G : e_0 \wedge (e_0 \leftrightarrow (p \leftrightarrow e_1)) \wedge (e_1 \leftrightarrow (q \rightarrow r))$

$$
\begin{aligned}
& e_0 \leftrightarrow (p \leftrightarrow e_1) \\
= \quad & (e_0 \rightarrow (p \leftrightarrow e_1)) \wedge ((p \leftrightarrow e_1)) \rightarrow e_0) \\
= \quad & (e_0 \rightarrow (p \rightarrow e_1)) \wedge (e_0 \rightarrow (e_1 \rightarrow p)) \wedge \\
& (((p \wedge e_1) \vee (\neg p \wedge \neg e_1)) \rightarrow e_0) \\
= \quad & (\neg e_0 \vee \neg p \vee e_1) \wedge (\neg e_0 \vee \neg e_1 \vee p) \wedge \\
& (\neg p \vee \neg e_1 \vee e_0) \wedge (p \vee e_1 \vee e_0)
\end{aligned}
$$

Tseitin Transformation: Example

$$G : p \leftrightarrow (q \rightarrow r)$$



$$G : e_0 \land (e_0 \leftrightarrow (p \leftrightarrow e_1)) \land (e_1 \leftrightarrow (q \rightarrow r))$$

$$G : e_0 \land (\neg e_0 \lor \neg p \lor e_1) \land (\neg e_0 \lor p \lor \neg e_1) \land (e_0 \lor p \lor e_1) \land (e_0 \lor \neg p \lor \neg e_1) \land (\neg e_1 \lor \neg q \lor \mathrm{r}) \land (e_1 \lor q) \land (e_1 \lor \neg \mathrm{r})$$

Tseitin Transformation [1968]

Used in practice

- No exponential blow-up
- CNF formula size is linear with respect to the original formula

Does not produce an equivalent CNF

However, given F, the following holds for the computed CNF F':

- F' is equisatisfiable to F
- Every model of F' can be translated (i.e., projected) to a model of F
- Every model of F can be translated (i.e., completed) to a model of F'

No model is lost or added in the conversion

MiniSat

MiniSat is one of the most famous modern SAT-solvers

- written in C++
- designed to be easily understandable and customizable
- many new SAT-solvers use MiniSAT as their base

Web page: http://minisat.se/

Good references for understanding SAT solving details

- MiniSat architecture: http://minisat.se/downloads/MiniSat.pdf
- Donald Knuth's SAT13 (also based on MiniSat)
  - http://www-cs-faculty.stanford.edu/~knuth/programs/sat13.w