Hoare Logic

Testing, Quality Assurance, and Maintenance
Winter 2020

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based on slides by Prof. Ruzica Piskac and others
I think you should be more specific here in step two.
History

Program verification is almost as old as computer science
• e.g., Checking A Large Routine by Alan Turing, 1949

In late 60s, Floyd proposed rules for flow-charts and Hoare for structured languages

Since then, there has been many axiomatic semantics for many substantial languages and many applications / automation
• ESC/Java, SLAM, PCC, SPARK/Ada, KeY, Dafny, Viper, SeaHorn, …
Tony Hoare said...

“Thus the practice of proving programs would seem to lead to solution of three of the most pressing problems in software and programming, namely, reliability, documentation, and compatibility. However, program proving, certainly at present, will be difficult even for programmers of high caliber; and may be applicable only to quite simple program designs.”

-- C.A.R Hoare, An Axiomatic Basis for Computer Programming, 1969
Tony Hoare also said…

“It has been found a serious problem to define these languages [ALGOL, FORTRAN, COBOL] with sufficient rigor to ensure compatibility among all implementations. ... one way to achieve this would be to insist that all implementations of the language shall satisfy the axioms and rules of inference which underlie proofs of properties of programs expressed in the language. In effect, this is equivalent to accepting the axioms and rules of inference as the ultimately definitive specification of the meaning of the language.”
Axiomatic Semantics

An axiomatic semantics consists of:

• a language for stating assertions about programs;
• rules for establishing the truth of assertions.

Some typical kinds of assertions:

• This program terminates.
• If this program terminates, the variables x and y have the same value throughout the execution of the program.
• The array accesses are within the array bounds.

Some typical languages of assertions

• First-order logic
• Other logics (temporal, linear, separation)
• Special-purpose specification languages (Z, Larch, JML)
Assertions for WHILE

The assertions we make about WHILE programs are of the form:

\[ \{ A \} c \{ B \} \]

with the meaning that:

- If \( A \) holds in state \( q \) and \( q \rightarrow q' \)
- then \( B \) holds in \( q' \)

\( A \) is the precondition and \( B \) is the post-condition

For example:

\[ \{ y \leq x \} z := x; z := z + 1 \{ y < z \} \]

is a valid assertion

These are called Hoare triples or Hoare assertions
{A} c {B} is a partial correctness assertion. It does not imply termination of c.

- If A holds in state $q$ and there exists $q'$ such that $q \rightarrow q'$
- then B holds in state $q'$

[A] c [B] is a total correctness assertion meaning that

- If A holds in state $q$
- then there exists $q'$ such that $q \rightarrow q'$
  and B holds in state $q'$
Now let’s be more formal

Formalize the language of assertions, $A$ and $B$

Define when an assertion holds in a state

Define rules for deriving valid Hoare triples
The Assertion Language

We use first-order predicate logic with WHILE expressions

\[ A ::= \text{true} \mid \text{false} \mid e_1 = e_2 \mid e_1 \geq e_2 \]
\[ \mid A_1 \land A_2 \mid A_1 \lor A_2 \mid A_1 \implies A_2 \mid \forall x. A \mid \exists x. A \]

We are somewhat sloppy and mix the logical variables and the program variables.

Implicitly, all WHILE variables range over integers.

All WHILE Boolean expressions are also assertions.
Semantics of Assertions

Notation $q \models A$ says that assertion $A$ holds in a given state $q$.

- This is well-defined when $q$ is defined on all variables occurring in $A$.

The $\models$ judgment is defined inductively on the structure of assertions.

It relies on the semantics of arithmetic expressions from WHILE.
Semantics of Assertions

\( q \vDash true \)
\( q \vDash e_1 = e_2 \)
\( q \vDash e_1 \geq e_2 \)
\( q \vDash A_1 \land A_2 \)
\( q \vDash A_1 \lor A_2 \)
\( q \vDash A_1 \Rightarrow A_2 \)
\( q \vDash \forall x.A \)
\( q \vDash \exists x.A \)

always

iff \( <e_1,q> \Downarrow = <e_2,q> \Downarrow \)
iff \( <e_1,q> \Downarrow \geq <e_2,q> \Downarrow \)
iff \( q \vDash A_1 \) and \( q \vDash A_2 \)
iff \( q \vDash A_1 \) or \( q \vDash A_2 \)
iff \( q \vDash A_1 \) implies \( q \vDash A_2 \)
iff \( \forall n \in \mathbb{Z}. \ q[x:=n] \vDash A \)
iff \( \exists n \in \mathbb{Z}. \ q[x:=n] \vDash A \)
Semantics of Hoare Triples

Now we can define formally the meaning of a *partial correctness* assertion:

\[ \models \{A\} c \{B\} \text{ iff } \forall q \in Q. \forall q' \in Q. q \models A \land q \rightarrow q' \Rightarrow q' \models B \]

and the meaning of a *total correctness* assertion:

\[ \models [A] c [B] \text{ iff } \forall q \in Q. q \models A \Rightarrow \exists q' \in Q. q \overset{c}{\rightarrow} q' \land q' \models B \]
Examples of Hoare Triples

\{ \text{true} \} x := 5 \{ \}

\{ \} x := x + 3 \{ x = y + 3 \}

\{ \} x := x \times 2 + 3 \{ x > 1 \}

\{ x = a \} \text{if } x < 0 \text{ then } x := -x \{ \}

\{ \text{false} \} x := 3 \{ \}

\{ x > 0 \} \text{while } x \neq 0 \text{ do } x := x - 1 \{ \}

\{ x < 0 \} \text{while } x \neq 0 \text{ do } x := x - 1 \{ \}
Inferring Validity of Assertions

We now have the formal mechanism to decide when \{A\} c \{B\}

- But it is not satisfactory,
- because \( \models \{A\} c \{B\} \) is defined in terms of the operational semantics.
- We practically have to run the program to verify an assertion.
- Thus, it is impossible to effectively verify the truth of a \( \forall x. A \) assertion (by using the definition of validity)

We need to define a symbolic technique for deriving valid assertions from others that are known to be valid
- We start with validity of first-order formulas
Inference Rules (First Order Logic)

We write $\vdash A$ when $A$ can be inferred from basic axioms.

The inference rules for $\vdash A$ are the usual ones from first-order logic with arithmetic.

Natural deduction style rules:

- $\vdash A$ and $\vdash B$ implies $\vdash A \land B$.
- $\vdash A$ implies $\vdash A \lor B$.
- $\vdash B$ implies $\vdash A \lor B$.
- $\vdash A \Rightarrow B$ and $\vdash A$ implies $\vdash B$.

- $\vdash A[e/x]$ implies $\vdash \exists x. A$.
- $\vdash \forall x. A$ implies $\vdash A[e/x]$.
- $\vdash A[a/x]$ implies $\vdash \forall x. A$ for $a$ fresh.

- $\vdash A$ implies $\vdash B$.
- $\vdash \exists x. A$ implies $\vdash B$ for $a$ fresh.
Inference Rules for Hoare Triples

We write $\vdash \{A\} \ c \ \{B\}$ when we can derive the triple using inference rules.

There is one inference rule for each command in the language.

Plus, the *rule of consequence*
- e.g., strengthen pre-condition, weaken post-condition

\[
\begin{align*}
\vdash A' & \implies A \\
\{A\} \ c \ \{B\} & \implies B \implies B' \\
\{A'\} \ c \ \{B'\} & \implies \text{Conseq}
\end{align*}
\]
Inference Rules for WHILE language

One rule for each syntactic construct:

\[ \vdash \{A\} \text{skip} \{A\} \]

\[ \vdash \{A\} s_1 \{B\} \quad \vdash \{B\} s_2 \{C\} \]

\[ \vdash \{A\} s_1 ; s_2 \{C\} \]

\[ \vdash \{A[e/x]\} x := e \{A\} \]

\[ \vdash \{A \land b\} s_1 \{B\} \quad \vdash \{A \land \neg b\} s_2 \{B\} \]

\[ \vdash \{A\} \text{if } b \text{ then } s_1 \text{ else } s_2 \{B\} \]

\[ \vdash \{I \land b\} s \{I\} \]

\[ \vdash \{I\} \text{while } b \text{ do } s \{I \land \neg b\} \]
Example: Conditional

D1 :: $\vdash \{true \land y \leq 0\} x := 1 \{x > 0\}$
D2 :: $\vdash \{true \land y > 0\} x := y \{x > 0\}$

$\vdash \{true\} \text{ if } y \leq 0 \text{ then } x := 1 \text{ else } x := y \{x > 0\}$

D1 is obtained by the rules of consequence and assignment

$\vdash true \land y \leq 0 \Rightarrow 1 > 0 \quad \vdash \{1 > 0\} x := 1 \{x > 0\}$

$\vdash \{true \land y \leq 0\} x := 1 \{x > 0\}$

D2 is obtained by the rules of consequence and assignment

$\vdash true \land y > 0 \Rightarrow y > 0 \quad \vdash \{y > 0\} x := y \{x > 0\}$

$\vdash \{true \land y > 0\} x := y \{x > 0\}$
Exercise: Hoare Rules

Is the following alternative rule for assignment still correct?

\[ \vdash \{ \text{true} \} \ x := e \ {x = e} \]
Hoare Rules

For some constructs, multiple rules are possible

alternative “forward axiom” for assignment:

\[ \vdash \{ A \} \ x := e \ \{ \exists x_0 \cdot x = e[x_0/x] \land A[x_0/x] \} \]

alternative rule for while loops:

\[ \vdash I \land b \implies C \quad \vdash \{ C \} \ c \ \{ I \} \quad \vdash I \land \neg b \implies B \]

\[ \vdash \{ I \} \ \text{while} \ b \ \text{do} \ c \ \{ B \} \]

These alternative rules are derivable from the previous rules, plus the rule of consequence.
\[
\vdash \{ I \land b \} s \{ I \}
\]

\[
\vdash \{ I \} \quad \text{while } b \text{ do } s \quad \{ I \land \neg b \}
\]
Example: a simple loop

We want to infer that

\[ \vdash \{ x \leq 0 \} \text{while } x \leq 5 \text{ do } x := x + 1 \{ x = 6 \} \]

Use the rule for while with invariant \( I \equiv x \leq 6 \)

\[ \vdash x \leq 6 \land x \leq 5 \Rightarrow x + 1 \leq 6 \quad \vdash \{ x + 1 \leq 6 \} \ x := x + 1 \{ x \leq 6 \} \]

\[ \vdash \{ x \leq 6 \land x \leq 5 \} \ x := x + 1 \{ x \leq 6 \} \]

\[ \vdash \{ x \leq 6 \} \text{while } x \leq 5 \text{ do } x := x + 1 \{ x \leq 6 \land x > 5 \} \]

Then finish-off with the rule of consequence

\[ \vdash x \leq 0 \Rightarrow x \leq 6 \quad \vdash x \leq 6 \land x > 5 \Rightarrow x = 6 \quad \vdash \{ x \leq 6 \} \text{while } \ldots \{ x \leq 6 \land x > 5 \} \]

\[ \vdash \{ x \leq 0 \} \text{while } \ldots \{ x = 6 \} \]
**Inductive Loop Invariants**

\[ \text{Inv is an inductive loop invariant if the following three conditions hold:} \]

- (Initiation) \text{Inv} holds \textit{initially} whenever the loop is reached. That is, it is true of the pre-condition \textit{Pre}

- (Consecution) \text{Inv} is \textit{preserved}: executing the loop body \textit{c} from any state satisfying \text{Inv} and loop condition \textit{b} ends in a state satisfying \text{Inv}

- (Safety) \text{Inv} is \textit{strong enough}: \text{Inv} and the negation of loop condition \textit{b} imply the desired post-condition \textit{Post}
A simple loop revisited

We want to infer that

$$\vdash \{ x \leq 0 \} \text{ while } x \leq 5 \text{ do } x := x + 1 \{ x = 6 \}$$

Using inductive invariant $x \leq 6$

$$\vdash x \leq 0 \Rightarrow x \leq 6 \quad \vdash \{ x \leq 6 \land x \leq 5 \} \ x := x + 1 \{ x \leq 6 \} \quad \vdash x > 5 \land x \leq 6 \Rightarrow x = 6$$

$$\{ x \leq 0 \} \text{ while } x \leq 5 \text{ do } x := x + 1 \{ x = 6 \}$$
Example: a more interesting program

We want to derive that
\{n \geq 0\}

\begin{verbatim}
p := 0;
x := 0;
while x < n do
    x := x + 1;
    p := p + m
\end{verbatim}

\{p = n * m\}
Example: a more interesting program

Only applicable rule (except for rule of consequence):

\[
\begin{align*}
\vdash & \{A\} \ c_1 \{C\} && \vdash & \{C\} \ c_2 \ \{B\} \\
\hline
\vdash & \{A\} \ c_1; \ c_2 \ \{B\}
\end{align*}
\]
Example: a more interesting program

What is $C$? Look at the next possible matching rules for $c_2$!

Only applicable rule (except for rule of consequence):

\[
\vdash \{I \land b\} \ c \ \{I\}
\]

\[
\vdash \{I\} \ while \ b \ do \ c \ \{I \land \neg b\}
\]

We can match $\{I\}$ with $\{C\}$ but we cannot match $\{I \land \neg b\}$ and $\{p = n \ast m\}$ directly. Need to apply the rule of consequence first!

\[
\vdash \{n \geq 0\} \ p::0; \ x::0 \ \{C\} \quad \vdash \{C\} \ while \ x < n \ do \ (x::x+1; \ p::p+m) \ \{p = n \ast m\}
\]

\[
\vdash \{n \geq 0\} \ p::0; \ x::0; \ while \ x < n \ do \ (x::x+1; \ p::p+m) \ \{p = n \ast m\}
\]

A \quad c_1 \quad c_2 \quad B
Example: a more interesting program

What is $C$? Look at the next possible matching rules for $c_2$!

Only applicable rule (except for rule of consequence):

$$\vdash \{I \land b\} \ c \ \{I\}$$

$$\vdash \{I\} \ \text{while } b \ \text{do} \ c \ \{I \land \neg b\}$$

Rule of consequence:

$$\vdash A' \Rightarrow A \quad \vdash \{A\} \ c' \ \{B\} \quad \vdash B \Rightarrow B'$$

$$\vdash \{A'\} \ c' \ \{B'\}$$

$I = A = A' = C$

$$\vdash \{n \geq 0\} \ p:=0; \ x:=0 \ \{C\}$$

$$\vdash \{C\} \ \text{while } x < n \ \text{do} \ (x:=x+1; \ p:=p+m) \ \{p = n \times m\}$$

$$\vdash \{n \geq 0\} \ p:=0; \ x:=0; \ \text{while } x < n \ \text{do} \ (x:=x+1; \ p:=p+m) \ \{p = n \times m\}$$
Example: a more interesting program

What is $I$? Let’s keep it as a placeholder for now!

Next applicable rule:

\[
\begin{align*}
\vdash \{A\} \ c_1 \{C\} & \quad \vdash \{C\} \ c_2 \ {B} \\
\vdash \{A\} \ c_1; \ c_2 \ {B} & \\
\end{align*}
\]

\[
\begin{align*}
\vdash \{I \land x<n\} \ x := x+1; \ p:=p+m \ {I} \\
\vdash \{I\} \ \text{while } x < n \ \text{do} \ (x:=x+1; \ p:=p+m) \ \{I \land x \geq n\} \\
\vdash \ I \land x \geq n \ \Rightarrow \ p = n \ast m \\
\vdash \{n \geq 0\} \ p:=0; \ x:=0 \ {I} \\
\vdash \{I\} \ \text{while } x < n \ \text{do} \ (x:=x+1; \ p:=p+m) \ \{p = n \ast m\} \\
\vdash \{n \geq 0\} \ p:=0; \ x:=0; \ \text{while } x < n \ \text{do} \ (x:=x+1; \ p:=p+m) \ \{p = n \ast m\}
\end{align*}
\]
Example: a more interesting program

What is \( C \)? Look at the next possible matching rules for \( c_2 \)!

Only applicable rule (except for rule of consequence):

\[
\vdash \{A[e/x]\} \ x:=e \ \{A\}
\]

\[
\begin{array}{c}
A \\
\vdash \{I \land x<n\} \ x:=x+1 \ \{C\}
\end{array}
\begin{array}{c}
c_1 \\
\vdash \{C\} \ p:=p+m \ \{I\}
\end{array}
\]

\[
\vdash \{I \land x<n\} \ x:=x+1; \ p:=p+m \ \{I\}
\]

\[
\vdash \{I\} \ \text{while} \ x < n \ \text{do} \ (x:=x+1; \ p:=p+m) \ \{I \land x \geq n\}
\]

\[
\vdash I \land x \geq n \Rightarrow p = n \ast m
\]

\[
\vdash \{n \geq 0\} \ p:=0; \ x:=0 \ \{I\}
\]

\[
\vdash \{I\} \ \text{while} \ x < n \ \text{do} \ (x:=x+1; \ p:=p+m) \ \{p = n \ast m\}
\]

\[
\vdash \{n \geq 0\} \ p:=0; \ x:=0; \ \text{while} \ x < n \ \text{do} \ (x:=x+1; \ p:=p+m) \ \{p = n \ast m\}
\]
Example: a more interesting program

What is $C$? Look at the next possible matching rules for $c_2$!

Only applicable rule (except for rule of consequence):

\[
\vdash \{A[e/x]\} \; x:=e \; \{A\}
\]

\[
\vdash \{I \land x<n\} \; x:=x+1 \; \{I[p+m/p]\} \quad \vdash \{I[p+m/p]\} \; p:=p+m \; \{I\}
\]

\[
\vdash \{I \land x<n\} \; x:=x+1; \; p:=p+m \; \{I\}
\]

\[
\vdash I \quad \text{while } x < n \; \text{do } (x:=x+1; \; p:=p+m) \; \{I \land x \geq n\}
\]

\[
\vdash I \quad \text{while } x < n \; \text{do } (x:=x+1; \; p:=p+m) \; \{p = n \ast m\}
\]

\[
\vdash \{n \geq 0\} \; p:=0; \; x:=0 \; \{I\}
\]

\[
\vdash \{I\} \quad \text{while } x < n \; \text{do } (x:=x+1; \; p:=p+m) \; \{p = n \ast m\}
\]

\[
\vdash \{n \geq 0\} \; p:=0; \; x:=0; \; \text{while } x < n \; \text{do } (x:=x+1; \; p:=p+m) \; \{p = n \ast m\}
\]
Example: a more interesting program

Only applicable rule (except for rule of consequence):
\[ \{A[e/x]\} x:=e \{A\} \]

Need rule of consequence to match \{I \land x<n\} and \{I[x+1/x, p+m/p]\}

\[\vdash \{I \land x<n\} x:=x+1 \{I[p+m/p]\} \quad \vdash \{I[p+m/p]\} p:=p+m \{I\}\]

\[\vdash \{I \land x<n\} x:=x+1; p:=p+m \{I\}\]

\[\vdash \{I\} \text{ while } x < n \text{ do (} x:=x+1; p:=p+m \) \{I \land x \geq n\}\]

\[\vdash I \land x \geq n \Rightarrow p = n \ast m\]

\[\vdash \{n \geq 0\} p:=0; x:=0 \{I\}\]

\[\vdash \{I\} \text{ while } x < n \text{ do (} x:=x+1; p:=p+m \) \{p = n \ast m\}\]

\[\vdash \{n \geq 0\} p:=0; x:=0; \text{ while } x < n \text{ do (} x:=x+1; p:=p+m \} \{p = n \ast m\}\]
Example: a more interesting program

Let’s just remember the open proof obligations!

\[\begin{align*}
\vdash \{1[x+1/x, \ p+m/p]\} \ x:=x+1 \ \{1[p+m/p]\} \\
\vdash 1\land x < n \Rightarrow 1[x+1/x, \ p+m/p] \\
\vdash \{1\land x<n\} \ x:=x+1 \ \{1[p+m/p]\} & \vdash \{1[p+m/p]\} \ p:=p+m \ \{1\} \\
\vdash \{1\land x<n\} \ x:=x+1; \ p:=p+m \ \{1\} & \vdash \{1\} \ \text{while } x < n \ \text{do } (x:=x+1; \ p:=p+m) \ \{1 \land x \geq n\} \\
\vdash \{n \geq 0\} \ p:=0; \ x:=0 \ \{1\} & \vdash \{1\} \ \text{while } x < n \ \text{do } (x:=x+1; \ p:=p+m) \ \{p = n \ast m\} \\
\vdash \{n \geq 0\} \ p:=0; \ x:=0; \ \text{while } x < n \ \text{do } (x:=x+1; \ p:=p+m) \ \{p = n \ast m\}
\end{align*}\]
Example: a more interesting program

Let's just remember the open proof obligations!
\[ \vdash \text{I} \land x < n \Rightarrow \text{I}[x+1/x, p+m/p] \]
\[ \vdash \text{I} \land x \geq n \Rightarrow p = n \times m \]

Continue with the remaining part of the proof tree, as before.

\[ \vdash n \geq 0 \Rightarrow \text{I}[0/p, 0/x] \]
\[ \vdash \{\text{I}[0/p, 0/x]\} \ p:=0 \ \{\text{I}[0/x]\} \]
\[ \vdash \{n \geq 0\} \ p:=0 \ {\text{I}[0/x]} \]
\[ \vdash \{\text{I}[0/x]\} \ x:=0 \ {\text{I}} \]
\[ \vdash \{n \geq 0\} \ p:=0; x:=0 \ {\text{I}} \]
\[ \vdash \{n \geq 0\} \ p:=0; x:=0; \text{while} \ x < n \text{ do} \ (x:=x+1; \ p:=p+m) \ \{p = n \times m\} \]

Now we only need to solve the remaining constraints!
Example: a more interesting program

Find $I$ such that all constraints are simultaneously valid:

$\vdash n \geq 0 \Rightarrow I[0/p, 0/x]$

$\vdash I \land x < n \Rightarrow I[x+1/x, p+m/p]$

$\vdash I \land x \geq n \Rightarrow p = n \ast m$

$I \equiv p = x \ast m \land x \leq n$

$\vdash n \geq 0 \Rightarrow 0 = 0 \ast m \land 0 \leq n$

$\vdash p = p \ast m \land x \leq n \land x < n \Rightarrow p+m = (x+1) \ast m \land x+1 \leq n$

$\vdash p = x \ast m \land x \leq n \land x \geq n \Rightarrow p = n \ast m$

All constraints are valid!
Back to the example: What did we just do?!

\{n \geq 0\}

p := 0;
x := 0;
while x < n \textbf{inv} p=x*m \land x\leq n \textbf{ do}
   x := x + 1;
   p := p + m
\{p = n * m\}
Using Hoare Rules

Hoare rules are mostly syntax directed

There are three obstacles to automation of Hoare logic proofs:

- When to apply the rule of consequence?
- What invariant to use for while?
- How do you prove the implications involved in the rule of consequence?

The last one is how theorem proving gets in the picture

- This turns out to be doable!
- The loop invariants turn out to be the hardest problem!
- Should the programmer give them?
Hoare Logic: Summary

We have a language for asserting properties of programs.

We know when such an assertion is true.

We also have a symbolic method for deriving assertions.
Software Size: Operating System

LOC OF AN OPERATING SYSTEM

- NT 3.1
- NT 3.5
- NT 4.0
- 2000
- XP
- 2003
- OSX
- Linux
Software Size: Software Distribution

LOC OF A SOFTWARE DISTRIBUTION

Verification must be automated!
Prove formulas automatically!