Solutions:

9.1 Refractive index and relative permittivity Using \( n = \sqrt{\epsilon_r} \), calculate the refractive index \( n \) of the materials in the table given their low frequency relative permittivities \( \epsilon_r \). What is your conclusion?

<table>
<thead>
<tr>
<th>Material →</th>
<th>a-Se</th>
<th>Ge</th>
<th>NaCl</th>
<th>MgO</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \epsilon_r ) (LF)</td>
<td>6.4</td>
<td>16.2</td>
<td>5.90</td>
<td>9.83</td>
</tr>
<tr>
<td>( n ) (~ 1 – 5 µm)</td>
<td>2.45</td>
<td>4.0</td>
<td>1.54</td>
<td>1.71</td>
</tr>
</tbody>
</table>

Solution

The results from the calculations are summarized in Table 9Q1-1.

For a-Se and Ge, there is an excellent agreement between \( \sqrt{\epsilon_r} \) (LF) and \( n \). Both are covalent solids in which electronic polarization (electronic bond polarization) is the only polarization mechanism at low and high frequencies. Electronic polarization involves the displacement of light electrons with respect to positive ions. This process can readily respond to the field oscillations up to optical frequencies.

For NaCl and MgO, \( \sqrt{\epsilon_r} \) (LF) is larger than \( n \). This is because at low frequencies both these solids possess a degree of ionic polarization. The bonding has a substantial degree of ionic character which contributes to polarization at frequencies below far-infrared wavelengths.

<table>
<thead>
<tr>
<th>Material</th>
<th>( \epsilon_r ) (LF)</th>
<th>( \sqrt{\epsilon_r} ) (LF)</th>
<th>( n ), optical for (1 - 5) µm range</th>
<th>Comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>a-Se</td>
<td>6.4</td>
<td>2.53</td>
<td>2.45</td>
<td>Electronic bond polarization</td>
</tr>
<tr>
<td>Ge</td>
<td>16.2</td>
<td>4.02</td>
<td>4.0</td>
<td>Electronic bond polarization</td>
</tr>
<tr>
<td>NaCl</td>
<td>5.9</td>
<td>2.43</td>
<td>1.54</td>
<td>Ionic polarization contributes to ( \epsilon_r ) (LF)</td>
</tr>
<tr>
<td>MgO</td>
<td>9.83</td>
<td>3.14</td>
<td>1.71</td>
<td>Ionic polarization contributes to ( \epsilon_r ) (LF)</td>
</tr>
</tbody>
</table>

Table 9Q1-1: Comparing \( \sqrt{\epsilon_r} \) (LF) with \( n \) for the given materials
9.7 Cauchy dispersion relation for zinc selenide  ZnSe is a II-VI semiconductor and a very useful optical material used in various applications such as optical windows (especially high power laser windows), lenses, prisms etc. It transmits over 0.50 to 19 µm. \( n \) in the 1 – 11 µm range described by a Cauchy expression of the form

\[
n = 2.4365 + \frac{0.0485}{\lambda^2} + \frac{0.0061}{\lambda^4} - 0.0003\lambda^2
\]

in which \( \lambda \) is in µm. What is ZnSe's refractive index \( n \) and group index \( N_g \) at 5 µm?

**Solution**

Given,

\[
n = 2.4365 + \frac{0.0485}{\lambda^2} + \frac{0.0061}{\lambda^4} - 0.0003\lambda^2 \quad (1)
\]

Differentiating equation (1) with respect to \( \lambda \), we obtain the following,

\[
\frac{dn}{d\lambda} = -\frac{0.097}{\lambda^3} - \frac{0.0244}{\lambda^5} - 0.0006\lambda
\]

where \( \lambda \) is in µm.

At \( \lambda = 5 \) µm:

Using equation (1), we calculate the refractive index \( n \) to be the following,

\[
n = 2.4365 + \frac{0.0485}{(5 \times 10^{-6})^2} + \frac{0.0061}{(5 \times 10^{-6})^4} - 0.0003(5 \times 10^{-6})^2 = 2.4309
\]

The group index is defined as (see Equation 9.23 on Pg 785 of the text)

\[
N_g = n - \lambda \frac{dn}{d\lambda}
\]

from (2) we see that,

\[
N_g (5 \text{µm}) = n(5 \text{µm}) - \lambda \frac{dn}{d\lambda} = 2.4309 - (5 \times 10^{-6}) \times [\frac{-0.097}{(5 \times 10^{-6})^3} - \frac{0.0244}{(5 \times 10^{-6})^5} - 0.0006 \times (5 \times 10^{-6})] = 2.449
\]

\[\therefore \quad N_g = 2.449\]
9.10 Electric and magnetic fields in light  The intensity (irradiance) of the red laser beam from a He-Ne laser in air has been measured to be about 1 mW cm\(^{-2}\). What are the magnitudes of the electric and magnetic fields? What are the magnitudes if this 1 mW cm\(^{-2}\) beam were in a glass medium with a refractive index \(n = 1.45\) and still had the same intensity?

**Solution**

The average irradiance (intensity) is given by (see Equation 9.29)

\[
I = \frac{1}{2} \frac{c \varepsilon_o n E_o^2}{n}
\]

For vacuum or air \(n = 1\) and we can calculate the magnitude of the electric field from the above relation as,

\[
E_o = \sqrt{\frac{2I}{ce_o n}} = \sqrt{\frac{2 \left(10 \text{ W m}^{-2}\right)}{\left(3 \times 10^8 \text{ m s}^{-1}\right) \left(8.85 \times 10^{-12} \text{ F m}^{-1}\right)(1)}} = 86.772 \text{ V m}^{-1}
\]

The corresponding magnetic field is

\[
B_o = \frac{nE_o}{c} = \frac{(1)\left(86.772 \text{ V m}^{-1}\right)}{\left(3 \times 10^8 \text{ m s}^{-1}\right)} = 2.892 \times 10^{-7} \text{ T}
\]

If this beam was traveling in a glass medium of \(n = 1.45\) and still had the same intensity (1 mW cm\(^{-2}\), then

\[
E_o = \sqrt{\frac{2I}{ce_o n}} = \sqrt{\frac{2 \left(10 \text{ W m}^{-2}\right)}{\left(3 \times 10^8 \text{ m s}^{-1}\right) \left(8.85 \times 10^{-12} \text{ F m}^{-1}\right)(1.45)}} = 72.06 \text{ V m}^{-1}
\]

and

\[
B_o = \frac{nE_o}{c} = \frac{(1.45)\left(86.772 \text{ V m}^{-1}\right)}{\left(3 \times 10^8 \text{ m s}^{-1}\right)} = 3.483 \times 10^{-7} \text{ T}
\]

9.11 Reflection of light from a less dense medium (internal reflection)  A ray of light which is traveling in a glass medium of refractive index \(n_1 = 1.450\) becomes incident on a less dense glass medium of refractive index \(n_2 = 1.430\). Suppose that the free space wavelength (\(\lambda\)) of the light ray is 1 \(\mu\)m.

a. What should be the minimum incidence angle for TIR?

b. What is the phase change in the reflected wave when \(\theta_i = 85^\circ\) and when \(\theta_i = 90^\circ\)?

c. What is the penetration depth of the evanescent wave into medium 2 when \(\theta_i = 85^\circ\) and when \(\theta_i = 90^\circ\)?
Solution

a. The critical angle $\theta_c$ for TIR is given by (see Equation 9.31)

$$\sin \theta_c = \frac{n_2}{n_1} = \frac{1.430}{1.450}$$

$$\therefore \quad \theta_c = 80.47^\circ$$

b. Since the incidence angle $\theta_i > \theta_c$, there is a phase shift in the reflected wave. The phase change in $E_{r,\perp}$ is given by $\phi_\perp$ (see Equation 9.44). With $n_1 = 1.450$, $n_2 = 1.430$ and $\theta_i = 85^\circ$, the phase change is

$$\phi_\perp = 2 \arctan \left[ \frac{(\sin^2 \frac{\theta_i}{2} - n^2)^{1/2}}{\cos \theta_i} \right] = 2 \arctan \left[ \frac{(\sin 85^\circ)^2 - \left(\frac{1.43}{1.45}\right)^2}{\cos 85^\circ} \right] = 116.45^\circ$$

For the $E_{r,\parallel}$ component (see Equation 9.45), the phase change is

$$\phi_\parallel = 2 \left\{ \arctan \left[ \frac{(\sin^2 \frac{\theta_i}{2} - n^2)^{1/2}}{n^2 \cos \theta_i} \right] - \frac{1}{2} \pi \right\}$$

$$\therefore \quad \phi_\parallel = 2 \arctan \left[ \frac{(\sin 85^\circ)^2 - \left(\frac{1.43}{1.45}\right)^2}{\left(\frac{1.43}{1.45}\right)^2 \cos 85^\circ} \right] - \frac{1}{2} \pi = 62.1^\circ$$

(Note: If we were to invert the reflected field, this phase change would be 117.9°).

We can repeat the calculation with $\theta_i = 90^\circ$ to find $\phi_\perp = 180^\circ$ and $\phi_\parallel = 0^\circ$.

Note that as long as $\theta_i > \theta_c$, the magnitude of the reflection coefficients are unity. There is a change only in the phase.

c. The amplitude of the evanescent wave as it penetrates into medium 2 is

$$E_{t,\perp}(y,t) \approx E_{t_0,\perp} \exp(-\alpha_2 y)$$

We ignore the $z$-dependence, $\exp(\omega t - k_z z)$, as this only gives a propagating property along $z$. The field strength drops to $e^1$ when $y = 1/\alpha_2 = \delta$, which is called the penetration depth. The attenuation constant $\alpha_2$ is given by (see Equation 9.47)

$$\alpha_2 = 2 \frac{\pi n_1}{\lambda} \left[ \left( \frac{n_1}{n_2} \right)^2 \sin^2 \theta_i - 1 \right]^{1/2}$$
\[
\alpha_2 = \frac{2\pi (1.43)}{(1 \times 10^{-6} \text{ m})} \left[ \left( \frac{1.45}{1.43} \right)^2 \left( \sin 85^\circ \right)^2 - 1 \right]^{1/2} = 1.28 \times 10^6 \text{ m}^{-1}
\]

so the penetration depth is, \( \delta = 1/\alpha_2 = 1/(1.28 \times 10^6 \text{ m}^{-1}) = 7.8 \times 10^{-7} \text{ m}, \) or 0.78 mm. For 90°, repeating the calculation we find, \( \alpha_2 = 1.5 \times 10^6 \text{ m}^{-1}, \) so that \( \delta = 1/\alpha_2 = 0.66 \text{ mm}. \) We see that the penetration is greater for smaller incidence angles.

9.12 Internal and external reflection at normal incidence

Consider the reflection of light at normal incidence on a boundary between a GaAs crystal medium of refractive index 3.6 and air of refractive index 1.

a. If light is traveling from air to GaAs, what is the reflection coefficient and the intensity of the reflected light in terms of the incident light?

b. If light is traveling from GaAs to air, what is the reflection coefficient and the intensity of the reflected light in terms of the incident light?

Solution

a. The light travels in air and becomes partially reflected at the surface of the GaAs crystal which corresponds to external reflection. Thus \( n_1 = 1 \) and \( n_2 = 3.6. \) Reflection coefficient is given by (see Equation 9.42),

\[
r_\parallel = r_\perp = \frac{n_1 - n_2}{n_1 + n_2} = \frac{1 - 3.6}{1 + 3.6} = 0.565
\]

This is negative which means that there is a 180° phase shift. The reflectance \( R \) (see Equation 9.49), which gives the fractional reflected power.

\[
\therefore \quad R = (r_\perp)^2 = (-0.565)^2 = 0.319 = 31.9 \%
\]

b. The light travels in GaAs crystal and becomes partially reflected at the crystal-air interface which corresponds to internal reflection. Thus \( n_1 = 3.6 \) and \( n_2 = 1. \) Then,

\[
r_\parallel = r_\perp = \frac{n_1 - n_2}{n_1 + n_2} = \frac{3.6 - 1}{3.6 + 1} = 0.565
\]

There is no phase shift. The reflectance is again 0.319 or 31.9%. In both cases, a and b, the amount of reflected light is the same.

9.13 Antireflection coating

a. Consider three dielectric media with flat and parallel boundaries with refractive indices \( n_1, \ n_2, \) and \( n_3. \) Show that for normal incidence the reflection coefficient between layers 1 and 2 is the same as that between layers 2 and 3 if \( n_2 = \sqrt{n_1 n_3}. \) What is the significance of this?

b. Consider a Si photodiode that is designed for operation at 900 nm. Given a choice of two possible antireflection coatings, SiO\(_2\) with a refractive index of 1.5 and TiO\(_2\) with
a refractive index of 2.3 which would you use and what would be the thickness of the antireflection coating you chose? The refractive index of Si is 3.5.

Solution

a. For light traveling in medium 1 when incident on the 1-2 interface at normal incidence the reflection coefficient is,

\[
r_{12} = \frac{n_1 - n_2}{n_1 + n_2} = \frac{n_1 - \sqrt{n_1 n_3}}{n_1 + \sqrt{n_1 n_3}} = \frac{1 - \sqrt{n_2}}{1 + \sqrt{n_2}} \left(1 + \frac{n_3}{n_1}\right)
\]

For light traveling in medium 2 when incident on the 2-3 interface at normal incidence the reflection coefficient is,

\[
r_{23} = \frac{n_2 - n_3}{n_2 + n_3} = \frac{\sqrt{n_1 n_3} - n_3}{\sqrt{n_1 n_3} + n_3} = \frac{n_1 - 1}{\sqrt{n_3} + 1} \frac{n_1}{n_1 - \sqrt{n_3}}
\]

\[
\because \quad r_{23} = r_{12}
\]

Significance: For an efficient antireflection effect, the waves A and B (see Figure 9Q13-1) should interfere destructively and to obtain a good degree of destructive interference between waves A and B, the two amplitudes must be comparable. This can be achieved by \( r_{12} = r_{23} \).

![Figure 9Q13-1: Illustration of how an antireflection coating reduces the reflected light intensity.](image)

b. The best antireflection coating should have a refractive index \( n_2 \) such that \( n_2 = (n_1 n_3)^{1/2} = [(1)(3.5)]^{1/2} = 1.87 \). Given a choice of two possible antireflection coatings, SiO\(_2\) with a refractive index of 1.5 and TiO\(_2\) with a refractive index of 2.3, SiO\(_2\) would be a good choice as it is closer to 1.87.

To find the thickness of the coating with a 900 nm wavelength (see Equation in example 9.10),
\[ d = m \left( \frac{\lambda}{4n^2} \right) \]

where \( m = 1, 3, 5, \ldots \) is an odd integer.

For \( \text{SiO}_2 \):
\[ d = \frac{(900 \ \text{nm})}{4(1.5)} \]
\[ \therefore \quad d = 150 \ \text{nm} \]
or odd multiples of \( d \).

For \( \text{TiO}_2 \) (if chosen):
\[ d = \frac{(900 \ \text{nm})}{4(2.3)} \]
\[ \therefore \quad d = 97.8 \ \text{nm} \]
or odd multiples of \( d \).

**9.17 Complex refractive index**  Spectroscopic ellipsometry measurements on a germanium crystal at a photon energy of 1.5 eV show that the real and imaginary parts of the complex relative permittivity are 21.56 and 2.772 respectively. Find the complex refractive index. What is the reflectance and absorption coefficient at this wavelength? How do your calculations match with the experimental values of \( n = 4.653 \) and \( K = 0.298 \), \( R = 0.419 \) and \( \alpha = 4.53 \times 10^6 \ \text{m}^{-1} \)?

**Solution**

We know that \( \varepsilon_\text{r} = 21.56 \) and that \( \varepsilon_\text{r}'' = 2.772 \). The real part \( n \) and the imaginary part \( K \) of the complex refractive index are solutions of the following system of equations (see Equation 9.60)
\[ n^2 - K^2 = 15.2254 \quad \text{and} \quad 2nK = 0.172 \]

We can take \( K \) from the second equation and substitute for it in the first equation,
\[ n^2 - \left( \frac{2.772}{2n} \right)^2 = 21.56 \]

This is a quadratic equation in \( n^2 \) that can be easily solved to find the four roots:
\[ n_{1,2} = \pm 4.653 \quad \text{and} \quad n_{3,4} = \pm 0.298 \]

Since \( n \) and \( K \) should be positive, and for photon energies of about 1.5 eV (more than the bandgap energy), \( n \) should be greater than one. Thus, \( n = 4.653 \) is the only root having any physical significance. Knowing \( n \), we can find \( K = \varepsilon_\text{r}''/2n = 2.772/[(2)(4.653)] = 0.298 \) (In fact, one of the roots above is \( K! \)). Both values compare very well with the experimental results.
Using Equations 9.62 and 9.57, we can calculate the reflectance $R$ and the absorption coefficient $\alpha$, respectively.

$$R = \frac{(n-1)^2 + K^2}{(n+1)^2 + K^2} = \frac{(4.653 - 1)^2 + 0.298^2}{(4.653 + 1)^2 + 0.298^2} = 0.42$$

$$\alpha = 2k'' = 2k_e K = 2 \frac{2\pi}{\lambda} K = 2 \frac{2\pi E_{ph}}{hc} K = 2 \frac{2\pi (1.5 \text{ eV})}{(4.135 \times 10^{-15} \text{ eV s})(3 \times 10^8 \text{ m s}^{-1})} (0.298)$$

$$\alpha = 4.52 \times 10^6 \text{ m}^{-1}$$

Since $n$ and $K$ were in good agreement with the experiment, $\alpha$ and $R$ are also very close to their experimental values.

### 9.21 Fundamental absorption

Consider the semiconductors in Figure 9.23, and those semiconductors listed in Table 9.3.

**a.** Which semiconductors can be candidates for a photodetector that can detect light in optical communications at 1550 nm?

**b.** For amorphous Si (a-Si), one definition of an optical gap is the photon energy that results in an optical absorption coefficient $\alpha$ of $10^4 \text{ cm}^{-1}$. What is the optical gap of a-Si in Figure 9.23?

**c.** Consider a solar cell from crystalline Si. What is the absorption depth of light at 1000 nm, and at 500 nm?

### Solution

**a.** Since $\lambda_{op} = 1550 \text{ nm}$ and only those semiconductor materials that have their $\lambda_g > \lambda_{op}$ can detect the light, we can only use, from Table 9.3, In$_{0.53}$Ga$_{0.47}$As, Ge, InAs, and InSb for our photodetector.

**b.** From Figure 9.23, when $\alpha = 10^4 \text{ cm}^{-1} = 10^6 \text{ m}^{-1}$, $\lambda_{op} \approx 650 \text{ nm}$

Thus, $E_{opt} = \frac{hc}{\lambda_{opt}} = 1.909 \text{ eV}$ or $1.91 \text{ eV}$

**c.** From Figure 9.23, when $\lambda_{op} = 500 \text{ nm}$, $\alpha = 1.3 \times 10^6 \text{ m}^{-1}$ and when $\lambda_{op} = 100 \text{ nm}$, $\alpha = 7 \times 10^3 \text{ m}^{-1}$. 
Thus, \[ \delta(500\text{ nm}) = \frac{1}{\alpha_2} = \frac{1}{1.3 \times 10^6} = 0.7692\,\mu\text{m} \text{ or } 0.77\,\mu\text{m} \]

and \[ \delta(1000\text{ nm}) = \frac{1}{\alpha_2} = \frac{1}{7 \times 10^3} = 142.86\,\mu\text{m} \text{ or } 143\,\mu\text{m} \]