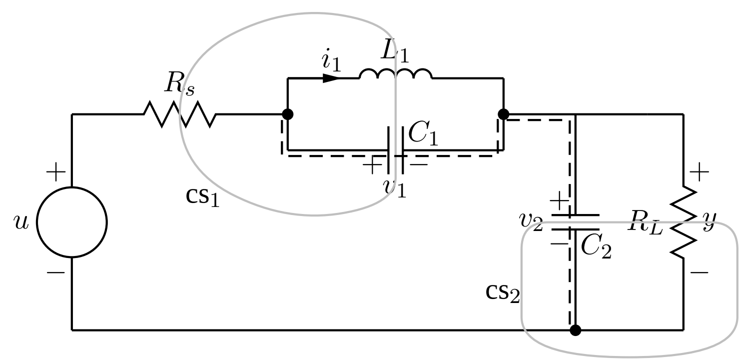

Fig. 4.1 A circuit with one input, one output, and three energy-storage elements.

1 Graph-based methods: Electric circuits

A method for finding state-space models for electric circuits will be given. Electric circuits are only one example of physical systems representable by graphs, and analogous graphs can be used to model mechanical, thermodynamic, hydraulic, and other systems. The method will be illustrated by the example in Figure 4.1 which contains a single independent input $u(t)$, a single output $y(t)$, and three elements that require derivatives in their characterizing equations.

An electric circuit is a graph, that is, a set of nodes connected by a set of branches. It is usually convenient to first perform source transformations as necessary so that every voltage source is in series with a nonsource, and every current source is in parallel with a nonsource. When identifying nodes and loops of the graph in the following procedure, a voltage source together with a nonsource in series are treated as a single branch, and a current source together with a nonsource in parallel are treated as a single branch. Then in Figure 4.1 there are five branches and three nodes.

1. A connected subgraph containing all the graph nodes but no loops is called a tree of the graph. Choose a tree that includes all the capacitors but no inductors. Such a tree is shown by the dashed line in Figure 4.1. The branches not in the tree are called links.

2. Choose the capacitor voltages and the inductor currents as state variables. Alternatively, charge can be substituted for one or more capacitor voltages, and flux linkages for one or more inductor currents.

3. If the removal of a set of branches, leaving all the nodes, reduces the original graph to exactly two subgraphs, the set of removed branches is called a cut-set. A cut-set containing links and exactly one tree branch is a fun-

damental cut-set. Two fundamental cut-sets are shown by the light lines cs_1 and cs_2 in the figure. By Kirchhoff's current law, the algebraic sum of the branch currents for any cut-set is zero. For each capacitor C_j, write the Kirchhoff equation for its fundamental cut-set,

(4.1) $$\text{capacitor current} = -\sum \text{link currents}$$

expressing all right-hand side quantities in terms of input and state variables. The links of the fundamental cut-set cs_1 for C_1 are the source branch and L_1, and the equation summing the currents to zero becomes, on division by C_1,

(4.2) $$\frac{dv_1}{dt} = -\frac{1}{C_1}\left(i_1 + \frac{v_2 + v_1 - u}{R_s}\right).$$

For C_2 the fundamental cut-set links are the source branch and R_L, giving

(4.3) $$\frac{dv_2}{dt} = -\frac{1}{C_2}\left(\frac{v_2}{R_L} + \frac{v_2 + v_1 - u}{R_s}\right).$$

4. A loop consisting of tree branches and one link is called a fundamental loop. For each inductor, sum the fundamental loop voltages to zero in an equation of the form

(4.4) $$\text{inductor voltage} = \sum \text{tree-branch voltages},$$

expressing all right-hand side quantities in terms of input and state variables. In the example the fundamental loop for L_1 contains tree branch C_1, giving, on division by L_1,

(4.5) $$\frac{di_1}{dt} = \frac{1}{L_1}v_1.$$

5. Write equations for the outputs as functions only of input or state variables. For the example the single required equation is

(4.6) $$y = v_2.$$

Example 1
Circuit

The equations of Figure 4.1, written in vector-matrix form, become

(4.7a) $$\begin{bmatrix} dv_1/dt \\ dv_2/dt \\ di_1/dt \end{bmatrix} = \begin{bmatrix} -1/(C_1R_s) & -1/(C_1R_s) & -1/C_1 \\ -1/(C_2R_s) & -1/(C_2R_s) - 1/(C_2R_L) & 0 \\ 1/L_1 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ i_1 \end{bmatrix} + \begin{bmatrix} 1/(C_1R_s) \\ 1/(C_2R_s) \\ 0 \end{bmatrix} u$$