Mathematical Background

$$\sum_{k=0}^{N} \log k \qquad \int_{0}^{\infty} e^{-x^2}\, dx$$

$$|a+b| \leqslant |a| + |b|$$

$$e^{\pi i} - 1 = 0$$

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These slides, the course material, and course web site are based on work by Douglas W. Harder

Mathematical Background

Standard reminder to set phones to silent/vibrate mode, please!

Mathematical Background

- Today's class:
  - Review of mathematical background, including:
    - Logarithms and some relevant properties
    - Arithmetic sums
    - Geometric sums
    - Recurrence relations
    - Permutations and Binomial expansion

Logarithms – Basic Properties

- Inverse of exponentials:

  If $y = \mathrm{e}^x$, then $x = \ln y$

  More in general, if $y = a^x$, then $x = \log_a y$

Logarithms – Basic Properties

- Interesting property: turns multiplicative expr. Into additive ones (*why?*):

$$\log(a \cdot b) = \log(a) + \log(b)$$

- This has an obvious, yet very interesting, consequence (example for log with base 2):

$$\lg(2x) = \lg x + 1$$

(*why is it that interesting?*)

Logarithms – Interesting Properties

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for every $\alpha > 0$

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for every $\alpha > 0$ (huh? Isn't it true for *every* α?)

Logarithms – Interesting Properties

- Exponentials grow faster than any polynomial:

$$\lim_{n \to \infty} \frac{e^n}{n^\alpha} = \infty$$

for every $\alpha > 0$

- Thus, logarithms grow slower than any polynomial:

$$\lim_{n \to \infty} \frac{\ln n}{n^\alpha} = 0$$

Logarithms – Interesting Properties

- If we start with a value N, divide it by 2, then that result we divide it by 2, and so on, until reaching 1 or less — Question: how many times did we divide before reaching 1 or less?

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- Follow-up question: How is this related to the idea of binary search?

Logarithms – Interesting Properties

- Given a value N, we write it as a decimal number (i.e. A sequence of 0 to 9 digits representing the value). Question: How many digits does it take to represent N?

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- Careful: the *exact* answer is non-trivial...

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Anyone remembers? Anyone ventures to obtain a solution? (yourselves, not Googling it!)

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Here's a thought: Do you think there should be any relationship between that sum and the following integral?

$$\int_{0}^{n} x^2 \, dx$$

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- *Solving* a recurrence relation consists of finding a closed-form expression for the sequence (that is, given the recurrence relation)

Recurrence Relations

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Actually, are we sure about that?

- Another thought: anyone sees a similarity between the above and the following?

$$\frac{dy}{dx} = 2$$

Recurrence Relations

- These recurrence relations show up very often when analyzing algorithms' performance; and we will prefer a notation that highlights the aspect of a function of the variable n, as opposed to a sequence. Thus, the previous example, for us in ECE-250, would be written as:

$$\mathrm{f}(n) = \mathrm{f}(n-1) + 2 \quad \Rightarrow \quad \mathrm{f}(n) = 2\mathrm{n}$$

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(remember that this is still wrong — *why?*)

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Permutations and Binomials

Permutations and Binomials

- You certainly remember factorials (right?)

$$n! = n \times (n-1) \times (n-2) \cdots 3 \times 2 \times 1$$

- Do you happen to remember what it means? (i.e., a physical or geometric or in some way practical interpretation of its meaning?)

- We'll look at this more in detail in class (additional details in the post-lecture slides)

Permutations and Binomials

- Binomial coefficients, the so-called "n choose k" and denoted

$$\binom{n}{k}$$

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Permutations and Binomials

- An important application is the *Binomial Expansion* — to obtain the n[th] power of (x+y):

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k \, y^{n-k}$$