Mathematical Background

$$\sum_{k=0}^{N} \log k \qquad \int_0^{\infty} e^{-x^2}\,dx$$

$$|a+b| \leqslant |a| + |b|$$

$$e^{\pi i} - 1 = 0$$

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These slides, the course material, and course web site are based on work by Douglas W. Harder

Mathematical Background

Standard reminder to set phones to silent/vibrate mode, please!

Mathematical Background

- Today's class:
  - Review of mathematical background, including:
    - Logarithms and some relevant properties
    - Arithmetic sums
    - Geometric sums
    - Recurrence relations
    - Permutations and Binomial expansion

Logarithms – Basic Properties

- Inverse of exponentials:

If $y = \mathrm{e}^x$, then $x = \ln y$

More in general, if $y = a^x$, then $x = \log_a y$

Logarithms – Basic Properties

- Interesting property: turns multiplicative expr. Into additive ones (*why?*):

$$\log(a \cdot b) = \log(a) + \log(b)$$

- This has an obvious, yet very interesting, consequence (example for log with base 2):

$$\lg(2x) = \lg x + 1$$

(*why is it that interesting?*)

Logarithms – Interesting Properties

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$$\lim_{n \to \infty} \frac{e^n}{n^\alpha} = \infty$$

for every $\alpha > 0$

Logarithms – Interesting Properties

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for every $\alpha > 0$ (huh? Isn't it true for *every* α?)

Logarithms – Interesting Properties

- Thus, logarithms grow slower than any polynomial:

$$\lim_{n \to \infty} \frac{\ln n}{n^{\alpha}} = 0$$

- The interesting aspect is: no matter how small α may be, log still grows slower
- This is for the exact same reason that no matter how *large* α, exponentials grow faster
(Hint: the inverse of $y = x^n$ is $x = y^{1/n}$)

Logarithms – Interesting Properties

- If we start with a value N, divide it by 2, then that result we divide it by 2, and so on, until reaching 1 or less — Question: How many times did we divide before reaching 1 or less?

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- To visualize the answer, consider: Start with 1, then double it, then double it again, until reaching N. If we doubled k times, then N = 2^k, or, k = lg(N)

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- Follow-up question: How is this related to the idea of binary search?

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- Follow-up question: How is this related to the idea of binary search?

  (I leave this one for you to think about it, as an exercise — the answer should be quite direct)

Logarithms – Interesting Properties

- Given a value x, we write it as a decimal number (i.e. A sequence of decimal digits representing the value). Question: How many digits does it take to represent x?

Logarithms – Interesting Properties

- The answer involves log... what a surprise! :-)
- Why? n digits represent values between 10^{n-1} and $10^{n} - 1$; $\log_{10}(x)$ is close to n (the number of digits), since $10^{n-1} \le x < 10^{n}$.

Logarithms – Interesting Properties

- The answer involves log... what a surprise! :-)
- Why? n digits represent values between 10^{n-1} and $10^n - 1$; $\log_{10}(x)$ is close to n (the number of digits), since $10^{n-1} \le x < 10^n$.

- Well... sort of ... I leave it as an exercise for you to complete the argument and get the exact answer.

Logarithms – Interesting Properties

- Follow-up question: How many *bits* does it take to represent x? (as in, if we write the binary representation of x)

- At this point, the answer should be trivial (as a random, entirely unrelated comment, remember that the notational convention for log to base 2 is lg :-))

Arithmetic Sums

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- We can easily see the solution if we represent it graphically/visually, adding twice, with one of the sequences written in inverse order:

Arithmetic Sums

$$\sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + (n-1) + n$$

$$= n + (n-1) + (n-2) + \cdots + 2 + 1$$

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$$= n + (n-1) + (n-2) + \cdots + 2 + 1$$

- Each of the pairs adds to (n+1); there are n of them, but we added the arithmetic sum twice. Thus:

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

Arithmetic Sums

- How about this variation?

$$\sum_{k=0}^{n} k^2$$

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- The solution is a little less trivial:

$$\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2\mathrm{n}+1)}{6}$$

Arithmetic Sums

- So, we ask again: Do you think there should be any relationship between that sum and the following integral?

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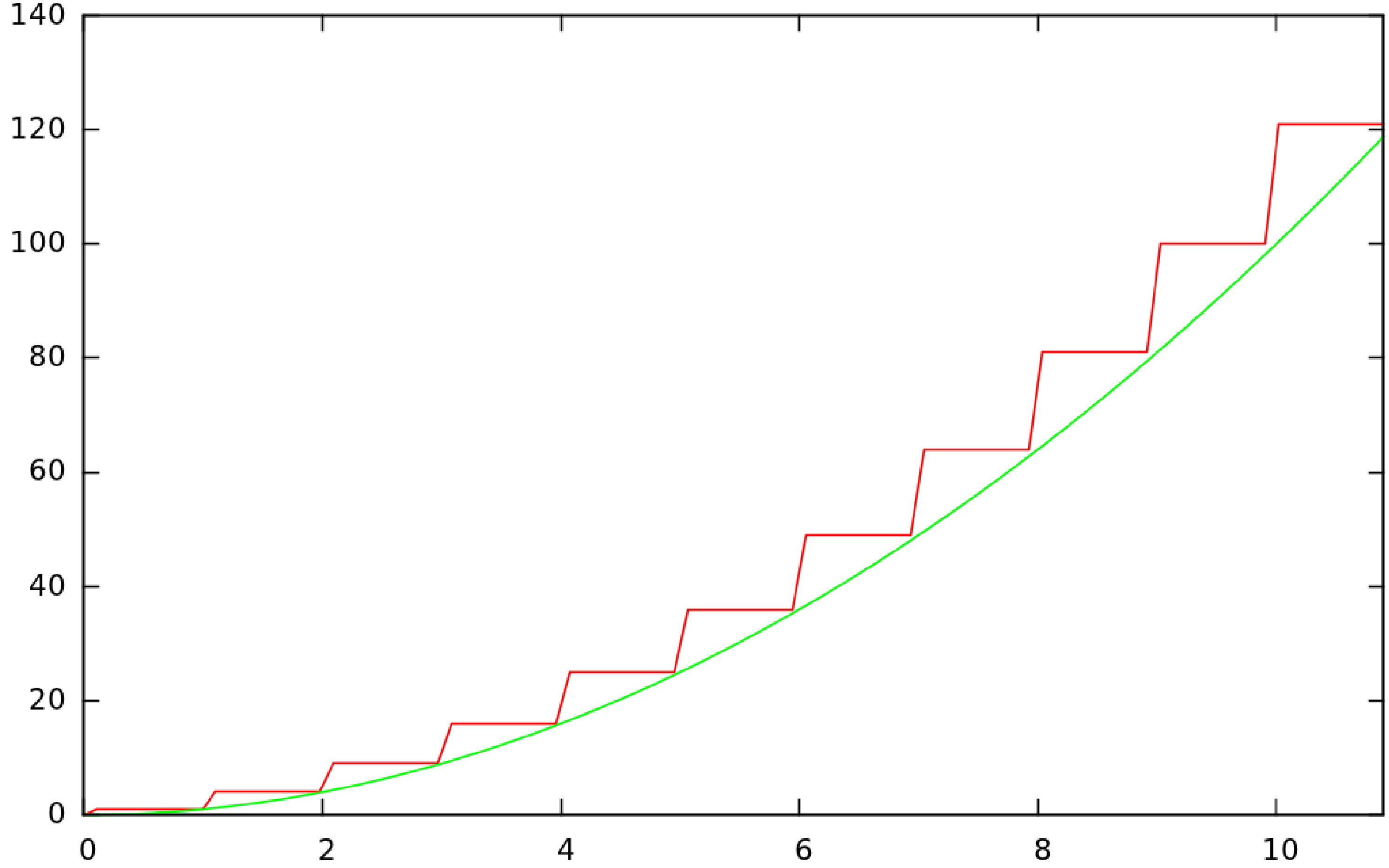
- Let's look at the geometric interpretation of each:

Arithmetic Sums



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- This one is also quite straightforward to solve...

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$$S = 1 + a + a^2 + a^3 + \cdots + a^{n-1} + a^n$$

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Geometric Sums

- From $aS = S + a^{n+1} - 1$, we solve for S, obtaining:

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- Often enough, you see it written with both signs reversed (makes more sense when |a| < 1)

Recurrence Relations

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- *Solving* a recurrence relation consists of finding a closed-form expression for the sequence (that is, given the recurrence relation)

Recurrence Relations

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Actually, are we sure about that? (doesn't this recurrence relation define sequences like $x_n = 2n + 1$ as well?)

Need to specify an “initial condition” ($x_0 = 0$ for the first example)

Recurrence Relations

- Another thought: anyone sees a similarity between the above and the following?

$$\frac{dy}{dx} = 2$$

(again, I'll leave this one for you to think about)

Recurrence Relations

- These recurrence relations show up very often when analyzing algorithms' performance; and we will prefer a notation that highlights the aspect of a function of the variable n, as opposed to a sequence. Thus, the previous example, for us in ECE-250, would be written as:

$$\mathrm{f}(n) = \mathrm{f}(n-1) + 2 \quad \Rightarrow \quad \mathrm{f}(n) = 2\mathrm{n}$$

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(along with an initial condition, of course)

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- For example, if you remember previous slides, you should easily solve this one:

$$\mathrm{f}(n) = \mathrm{f}(n/2) + 1, \quad \textit{with} \ \ \mathrm{f}(1) = 0$$

Permutations and Binomials

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- You certainly remember factorials (right?)

$$n! = n \times (n-1) \times (n-2) \cdots 3 \times 2 \times 1$$

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- They represent the number of permutations for a group of n elements (easy to visualize why... right?)

Permutations and Binomials

- Binomial coefficients, the so-called "n choose k" and denoted

$$\binom{n}{k}$$

are closely related to factorials and permutations; the value of that expression is given by:

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

Permutations and Binomials

- They represent the number of choices of k elements out of a set of n elements, disregarding the order in which they were given.
- Classical example is the number of different hands that you can get in a cards game (in this example, if it is a hand of, say, 5 cards, and the deck is 52 cards, we would have a total of 52 choose 5 different hands.

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- Classical example is the number of different hands that you can get in a cards game (in this example, if it is a hand of, say, 5 cards, and the deck is 52 cards, we would have a total of 52 choose 5 different hands.
- (Is it clear why this is the case?)

Permutations and Binomials

- An important application is the *Binomial Expansion* — to obtain the n^{th} power of (x+y):

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k \, y^{n-k}$$

Permutations and Binomials

- An interesting (though extremely obvious) property of this binomial coefficient is its symmetry:

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(you may have noticed how polynomial expansions for $(x+a)^n$ are always symmetric; try it out if you haven't!)

Permutations and Binomials

- When you get to your course of Probabilities and Random Processes, you will see that this has many applications in that area, including the very important Binomial distribution, related to Bernoulli trials

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Summary

- During today's lesson, we discussed:
  - A review of some of the relevant mathematical tools that we'll use during this course, including:
    - Logarithms and some interesting properties
    - Arithmetic sums (and relationship to integrals)
    - Geometric sums
    - Recurrence relations and a brief overview of their relevance in this course.
    - Permutations and Binomial expansion