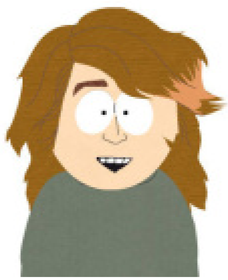
Proofs and Mathematical Induction



$$\sum_{k=0}^{N} \log k \qquad \int_0^\infty e^{-x^2}\, dx$$

$$|a+b| \leqslant |a| + |b|$$

$$e^{\pi i} - 1 = 0$$

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Proofs and Mathematical Induction

Standard reminder to set phones to silent/vibrate mode, please!

Proofs and Mathematical Induction

- Today's class:
  - Discuss the importance of proofs for us, engineers
  - Introduce some basic notions — but...
  - We'll mainly focus on mathematical induction
    - Including a (rather neat) example of incorrect use of the technique!
  - (Next class, we'll look at some other typical techniques and tricks)

Proofs and Mathematical Induction

- Mathematical proof:
  - Rough / informal definition:
    An argument, typically based on logic/deductive steps, that shows, in a verifiable and non-disputable way, that a given statement is true.
  - Typically, proofs rely on some "background rules" to be true (usually called "axioms").
    - For example, algebraic manipulations and basic properties of functions, etc., may be simply used as part of a proof, since they are assumed to be true — we don't need to extend our argument to cover those as well.

Proofs and Mathematical Induction

- Mathematical proof:
  - It is essential that a proof uses deductive arguments rather than inductive or intuition-based arguments:
    - The proof must show that the statement holds under all possible conditions/scenarios, rather than showing a (no matter how large) number of cases that confirm it.
    - For example, to prove that $|a+b| \leq |a| + |b|$, it's not enough to show a number of examples of values for *a* and *b* where the inequality holds.

Proofs and Mathematical Induction

- Mathematical proof:
  - Bottom line — our arguments have to be carefully chosen and we have to be very strict about what they say and what we conclude about them.

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Proofs and Mathematical Induction

- Mathematical proof:
  - Bottom line — our arguments have to be carefully chosen and we have to be very strict about what they say and what we conclude about them.
  - As an example/analogy of how proofs should work (including how shouldn't they work!), I'm going to *prove* to you that my son, last year when he was 1.75 years old, recognized and loved Chopin's music.
  - The proof is a video that, well, I claim proves it !

Proofs and Mathematical Induction

- Mathematical proof:
  - Bottom line — our arguments have to be carefully chosen and we have to be very strict about what they say and what we conclude about them.
  - As the video shows, it may be very easy for our minds to inadvertently jump to *inductive* reasoning (instead of the required *deductive* reasoning) and draw general conclusions/statements from particular cases.
  - Also, maybe keep this silly example in mind when writing test cases/protocols for boolean functions!!

Proofs and Mathematical Induction

- Why are proofs important for us?
  - If you happen to end up doing research, or even as early as Grad school, you will need a reasonable level of skills at proving things — remember that you would be creating new knowledge! Because it is new, you have to convince your audience that it is correct!
  - But also while practicing engineering, coming up with a proof may be a way to convince yourselves (or your boss!) that some trick you just came up with works!

Proofs and Mathematical Induction

- Why are proofs important for us?
  - Additionally, proofs (at least the “good” ones) may be quite insightful — often enough, they tell you a lot about the statement being proven (such as why it is true, or why exactly it works the way it works)

Proofs and Mathematical Induction

- Many “standard” techniques for proofs.
- We'll see some of the basic ones next class
- Today, we'll focus on Proofs by Induction.

Mathematical Induction

- This technique is applicable to “discrete” cases; typically, statements involving either an integer number, or a group of objects (the number of objects is restricted to be an integer).
- Statements to be proven may be a formula for *n*, purported to hold for every value of $n > 0$, or it could be a statement about a property of a set of values, etc.

Mathematical Induction

- It could be an algorithm, or some statement about a data structure!! (the "statement" to be proven about the algorithm typically being its correctness, or the fact that it requires whatever number of operations to complete, etc.)

Mathematical Induction

- It could be an algorithm, or some statement about a data structure!! (the "statement" to be proven about the algorithm typically being its correctness, or the fact that it requires whatever number of operations to complete, etc.)
- Why is mathematical induction applicable here?
  - An algorithm is discrete by nature, in that it runs in an integer number of steps.
  - Data structures are discrete by nature.

Mathematical Induction

- How it works:
  - We want to prove a statement about *n*, or which is stated for a number of things, or for something with integer size, that we denote *n*.
  - The statement is supposed to be valid for $n \geq n_0$, for some n_0 (for things with a size, or with a number of elements, it is somewhat implicit that $n_0 = 1$)
  - We proceed in two steps:
    - Base case
    - Induction step

Mathematical Induction

- How it works:
  - Base case:
    We start by showing that the statement is true for $n = n_0$ (this is not hard as a general proof, since it is a particular case).
  - Induction step:
    We show that if the statement holds for n = k, then the statement holds for n = k+1 as well. Notice that it is an implication that we're trying to show; thus, we start by assuming (this is called the *induction hypothesis*) that the statement holds for n = k

Mathematical Induction

- How it works:
  - Errm.... huh? Isn't *k* supposed to be an arbitrary value? So, we're assuming that the statement (what we need to prove, remember!) is true.... How is this not a logical fallacy of assuming the result??

Mathematical Induction

- How it works:
  - Errm.... huh? Isn't *k* supposed to be an arbitrary value? So, we're assuming that the statement (what we need to prove, remember!) is true.... How is this not a logical fallacy of assuming the result??

  - The important detail to keep in mind is that we need to prove *an implication*! (that statement true for n=k *implies* that it is true for n=k+1 as well). To prove an implication, *we need to assume* the antecedent.

Mathematical Induction

- How it works:
  - So, the trick: we're showing that it is true for the lowest possible value (say that it is $n = 1$). And we're also showing that the fact that it is true for $n = k$ implies that it is also true for $n = k+1$. These two things together imply that the statement is true for all values of n.
  - True for $n=1 \Rightarrow$ True for $n=2$. But now, true for $n=2 \Rightarrow$ true for $n=3$, and so on, for every value of $n \geq 1$.

Mathematical Induction

- An example:
  - Last class, we saw, without much supporting evidence, the sum of squares. Let's prove that one by induction:

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2\mathrm{n}+1)}{6}$$

(for all $n \geq 1$)

Mathematical Induction

- An example:
  - Step 1: Base case — we show that the statement holds for $n = 1$

$$\sum_{k=1}^{1} k^2 = 1^2 = 1 = \frac{1(1+1)(2\times 1+1)}{6}$$

Mathematical Induction

- An example:
  - Step 2: Induction step — we show that if the statement holds for *n*, then it also holds for *n*+1 (notice that we don't really need to call it *k* and *k*+1; *n* is as much an arbitrary name as *k* is — we are, in a sense, disconnected from the original statement, so *n* in here does not correspond to *n* in the original statement)

Mathematical Induction

- An example:
  - Step 2: Induction step — we show that if the statement holds for *n*, then it also holds for *n*+1
  - To this end, we define our *induction hypothesis*, which is that the statement is true for *n*:

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

Mathematical Induction

- An example:
  - Based on this induction hypothesis, we need to show that the statement holds for *n*+1; that is:

$$\sum_{k=1}^{n+1} k^2 = \frac{(n+1)(n+2)(2n+3)}{6}$$

Mathematical Induction

- An example:
  - How do we proceed? Two details: we know that the sum up to $n+1$ is the sum up to n, plus $(n+1)^2$ (this is "axiomatically" true — plain old algebra).
  - But we also know what the sum up to n is.... (right?)
  - Let's do the derivation on the board first (it's in the next few slides, so don't worry about taking notes):

Mathematical Induction

- An example:

$$\sum_{k=1}^{n+1} k^2 = \sum_{k=1}^{n} k^2 + (n+1)^2$$

$$= \frac{n(n+1)(2\mathrm{n}+1)}{6} + (n+1)^2$$

$$= \frac{n(n+1)(2\mathrm{n}+1) + 6(n+1)^2}{6}$$

Mathematical Induction

- Working on the numerator, take common factor (n+1) to get:

$$n(n+1)(2n+1)+6(n+1)^2$$
$$=(n+1)(n(2n+1)+6(n+1))$$
$$=(n+1)(2n^2+7n+6)$$
$$=(n+1)(n+2)(2n+3)$$

Mathematical Induction

- Working on the numerator, take common factor (n+1) to get:

$$n(n+1)(2\text{n}+1)+6(n+1)^2$$
$$=(n+1)\left(n(2\text{n}+1)+6(n+1)\right)$$
$$=(n+1)(2\text{n}^2+7\text{n}+6)$$
$$=(n+1)(n+2)(2\text{n}+3)$$

- Thus, we're obtaining:

Mathematical Induction

$$\sum_{k=1}^{n+1} k^2 = \frac{(n+1)(n+2)(2n+3)}{6}$$

$$= \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6}$$

And this completes the induction step (showing that statement true for *n* implies that statement is true for *n*+1), completing the proof

Mathematical Induction

- In the previous example, we (sort of) started from the induction hypothesis and worked our way to the expression corresponding to the case $n+1$ (which is the one that we need to show to be true, based on the assumed induction hypothesis)
- We could have worked in the opposite direction; in the next example we'll proceed that way.

Mathematical Induction

- Example 2:

  Prove that $8^n - 3^n$ is divisible by 5 for all n ≥ 1.

Mathematical Induction

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  Prove that $8^n - 3^n$ is divisible by 5 for all n ≥ 1.

- Step 1: Base case: $n = 1$, $8 - 3 = 5$, which is of course divisible by 5.

Mathematical Induction

- Example 2:
  Prove that $8^n - 3^n$ is divisible by 5 for all n ≥ 1.

- Step 1: Base case: $n = 1$, $8 - 3 = 5$, which is of course divisible by 5.

- Induction step: Based on the induction hypothesis that the statement holds for *n*, we want to show that it holds for $n+1$ as well.

Mathematical Induction

- Example 2:
- That is, we want to show that $8^{n+1} - 3^{n+1}$ is divisible by 5 *provided that* $8^n - 3^n$ is divisible by 5.
- We observe (somewhat trivial observation) that being divisible by 5 is the same as being a multiple of 5.
- We'll work on the fun part on the board first ...

Mathematical Induction

- Example 2:

$$\begin{aligned} 8^{n+1} - 3^{n+1} &= 8 \cdot 8^n - 3 \cdot 3^n \\ &= 8 \cdot 8^n - 8 \cdot 3^n + 5 \cdot 3^n \\ &= 8\left(8^n - 3^n\right) + 5 \cdot 3^n \end{aligned}$$

Mathematical Induction

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$$\begin{aligned} 8^{n+1} - 3^{n+1} &= 8 \cdot 8^n - 3 \cdot 3^n \\ &= 8 \cdot 8^n - 8 \cdot 3^n + 5 \cdot 3^n \\ &= 8\left(8^n - 3^n\right) + 5 \cdot 3^n \end{aligned}$$

- By induction hypothesis, $8^n - 3^n$ is divisible by 5, or equivalently, is a multiple of 5; say, $5 \cdot m$ for some m.

Mathematical Induction

- Example 2:

  Thus,

$$8^{n+1} - 3^{n+1} = 5 \cdot m + 5 \cdot 3^n = 5\left(m + 3^n\right)$$

- That is, $8^{n+1} - 3^{n+1}$ is divisible by 5, completing the proof.

Mathematical Induction

- And again, $8^n - 3^n$ being divisible by 5 is what we want to prove, so how come we can just assume it, simply because we add the “by induction hypothesis” ??

Mathematical Induction

- And again, $8^n - 3^n$ being divisible by 5 is what we want to prove, so how come we can just assume it, simply because we add the “by induction hypothesis” ??
- Again, we're proving the *implication* $8^n - 3^n$ divisible by 5 implies that $8^{n+1} - 3^{n+1}$ is divisible by 5. We need to assume it in the context of proving this implication, which is unrelated to the $8^n - 3^n$ in the original statement.

Mathematical Induction

- There's an alternative form of Mathematical Induction, known as *Strong* or *Complete* Induction, in which the induction hypothesis is not simply the statement being true for n (or for $n=k$), but the statement being true for all values between n_0 and n. (n_0 being the value for the base case).
- The next example illustrates this approach.

Mathematical Induction

- Example 3:
  Prove that every integer number > 1 is either prime, or it is divisible by a prime number.

Mathematical Induction

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- Step 1: Base case, in this case $n = 2$. The statement trivially holds, since 2 is prime.

Mathematical Induction

- Example 3:
  Prove that every integer number > 1 is either prime, or it is divisible by a prime number.

- Step 1: Base case, in this case $n = 2$. The statement trivially holds, since 2 is prime.

  (errr, don't we need to verify the "or is divisible by a prime number?")

Mathematical Induction

- Example 3:
  Prove that every integer number > 1 is either prime, or it is divisible by a prime number.

- Step 1: Base case, in this case $n = 2$. The statement trivially holds, since 2 is prime.

  (errr, don't we need to verify the “or is divisible by a prime number?” .... I'll leave it to you to think about why not)

Mathematical Induction

- Example 3:
  Prove that every integer number > 1 is either prime, or it is divisible by a prime number.

- Step 2: Induction step. The induction hypothesis now is that *every number* between 2 and n is either prime, or is divisible by a prime.
- We need to show that $n+1$ also is either prime or divisible by a prime.

Mathematical Induction

- Example 3:

  Clearly, for n+1 there are only two possibilities: either it is prime, or it is not prime. If it is prime, the statement holds.

Mathematical Induction

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  Clearly, for n+1 there are only two possibilities: either it is prime, or it is not prime. If it is prime, the statement holds.

  If it is not prime, then by definition, it is divisible by some number (say, $n_1 \neq 1$). If n+1 is divisble by n_1, then $n_1 < n+1$ — therefore, $2 \leq n_1 \leq n$

Mathematical Induction

- Example 3:

By induction hypothesis, if $2 \leq n_1 \leq n$, then n_1 is either prime, or divisible by a prime (say, p_1), and we're pretty much done: If n_1 is prime, then the statement holds (n+1 is divisible by a prime). If not.... then it is divisible by p_1 times something — therefore, it is divisible by a prime (namely, by p_1)

Mathematical Induction

- So, what could go wrong with Mathematical Induction?
- You have to apply the technique carefully, as the next example shows by proving a (blatantly false) statement!!

Mathematical Induction

- Example 4:

  Prove that in every group of n numbers, all numbers are equal to each other.

Mathematical Induction

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  Prove that in every group of n numbers, all numbers are equal to each other. (I'm not kidding!! We're going to prove this!!!)

Mathematical Induction

- Example 4:

  Prove that in every group of n numbers, all numbers are equal to each other.

- Step 1: Base case, n=1. The statement is trivially true: In a group with only one number, all numbers are equal.

Mathematical Induction

- Example 4:

  Prove that in every group of *n* numbers, all numbers are equal to each other.

- Induction step: Our induction hypothesis is that the statement holds for *n*; that is, in every group of *n* numbers, all numbers are equal, and we want to show that this implies that in every group of *n*+1 numbers all numbers must also be equal.

Mathematical Induction

- Example 4:

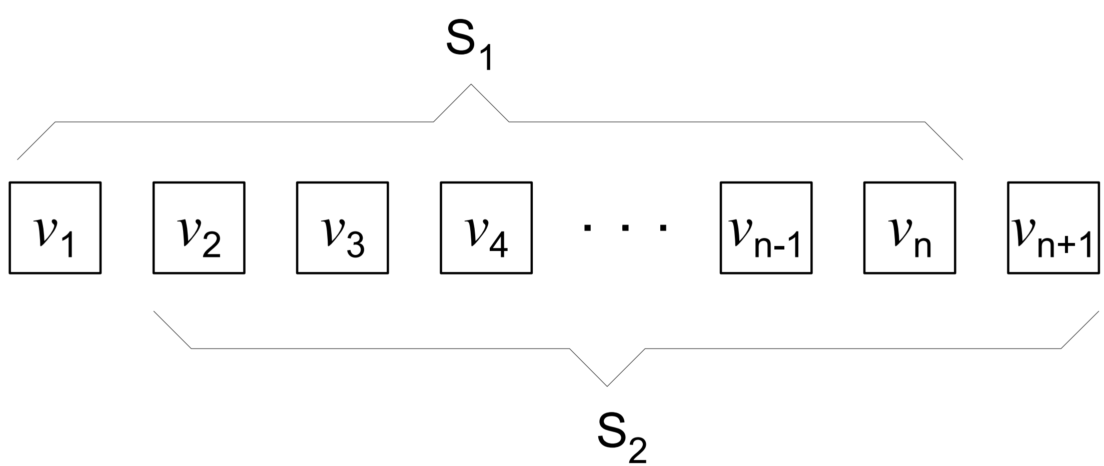
  The argument in the induction step will not be algebraic (but then, no-one said that induction proofs are restricted to algebraic / arithmetic arguments). Consider a group of *n*+1 values:

v_1	v_2	v_3	v_4	. . .	v_{n-1}	v_n	v_{n+1}

Mathematical Induction

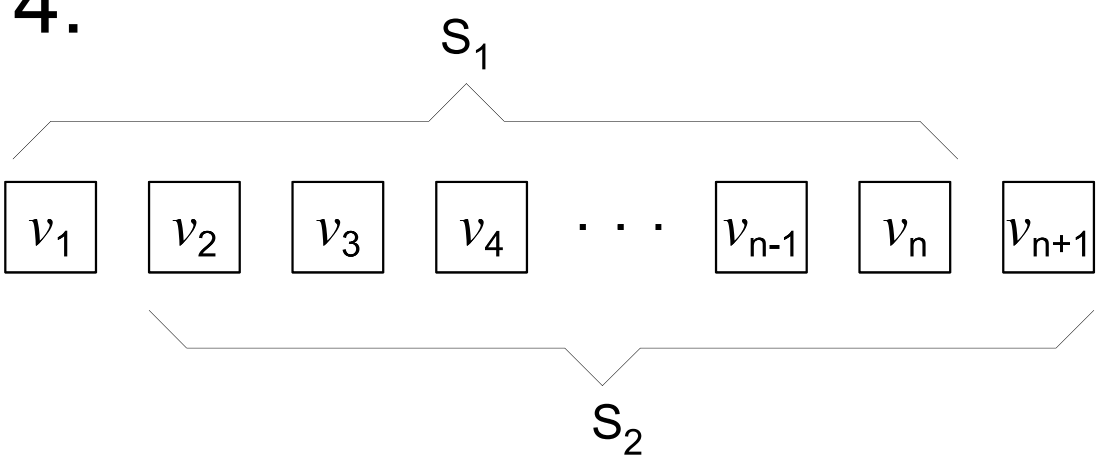
- Example 4:

  The argument in the induction step will not be algebraic (but then, no-one said that induction proofs are restricted to algebraic / arithmetic arguments). Consider a group of *n*+1 values. And consider the two "subsets" indicated below:



Mathematical Induction

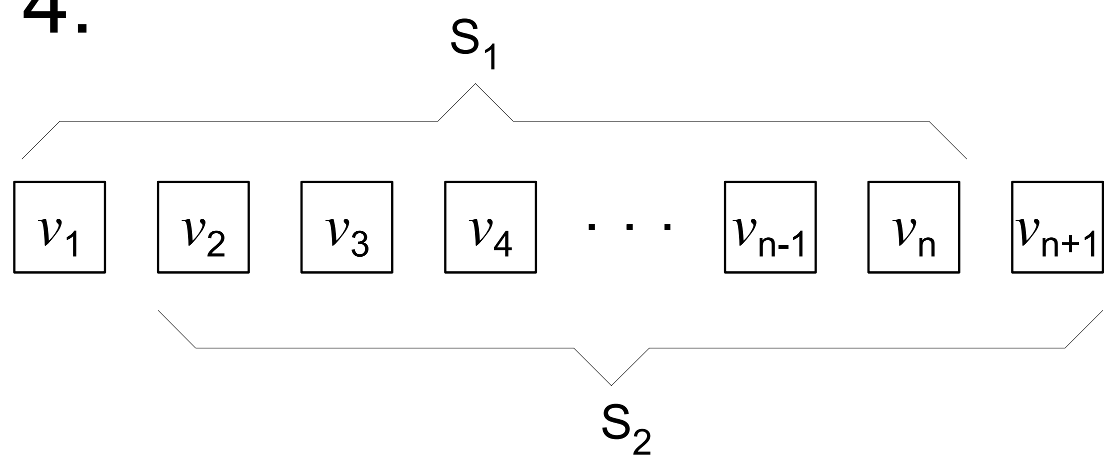
- Example 4:



Since both S_1 and S_2 are groups of *n* numbers, by induction hypothesis, both consist of *n* times the same value.

Mathematical Induction

- Example 4:



That is, by induction hypothesis for S_1, we obtain $v_1=v_2= \cdots =v_n$, and by induction hypothesis for S_2, we obtain $v_2=v_3= \cdots =v_n=v_{n+1}$.

By combining the two sets of equalities, we obtain $v_1=v_2= \cdots =v_n=v_{n+1}$, completing the proof!!

Mathematical Induction

- Example 4:

  So... What went wrong?

  You will understand that I will not accept the argument that this example shows that Mathematical Induction does not work... Right?

Mathematical Induction

- Example 4:

  So... What went wrong?

  Notice also that we're not talking about an unsubstantiated argument, assuming the result or something like that — the argument is actually quite clever — the “cleverness” being in the fact that the overlap in the two subsets is what allows the induction step to work!

Mathematical Induction

- Example 4:

  So... I ask again, and hope you guys will be intrigued and love solving puzzles and will think about it: *What went wrong?*

  Please think about it, and try to figure out the flaw in this proof by yourself!

  (maaaybe we'll discuss the solution next class)

Mathematical Induction

- Example 4:

BTW, when I say “the argument is quite clever”, I'm not bragging / being blatantly immodest — I clarify: the argument is not mine. In fact, it is much more clever than it appears — the real cleverness being on how an incorrect argument can look so much like a correct and clever one!

The puzzle was originally proposed by George Pólya, a Hungarian Mathematician.

Summary

- During today's lesson, we discussed:
  - Basic notions about mathematical proofs
  - Proofs by induction and examples
  - An example of strong induction
  - An example of a flawed proof by induction