

Proofs and Mathematical Induction



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$$\sum_{k=0}^N \log k \int_0^{\infty} e^{-x^2} dx$$

$$|a+b| \leq |a| + |b|$$

$$e^{\pi i} - 1 = 0$$

<https://ece.uwaterloo.ca/~cmoreno/ece250>

Proofs and Mathematical Induction

- Today's class:
 - Discuss the importance of proofs for us, engineers
 - Introduce some basic notions — but...
 - We'll mainly focus on mathematical induction
 - Including some (rather neat) examples of incorrect use of the technique!
 - (Next class, we'll look at some other typical techniques and tricks)

Proofs and Mathematical Induction

- Mathematical proof:
 - Rough / informal definition:
An argument, typically based on logic/deductive steps, that shows, in a verifiable and non-disputable way, that a given statement is true.
 - Typically, proofs rely on some “background rules” to be true (usually called “axioms”).
 - For example, algebraic manipulations and basic properties of functions, etc., may be simply used as part of a proof, since they are assumed to be true — we don't need to extend our argument to cover those as well.

Proofs and Mathematical Induction

- Mathematical proof:
 - It is essential that a proof uses deductive arguments rather than inductive or intuition-based arguments:
 - The proof must show that the statement holds under all possible conditions/scenarios, rather than showing a (no matter how large) number of cases that confirm it.
 - For example, to prove that $|a+b| \leq |a| + |b|$, it's not enough to show a number of examples of values for a and b where the inequality holds.

Proofs and Mathematical Induction

- Mathematical proof:
 - Often enough, the statement being proven is an implication — if A holds, then B must hold as well (this is a “rephrased” way of an implication; the above is the same as saying “A implies B”).
 - In those cases, as part of the argument, we must assume A to be true (and this assumption is indisputable — the important detail to understand is that *in the context of proving the implication*, A is indisputably true).
 - However, careful! One mistake would be to assume B (either directly or indirectly) — this is one of the standard categories of “logical fallacies”: assuming the conclusion!

Proofs and Mathematical Induction

- Why are proofs important for us?
 - If you happen to end up doing research, or even as early as Grad school, you will need a reasonable level of skills at proving things — remember that you would be creating new knowledge! Because it is new, you have to convince your audience that it is true!
 - But also while practicing engineering, coming up with a proof may be a way to convince yourselves (or your boss!) that some trick you just came up with works!

Proofs and Mathematical Induction

- Why are proofs important for us?
 - Additionally, proofs (at least the “good” ones) may be quite insightful — often enough, they tell you a lot about the statement being proven (such as why it is true, or why exactly it works the way it works)

Proofs and Mathematical Induction

- Many “standard” techniques for proofs.
- We'll see some of the basic ones next class
- Today, we'll focus on Proofs by Induction.

Mathematical Induction

- This technique is applicable to “discrete” cases; typically, statements involving either an integer number, or a group of objects (the number of objects is restricted to be an integer).
- Statements to be proven may be a formula for n , purported to hold for every value of $n > 0$, or it could be a statement about a property of a set of values, etc.

Mathematical Induction

- It could be an algorithm, or some statement about a data structure!! (the “statement” to be proven about the algorithm typically being its correctness, or the fact that it requires whatever number of operations to complete, etc.)
- Why is mathematical induction applicable here?
 - An algorithm is discrete by nature, in that it runs in an integer number of steps.
 - Data structures are discrete by nature.

Mathematical Induction

- How it works:
 - We want to prove a statement about n , or which is stated for a number of things, or for something with integer size, that we denote n .
 - The statement is supposed to be valid for $n \geq n_0$, for some n_0 (for things with a size, or with a number of elements, it is somewhat implicit that $n_0 = 1$)
 - We proceed in two steps:
 - Base case
 - Induction step

Mathematical Induction

- How it works:

- Base case:

We start by showing that the statement is true for $n = n_0$ (this is not hard as a general proof, since it is a particular case).

- Induction step:

We show that if the statement holds for $n = k$, then the statement holds for $n = k+1$ as well. Notice that it is an implication that we're trying to show; thus, we start by assuming (this is called the *induction hypothesis*) that the statement holds for $n = k$.

Mathematical Induction

- How it works:
 - Errm.... huh? Isn't k supposed to be an arbitrary value? So, we're assuming that the statement (what we need to prove, remember!) is true.... How is this not a logical fallacy of assuming the result??

Mathematical Induction

- How it works:
 - So, the trick: we're showing that it is true for the lowest possible value (say that it is $n = 1$). And we're also showing that the fact that it is true for $n = k$ implies that it is also true for $n = k+1$. These two things together imply that the statement is true for all values of n .
 - True for $n=1 \Rightarrow$ True for $n=2$. But now true for $n=2 \Rightarrow$ true for $n=3$, and so on, for every value of $n \geq 1$.

Mathematical Induction

- An example:
 - Last class, we saw, without much supporting evidence, the sum of squares. Let's prove that one by induction:

$$\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$$

(for all $n \geq 1$)

Mathematical Induction

- An example:
 - Step 1: Base case — we show that the statement holds for $n = 1$

$$\sum_{k=1}^1 k^2 = 1^2 = 1 = \frac{1(1+1)(2 \times 1 + 1)}{6}$$

Mathematical Induction

- An example:
 - Step 2: Induction step — we show that if the statement holds for n , then it also holds for $n+1$ (why are we using, or rather, are we allowed to say for n , instead of for $n=k$? I mean, isn't this *really* assuming the result??)
 - To this end, we define our *induction hypothesis*, which is that the statement is true for n :

$$\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$$

Mathematical Induction

- An example:
 - Based on this induction hypothesis, we need to show that the statement holds for $n+1$; that is:

$$\sum_{k=1}^{n+1} k^2 = \frac{(n+1)(n+2)(2n+3)}{6}$$

Mathematical Induction

- An example:
 - How do we proceed? Two details: we know that the sum up to $n+1$ is the sum up to n , plus $(n+1)^2$ (this is “axiomatically” true — plain old algebra).
 - But we also know what the sum up to n is.... (right?)
 - I'll leave it (for now) for you to think about what exactly is going on, and will show you the necessary derivation/algebra:

Mathematical Induction

- An example:

$$\begin{aligned}\sum_{k=1}^{n+1} k^2 &= \sum_{k=1}^n k^2 + (n+1)^2 \\&= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\&= \frac{n(n+1)(2n+1) + 6(n+1)^2}{6}\end{aligned}$$

Mathematical Induction

- Working on the numerator, take common factor $(n+1)$ to get:

$$\begin{aligned} n(n+1)(2n+1) + 6(n+1)^2 &= (n+1)\left(n(2n+1) + 6(n+1)\right) \\ &= (n+1)(2n^2 + 7n + 6) \\ &= (n+1)(n+2)(2n+3) \end{aligned}$$

Mathematical Induction

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- Thus, we're finally obtaining:

Mathematical Induction

Mathematical Induction

$$\begin{aligned}\sum_{k=1}^{n+1} k^2 &= \frac{(n+1)(n+2)(2n+3)}{6} \\ &= \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6}\end{aligned}$$

And this completes the induction step (showing that statement true for n implies that statement is true for $n+1$), completing the proof

Mathematical Induction

- In the previous example, we (sort of) started from the induction hypothesis and worked our way to the expression corresponding to the case $n+1$ (which is the one that we need to show to be true, based on the assumed induction hypothesis)
- We could have worked in the opposite direction; in the next example we'll proceed that way.

Mathematical Induction

- Example 2:

Prove that $8^n - 3^n$ is divisible by 5 for all $n \geq 1$.

Mathematical Induction

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Prove that $8^n - 3^n$ is divisible by 5 for all $n \geq 1$.
- Step 1: Base case: $n = 1$, $8 - 3 = 5$, which is of course divisible by 5.

Mathematical Induction

- Example 2:
Prove that $8^n - 3^n$ is divisible by 5 for all $n \geq 1$.
- Step 1: Base case: $n = 1$, $8 - 3 = 5$, which is of course divisible by 5.
- Induction step: Based on the induction hypothesis that the statement holds for n , we want to show that it holds for $n+1$ as well.

Mathematical Induction

- Example 2:
- That is, we want to show that $8^{n+1} - 3^{n+1}$ is divisible by 5 provided that $8^n - 3^n$ is divisible by 5.
- We observe (somewhat trivial observation) that being divisible by 5 is the same as being a multiple of 5.
- We proceed as follows:

Mathematical Induction

- Example 2:

$$\begin{aligned}8^{n+1} - 3^{n+1} &= 8 \cdot 8^n - 3 \cdot 3^n \\&= 8 \cdot 8^n - 8 \cdot 3^n + 5 \cdot 3^n \\&= 8(8^n - 3^n) + 5 \cdot 3^n\end{aligned}$$

Mathematical Induction

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- By induction hypothesis, $8^n - 3^n$ is divisible by 5, so we're pretty much done (you complete the argument)

Mathematical Induction

- And again, $8^n - 3^n$ being divisible by 5 is what we want to prove, so how come we can just assume it, simply because we add the “by induction hypothesis” ??

Mathematical Induction

- And again, $8^n - 3^n$ being divisible by 5 is what we want to prove, so how come we can just assume it, simply because we add the “by induction hypothesis” ?? (so, really: what's up with that!!)

Mathematical Induction

- There's an alternative form of Mathematical Induction, known as *Strong* or *Complete* Induction, in which the induction hypothesis is not simply the statement being true for n (or for $n=k$), but the statement being true for all values between n_0 and n . (n_0 being the value for the base case).
- The next example illustrates this approach.

Mathematical Induction

- Example 3:
Prove that every integer number > 1 is either prime, or it is divisible by a prime number.

Mathematical Induction

- Example 3:
Prove that every integer number > 1 is either prime, or it is divisible by a prime number.
- Step 1: Base case, in this case $n = 2$. The statement trivially holds, since 2 is prime.

Mathematical Induction

- Example 3:
Prove that every integer number > 1 is either prime, or it is divisible by a prime number.
- Step 1: Base case, in this case $n = 2$. The statement trivially holds, since 2 is prime.
(errr, don't we need to verify the “or is divisible by a prime number?”)

Mathematical Induction

- Example 3:
Prove that every integer number > 1 is either prime, or it is divisible by a prime number.
- Step 2: Induction step. The induction hypothesis now is that *every number* between 2 and n is either prime, or is divisible by a prime.
- We need to show that $n+1$ also is either prime or divisible by a prime.

Mathematical Induction

- Example 3:

Clearly, for $n+1$ there are only two possibilities: either it is prime, or it is not prime. If it is prime, the statement holds.

Mathematical Induction

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Clearly, for $n+1$ there are only two possibilities: either it is prime, or it is not prime. If it is prime, the statement holds.

If it is not prime, then by definition, it is divisible by some number (say, $n_1 \neq 1$). If $n+1$ is divisible by n_1 , then $n_1 < n+1$ — therefore, $2 \leq n_1 \leq n$

Mathematical Induction

- Example 3:

By induction hypothesis, if $2 \leq n_1 \leq n$, then n_1 is either prime, or divisible by a prime (say, p_1), and we're pretty much done: If n_1 is prime, then the statement holds ($n+1$ is divisible by a prime). If not.... well, you complete the argument.

Mathematical Induction

- So, what could go wrong with this ?
- You have to apply the technique carefully, as the next example shows by proving a (blatantly false) statement!!

Mathematical Induction

- Example 4:

Prove that in every group of n numbers, all numbers are equal to each other.

Mathematical Induction

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Prove that in every group of n numbers, all numbers are equal to each other. (I'm not kidding!! We're going to prove this!!!)

Mathematical Induction

- Example 4:

Prove that in every group of n numbers, all numbers are equal to each other.
- Step 1: Base case, $n=1$. The statement is trivially true: In a group with only one number, all numbers are equal.

Mathematical Induction

- Example 4:

Prove that in every group of n numbers, all numbers are equal to each other.
- Induction step: Our induction hypothesis is that the statement holds for n ; that is, in every group of n numbers, all numbers are equal, and we want to show that this implies that in every group of $n+1$ numbers all numbers must also be equal.

Mathematical Induction

- Example 4:

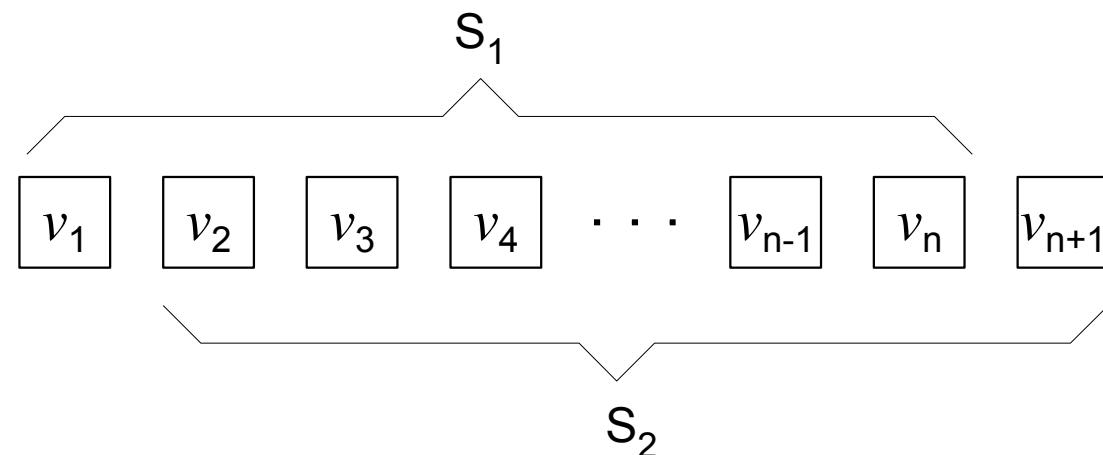
The argument in the induction step will not be algebraic (but then, no-one said that induction proofs are restricted to algebraic / arithmetic arguments). Consider a group of $n+1$ values:

$$\boxed{v_1} \quad \boxed{v_2} \quad \boxed{v_3} \quad \boxed{v_4} \quad \dots \quad \boxed{v_{n-1}} \quad \boxed{v_n} \quad \boxed{v_{n+1}}$$

Mathematical Induction

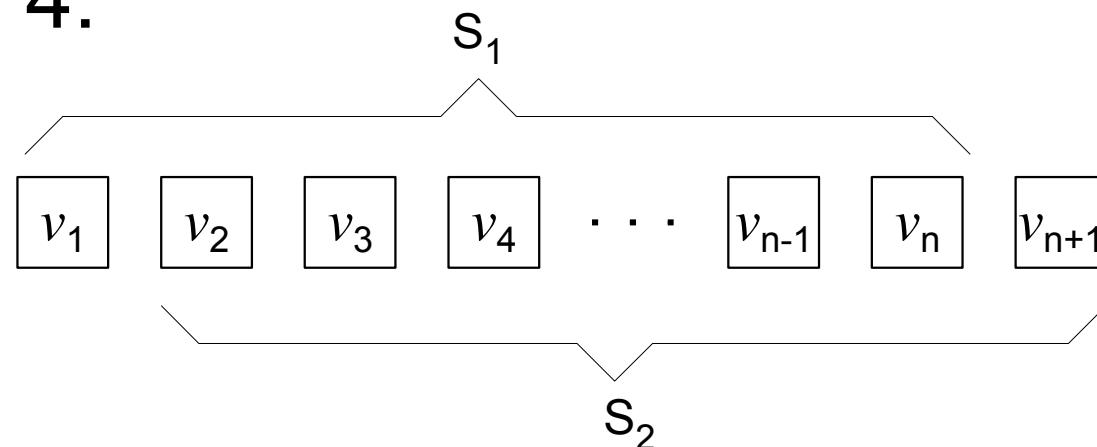
- Example 4:

The argument in the induction step will not be algebraic (but then, no-one said that induction proofs are restricted to algebraic / arithmetic arguments). Consider a group of $n+1$ values. And consider the two “subsets” indicated below:



Mathematical Induction

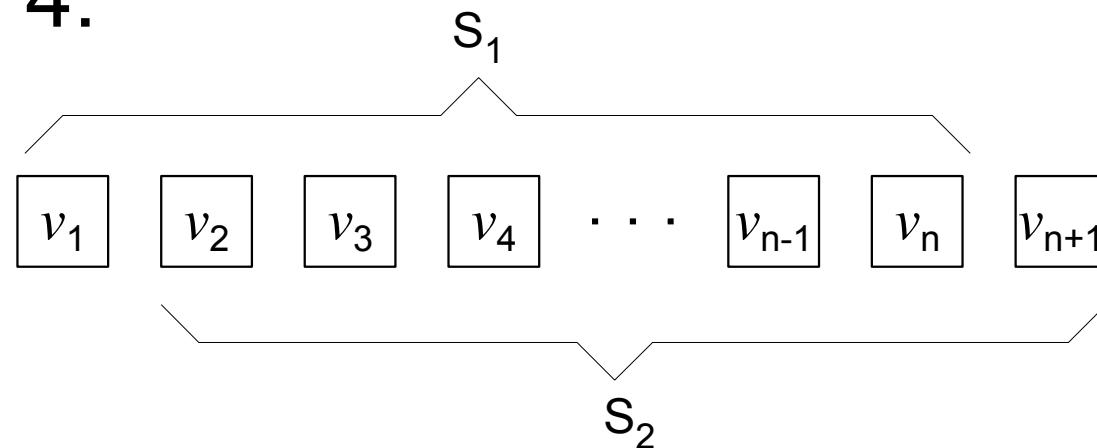
- Example 4:



Since both S_1 and S_2 are groups of n numbers, by induction hypothesis, both consist of n times the same value.

Mathematical Induction

- Example 4:



That is, by induction hypothesis for S_1 , we obtain $v_1=v_2=\dots=v_n$, and by induction hypothesis for S_2 , we obtain $v_2=v_3=\dots=v_n=v_{n+1}$.

By combining the two sets of equalities, we obtain $v_1=v_2=\dots=v_n=v_{n+1}$, completing the proof!!

Mathematical Induction

- Example 4:

So... What went wrong?

You will understand that I will not accept the argument that this example shows that Mathematical Induction does not work... Right?

Mathematical Induction

- Example 4:

So... What went wrong?

Notice also that we're not talking about an unsubstantiated argument, assuming the result or something like that — the argument is actually quite clever — the “cleverness” being in the fact that the overlap in the two subsets is what allows the induction step to work!

Mathematical Induction

- Example 4:

So... I ask again, and hope you guys will be intrigued and love solving puzzles and will think about it: *What went wrong?*

Mathematical Induction

- Example 4:

So... I ask again, and hope you guys will be intrigued and love solving puzzles and will think about it: *What went wrong?*

BTW, when I say “the argument is quite clever”, I'm not bragging / being blatantly immodest — I clarify: the argument is not mine. In fact, it is much more clever than it appears — the real cleverness being on how an incorrect argument can look so much like a correct and clever one!