Mathematical Proofs

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$$\sum_{k=0}^{N} \log k \quad \int_0^{\infty} e^{-x^2}\, dx$$

$$|a+b| \leqslant |a| + |b|$$

$$e^{\pi i} - 1 = 0$$

https://ece.uwaterloo.ca/~cmoreno/ece250

These slides, the course material, and course web site are based on work by Douglas W. Harder

Mathematical Proofs

Standard reminder to set phones to silent/vibrate mode, please!

Mathematical Proofs

- Today's class:
  - We'll investigate a few additional techniques to prove statements, including:
    - Direct proof
    - Proof by construction
    - Proof by enumeration (or exhaustion)
    - Proving the contrapositive
    - Proof by contradiction (very useful in general)
    - Proof by reduction (very useful in the context of algorithms)

Mathematical Proofs

- Direct proof
  - In general the simplest form to prove statements.
  - The result is directly obtained from the hypothesis (along with the basic axioms, etc.)
  - Example: proving that the square of an even number is even:

    Proof: An even number has the form $2k$. Its square is $(2k)^2 = 4k^2 = 2(2k^2)$, and so it is even.

Mathematical Proofs

- Proof by construction
  - Usually applicable to statements about existence of some entity — by explicitly constructing an example of such entity, the statement is proven.

Mathematical Proofs

- Proof by enumeration (or exhaustion)
  - The statement is proved by going over all possible conditions, and proving the statement for each one individually. Obviously, this is feasible when there is a reasonable number of conditions.
  - Example: prove the triangle inequality for real numbers (we won't really prove it — just sketch the procedure to illustrate the technique)

Mathematical Proofs

- Proof by enumeration (or exhaustion)

  In this case, the relevant possibilities are *a* and *b* being negative or non-negative, with the combinations of each one's magnitude being the larger.

  For example, if *a* is non-negative and *b* is negative, then we have: $|a| = a$ and $|b| = -b$, and $|a+b| = a+b$ if $|a| \geq |b|$ or $-(a+b)$ otherwise. In the first case, the inequality leads to $a+b \leq a-b$, or $b \leq -b$, which is true since *b* is negative. (etc. etc.)

Mathematical Proofs

- Proving the contrapositive
  - The contrapositive of the statement A ⇒ B, is the *strictly equivalent* statement Not B ⇒ Not A.
  - Since the two forms are equivalent, proving one proves the other one.
  - Example: Prove that if n^2 is even, then n is even.

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    Careful!! A ⇒ B is not the same as B ⇒ A

Mathematical Proofs

- Proving the contrapositive
  - The contrapositive of this statement is: if n is odd, then n^2 is odd (if Not{n is even}, then Not{n^2 is even}), and this one is easy to show by a direct argument:

    An odd number has the form $2k+1$, and its square is $(2k+1)^2 = (4k^2 + 4k + 1) = 2(2k^2 + 2k) + 1$, and so is odd.

Proof by Contradiction

- There's this old (colloquial) adage that “you can't prove a negative”
- The rationale being more or less that proving something to be impossible or not to exist requires exhausting all possibilities in the Universe. Any attempt to show that it's not possible might show instead that one is unable to, and not that it can not be done.

Proof by Contradiction

- However, that old adage could not be more wrong!
- Arguments by contradiction are a quite remarkable way around that rationale!

Proof by Contradiction

- To prove a statement by contradiction, you assume that the statement is false (or equivalently, assume the negation of the statement to be true), and show, using deductive steps and axiomatically true arguments, that such assumption leads to some inconsistency (contradiction).

Proof by Contradiction

- If every step is correct and you did everything right (which is an obvious condition for any proof to be correct), then the only item in question is the initial assumption, so it must be the one that is proven false.
- But the initial assumption is that what you want to prove is false, and you then proved that *that* is false — thus, what we wanted to prove *is* proved.

Proof by Contradiction

- One of the most remarkable examples (IMHO) is the proof that $\sqrt{2}$ is not a rational number.
- It would seem (if we follow that old adage's line of thought) impossible to prove.... No matter how many digits you show (showing that there is no periodicity), you still don't know if later on periodicity will appear...
- The argument is completely different ... Let's assume that it is a rational number, and let p/q be the *reduced form* of that fraction.

Proof by Contradiction

- That is, p and q share no common factors, and it holds, of course, that

$$\left(\frac{p}{q}\right)^2 = 2 \Rightarrow p^2 = 2q^2$$

Proof by Contradiction

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$$\left(\frac{p}{q}\right)^2 = 2 \;\Rightarrow p^2 = 2q^2$$

So, p^2 is even... We already proved that this means that *p* is even; say, $p = 2k$ for some *k*. Then, $q^2 = p^2 / 2 = 4k^2 / 2 = 2k^2$.

Proof by Contradiction

- So, *q* is also even — but this contradicts part of the initial assumption, that *p* and *q* share no common factors.
- Thus, our initial assumption that $\sqrt{2}$ is rational must be false.

Proof by Contradiction

- Another reasonably neat example:
- Prove that there are infinitely many prime numbers

Proof by Contradiction

- Another reasonably neat example:
- Prove that there are infinitely many prime numbers (and again, talk about a remarkably powerful argument — statements like these may sound like the perfect example for those that defend that old adage .... How can you convincingly argue that there are infinitely many of something without showing them all? How can you know that they won't stop after a certain value??)

Proof by Contradiction

- Philosophy aside, let's prove it!
- Assume there are finitely many primes:

$$p_1, p_2, p_3, \cdots, p_{n-1}, p_n$$

And consider the value obtained as the product of all primes + 1:

$$d = 1 + p_1 p_2 p_3 \cdots p_{n-1} p_n$$

Proof by Contradiction

- The value *d* is larger than the largest prime (since it is the largest prime times all the other primes + 1), so it can not be prime.
- But, as we proved last class, this means that *d* must be divisible by a prime — say, p_k (one of the *n* primes that exist). It could be divisible by more than one prime, but we only consider one (which anyway is all we proved last time!)
- So, $d = p_k \cdot m$, for some *m*.

Proof by Contradiction

- Then, we have:

$$d = p_k\, m = 1 + p_1 p_2 p_3 \cdots p_k \cdots p_{n-1} p_n$$

$$\Rightarrow \quad p_k(m - p_1 p_2 p_3 \cdots p_{k-1} p_{k+1} \cdots p_{n-1} p_n) = 1$$

- But this means that p_k divides 1, which is not possible, since $p_k > 1$.
- Thus, our initial assumption (that there are finitely many primes) must be false.

Proof by Contradiction

- A perhaps interesting observation:
  - This proof can be seen as an example of a proof by construction — if you think about it, what happens is that the number *d* is prime; so, we can rephrase the argument as: we're showing that given the first *n* primes (no matter how large *n*), we're explicitly *constructing* a prime number that is larger than the largest of the first *n* primes, and that shows that there are infinitely many of them.

Proof by Contradiction

- In both examples so far, the argument leads to a contradiction of some of the "background" assumptions — something that we know to be axiomatically true is contradicted by our assumption that the given statement is false, leading to the conclusion that the given statement then must be true.

Proof by Contradiction

- In both examples so far, the argument leads to a contradiction of some of the “background” assumptions — something that we know to be axiomatically true is contradicted by our assumption that the given statement is false, leading to the conclusion that the given statement then must be true.
- In some cases, we might end up contradicting the assumption itself, like in the next example.

Proof by Contradiction

- BTW, in some cases, we might end up contradicting the hypothesis.
- In these cases, we're essentially proving the contrapositive — if A, then B; let's assume that A holds, and assume for a contradiction that B is false; then, we prove that that implies that A is false, but that contradicts the assumption that A holds.
- Offten enough, the argument by contradiction looks “better articulated”.

Proof by Contradiction

- Example: (multi-processor parallel processing)

  You have some processing totalling time *T*, and it can be split, in a continuous and arbitrary way, into *N* chunks, to be executed in parallel in *N* processors.

  Prove that the optimal execution time is reached when the task is split into *N* equal-size subtasks.

Proof by Contradiction

- This example illustrates an additional interesting aspect that sometimes makes the technique applicable: when we have to prove some condition about one particular combination among many, if we negate the condition, it may be easy to focus on one aspect of the negated condition that covers all other combinations. Trying to prove the "positive" form of the statement may end up being quite challenging by comparison!

Proof by Contradiction

- Anyway.... the proof:
- Assume, for a contradiction, that the the optimal time T_O is reached for a splitting into N subtasks that are not all equally-sized.
- If they're not equally sized, then we can pick the task of largest size (let's call it T_{MAX}) and the task of smallest size (let's call it T_{MIN})
- Notice that in this case, $T_O = T_{MAX}$ (the task is not complete until the subtask for T_{MAX} is done)

Proof by Contradiction

- Given these conditions, we can find a different splitting with lower execution time:
- Since $T_{\mathrm{MAX}} > T_{\mathrm{MIN}}$ (they can't be equal, since by assumption, we're splitting into tasks that are not equally-sized), then we can rearrange these two tasks so that each one takes $(T_{\mathrm{MAX}} + T_{\mathrm{MIN}})/2$
- But then, the T_{O} that we obtain (let's call it T'_{O} is lower than T_{MAX} (the new max will be either the second highest T, or $(T_{\mathrm{MAX}} + T_{\mathrm{MIN}})/2$)

Proof by Contradiction

- Summarizing: $T'_0 < T_0$, but this contradicts the assumption that the splitting that we had was optimal (since we're showing another one that gives a better execution time)
- Thus, the assumption must be incorrect, and the optimal splitting must be the one with N equally-sized tasks.

Proof by Contradiction

- The (easy to fix) flaw in the argument is that there may not be a single max, or a single min, for that matter!
- I will leave it to you to fix the argument to account for this detail!

Proof by Reduction

Proof by Reduction

- In the context of algorithms, *reduction* refers to implementing one algorithm in terms of another one.
- If we implement algorithm A in terms of algorithm B, we say that A *reduces to* B.
- This has obvious theoretical applications, but also practical ones — maybe an algorithm is readily available, and it may be easier to reduce to that one rather than implementing from scratch some new algorithm that we require.

Proof by Reduction

- It may be helpful for proofs about execution time of some algorithms.
- For example, if algorithm A is known to be "hard" or "slow" and we can find a reduction from A to B, then that proves that B can not be easy (in any case, it can not be easier/faster than A).
- Right? If B was easy, then A would also be easy, contradicting the known fact (or in any case the assumption) that A is hard

Proof by Reduction

- There's actually a subtlety in here, that ruins that argument (as stated)... We'll come back to this detail ...

Proof by Reduction

- There's actually a subtlety in here, that ruins that argument (as stated)... We'll come back to this detail ...
- Subtlety aside, let's see one rather neat example (the subtlety I'm referring to is really one extra condition that needs to be added, and that extra condition doesn't ruin the applicability of the technique):

Proof by Reduction

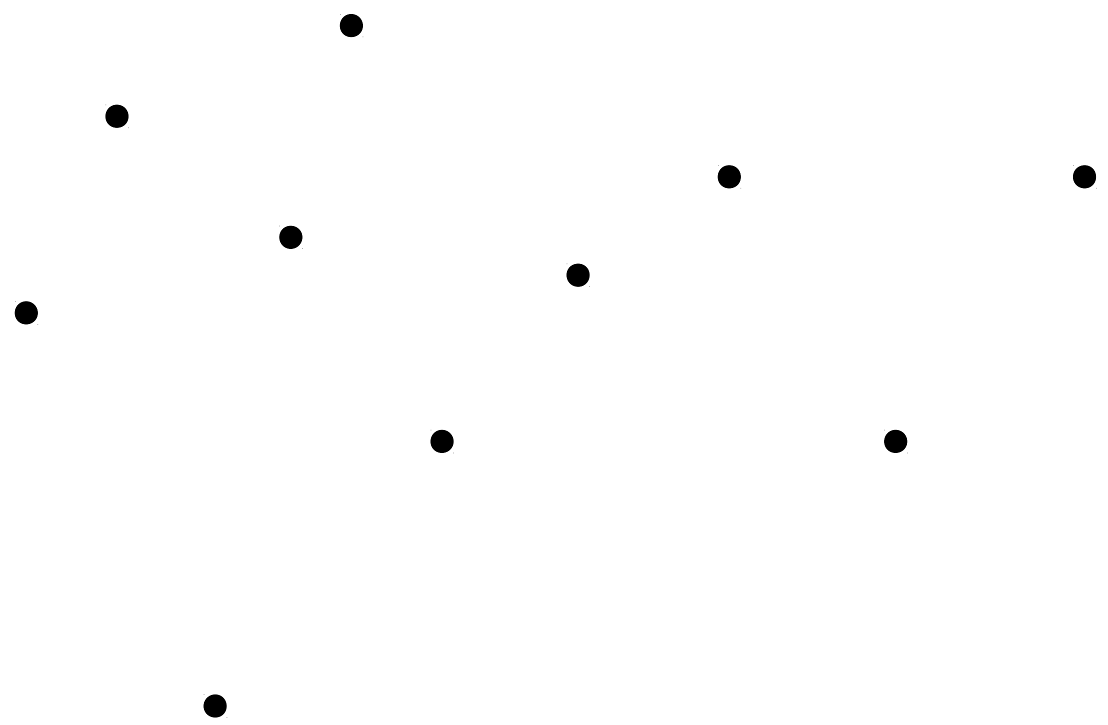
- Prove that an algorithm to find the convex hull of *n* points in the plane can not run in a worst-case better than an amount proportional to *n* log(*n*).

Proof by Reduction

- Preliminary definitions:
- Convex hull of a set of points: the smallest (in area) convex region that includes all the points.
- It is a known fact that for a set of points, the convex hull is always a convex polygon — the following example illustrates this:

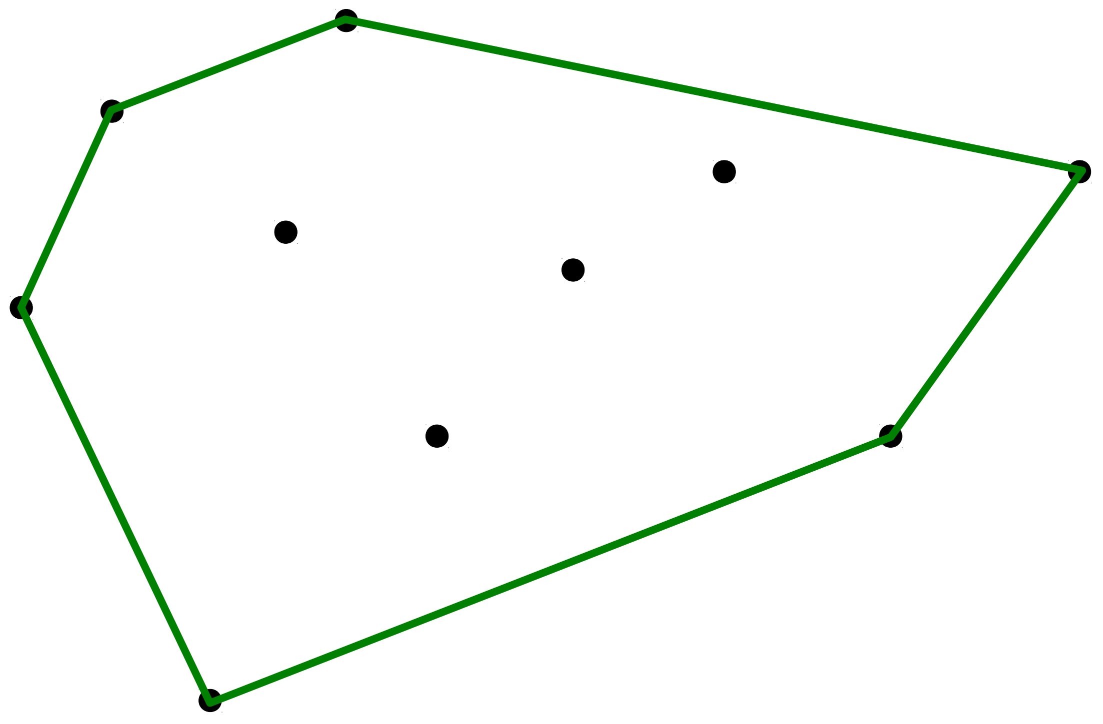
Proof by Reduction

Consider the following set of points:



Proof by Reduction

Their convex hull:



Proof by Reduction

- Preliminary definitions:
- An important detail is that an algorithm to compute the convex hull must output the vertices of the polygon *in sequence*. If not, we don't really have the convex hull — we would have a set of points that requires extra work to determine what the convex hull really is.

Proof by Reduction

- Another preliminary:

  It is a proven fact that (under certain conditions, operating on a single processor computer with standard assumptions on memory, assembler instruction set, etc.) that no sort algorithm can sort a set of *n* arbitrary values in a worst-case time better than an amount of time proportional to *n* log(*n*)

Proof by Reduction

- We're now ready to prove the statement.
- As you already suspect, we're going to reduce sorting to computing a convex hull.
- That is, we're going to find a way to sort a set of values given an algorithm that computes a convex hull.
- This will clearly require to somehow construct points from the given values (not a big challenge, really...)

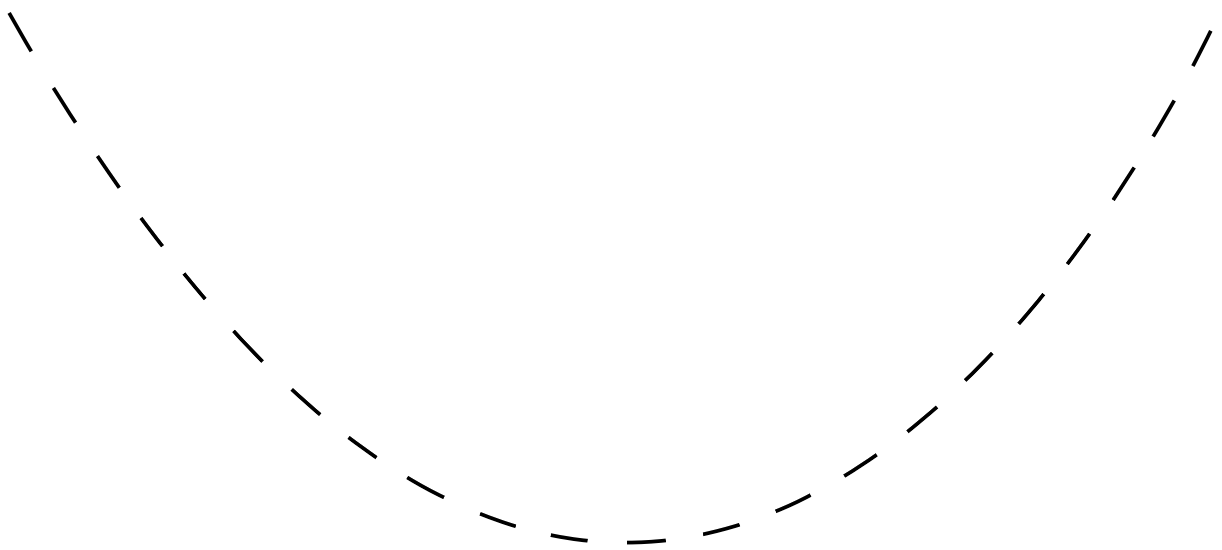
Proof by Reduction

- The real challenge is finding a way to construct points in a way that the convex hull will output something that we know will be useful for the purpose of sorting (and thus, useful for the purpose of our argument / proof)

- Hint: What is the convex hull of a bunch of points that lie on the parabola given by $y = x^2$?

Proof by Reduction

- Let's try and answer that question with an example:



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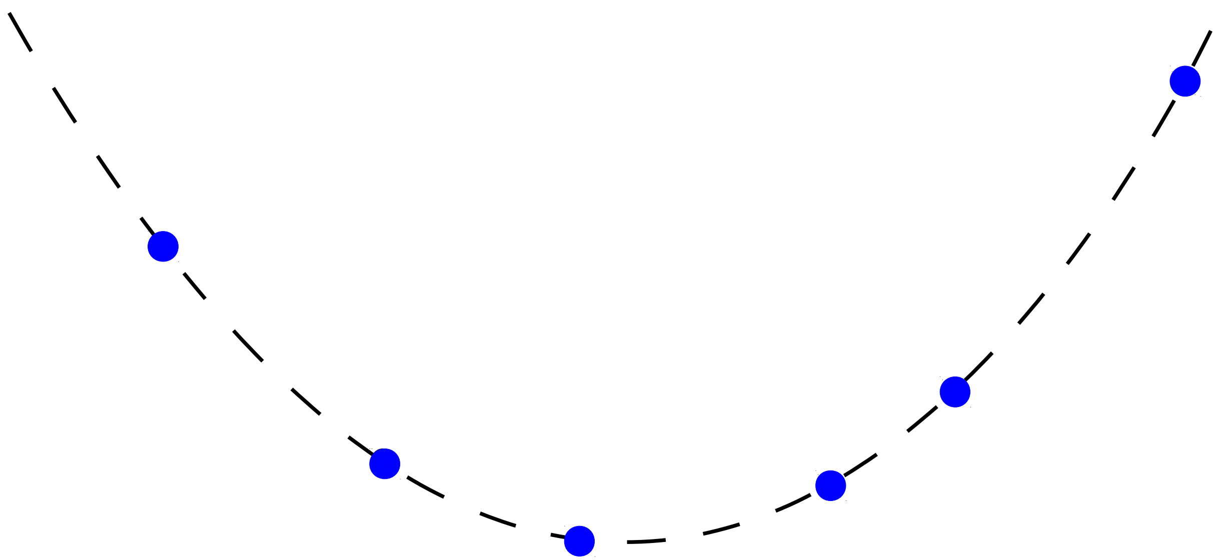

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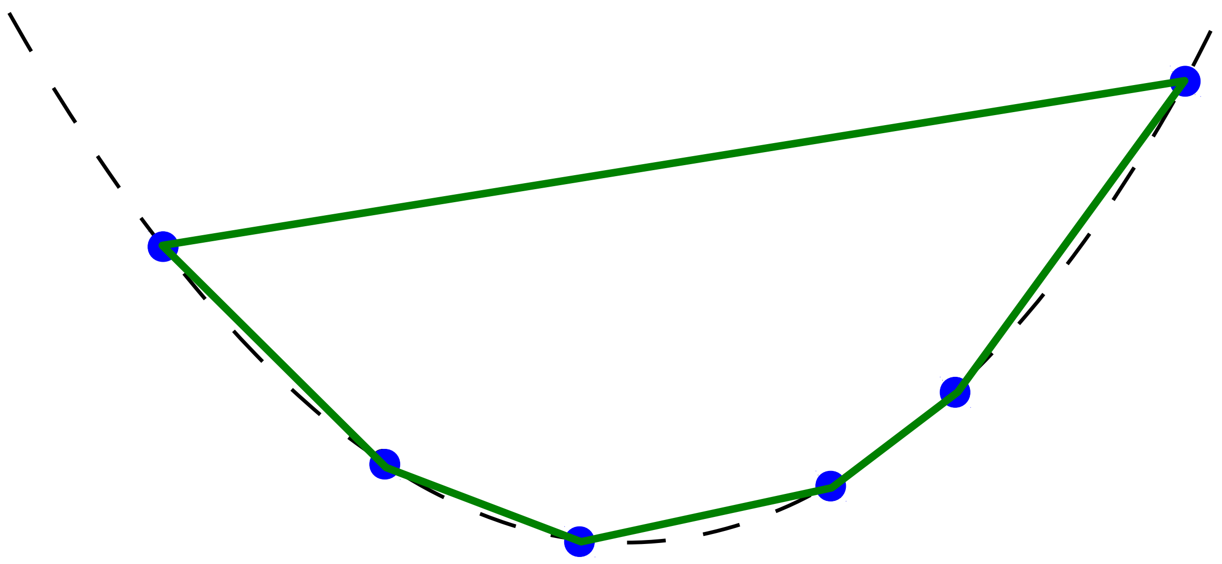
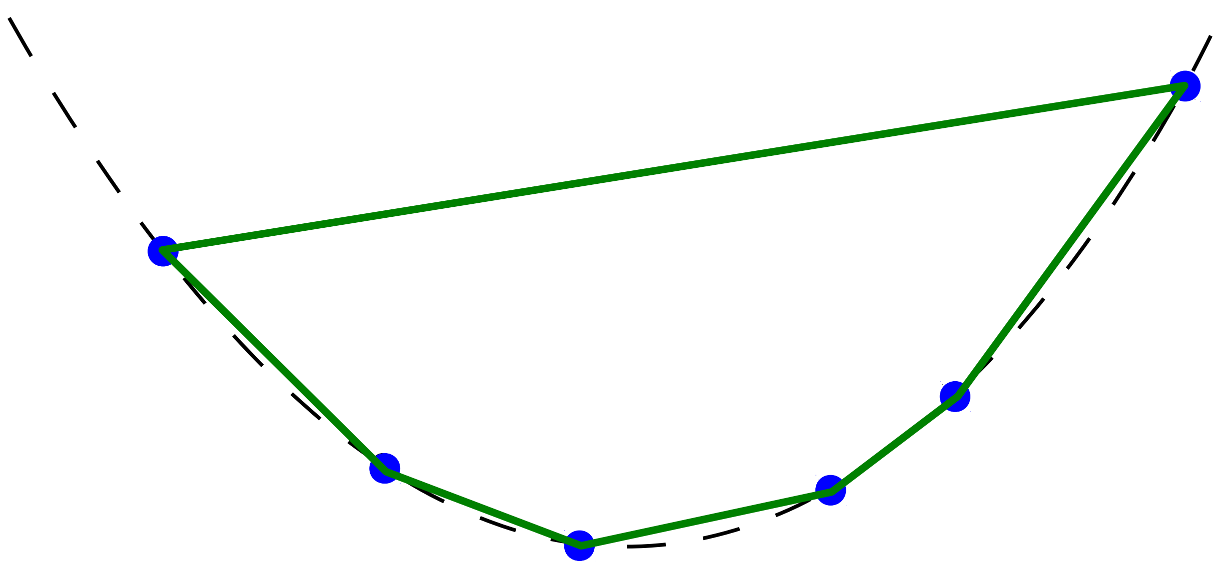


Proof by Reduction

- Key detail being: the convex hull always contains every point (the parabola is convex!)



Proof by Reduction

- The other key detail being what we already said: any convex hull algorithm would have to output those points in sequence.
- In this case, such output sequence involves the points in order by their x-coordinate (right?)

Proof by Reduction

- The other key detail being what we already said: any convex hull algorithm would have to output those points in sequence.
- In this case, such output sequence involves the points in order by their x-coordinate (right?)
- That probably means that the input values (the ones that need to be sorted) should be the x-coordinates of the points that we're going to feed as input to the convex hull algorithm....

Proof by Reduction

- But the useful aspect was that the points should be on the parabola $y = x^2$

- So, the reduction is:
  - Given the values $\{x_1, x_2, x_3, \cdots, x_n\}$, compute the set of points $\{(x_1,x_1^2), (x_2,x_2^2), (x_3,x_3^2), \cdots, (x_n,x_n^2)\}$ and feed those points to the convex-hull algorithm.

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  - Take the output from the convex-hull algorithm, extract and output the x-coordinates of each point

Proof by Reduction

- Example: Input values: {-3, 5, 2, 4,-2,1}
- Input to Convex-Hull algorithm: {(-3,9), (5,25), (2,4), (4,16), (-2,4), (1,1)}

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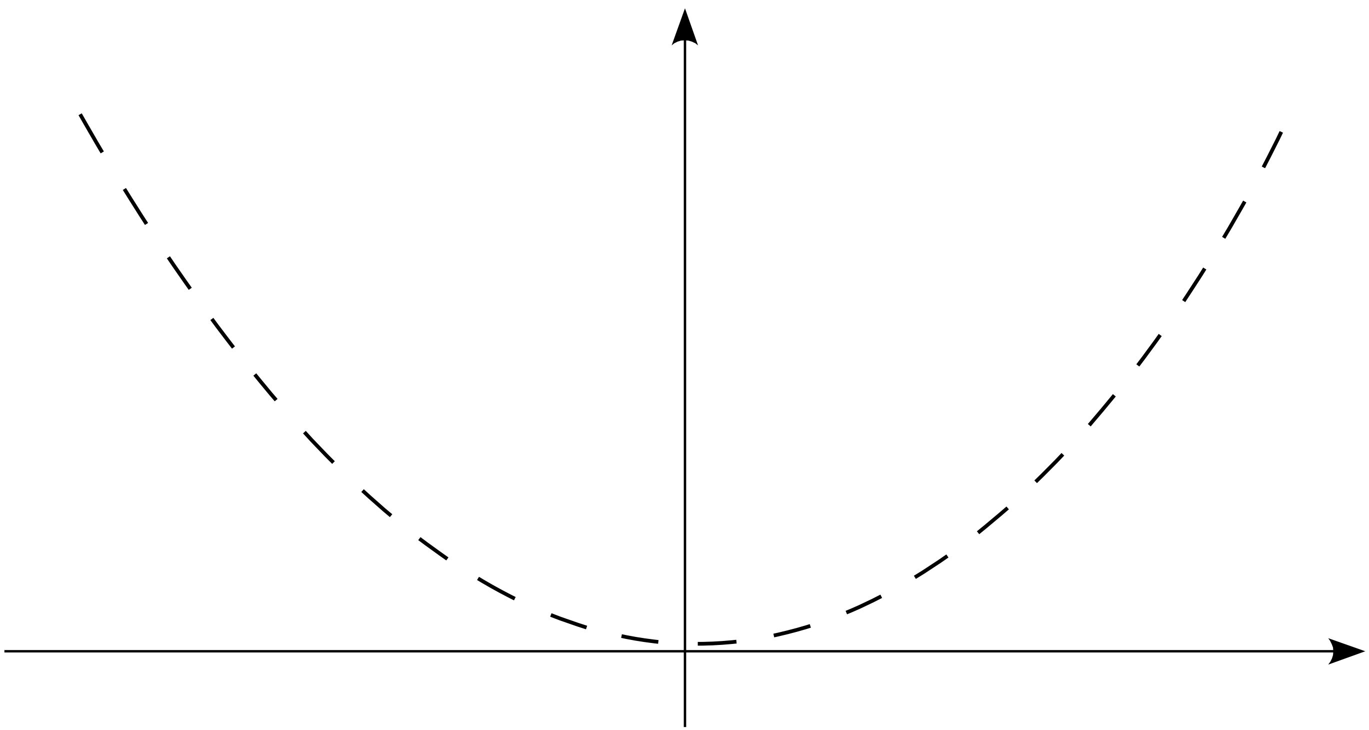

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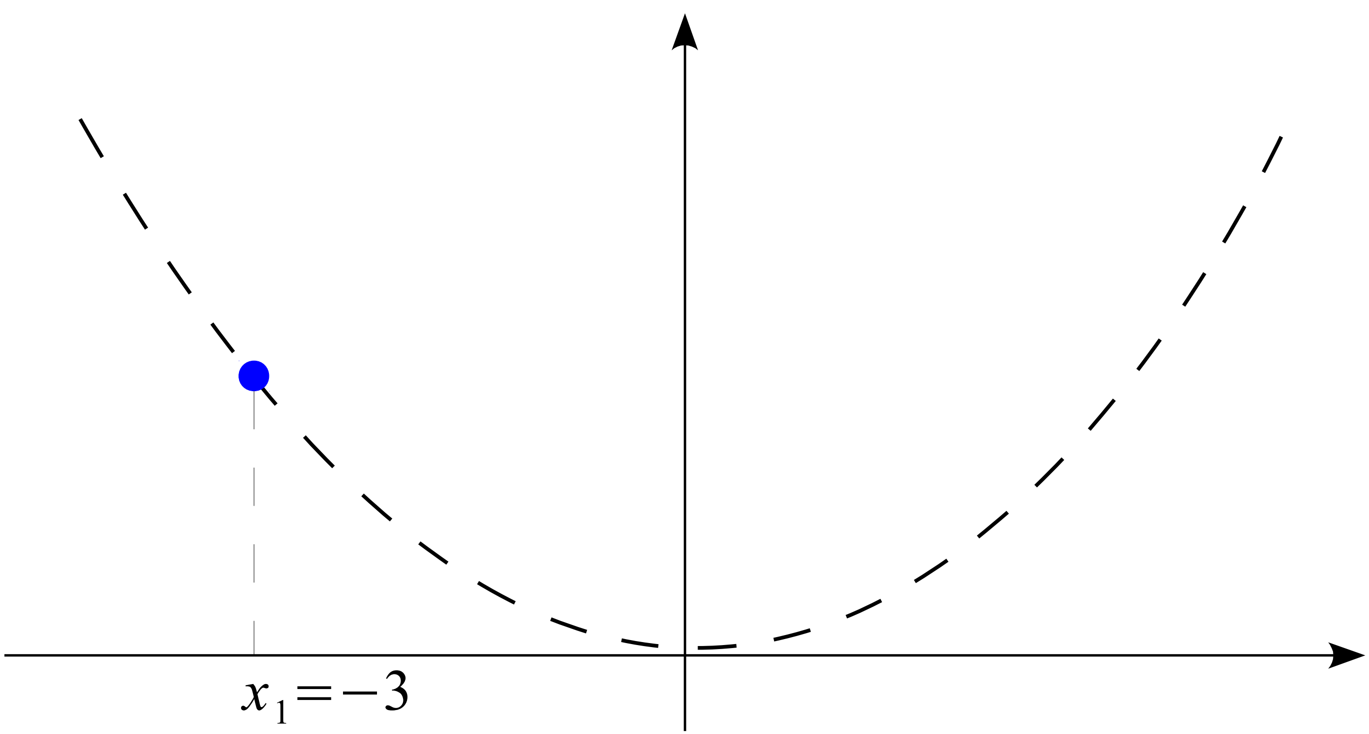

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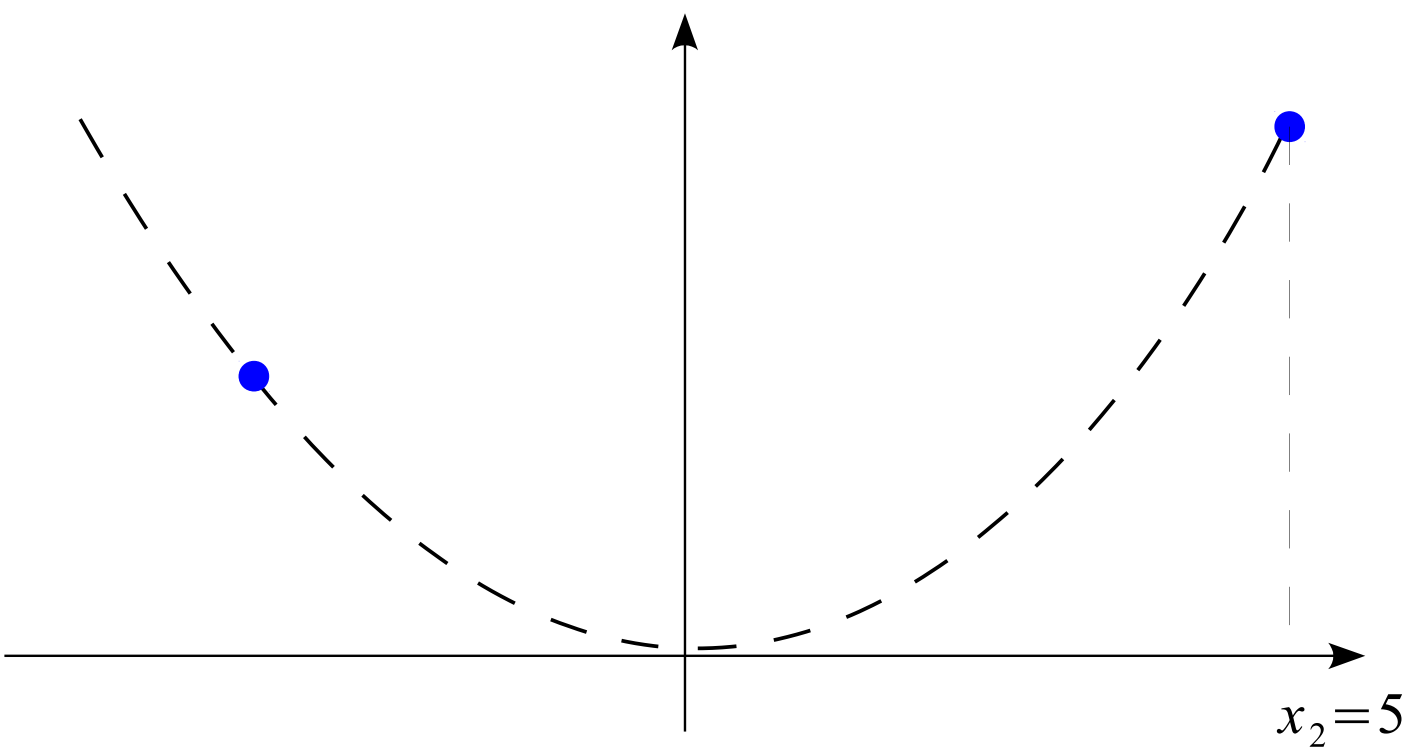

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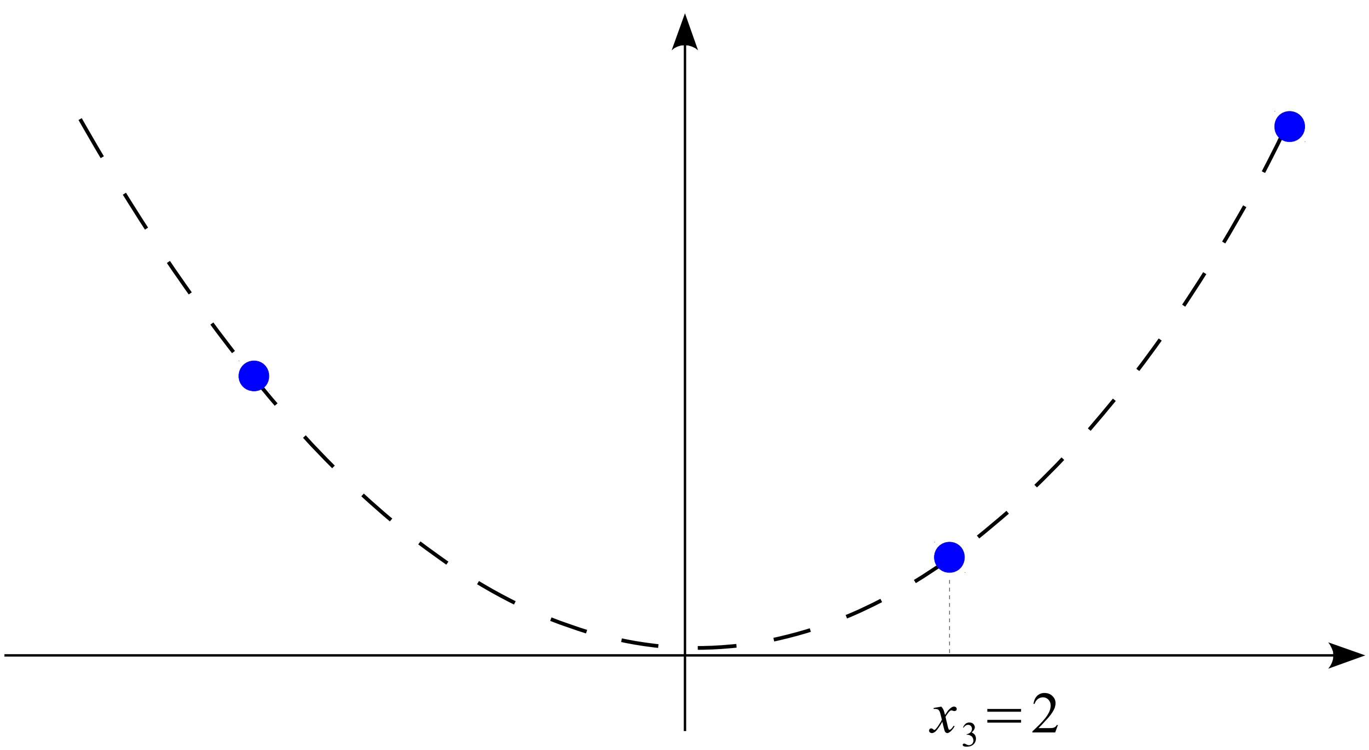

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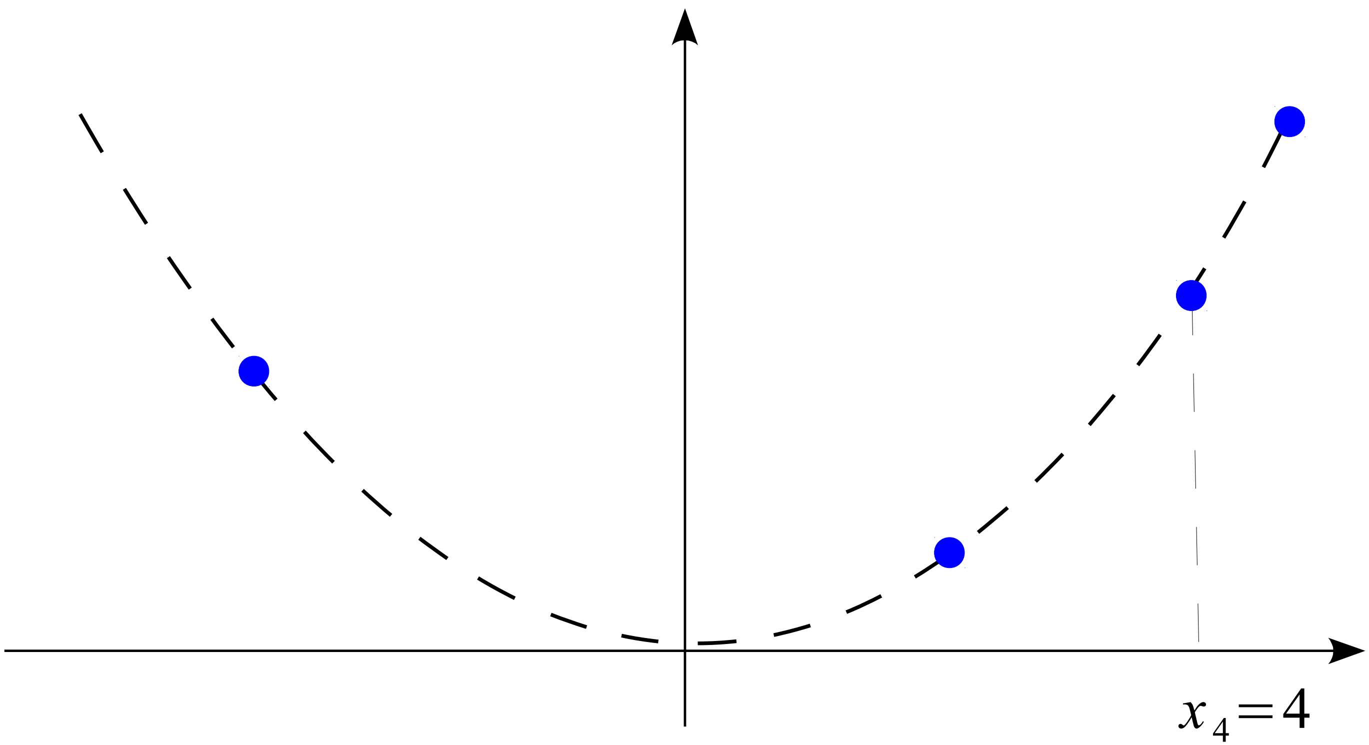

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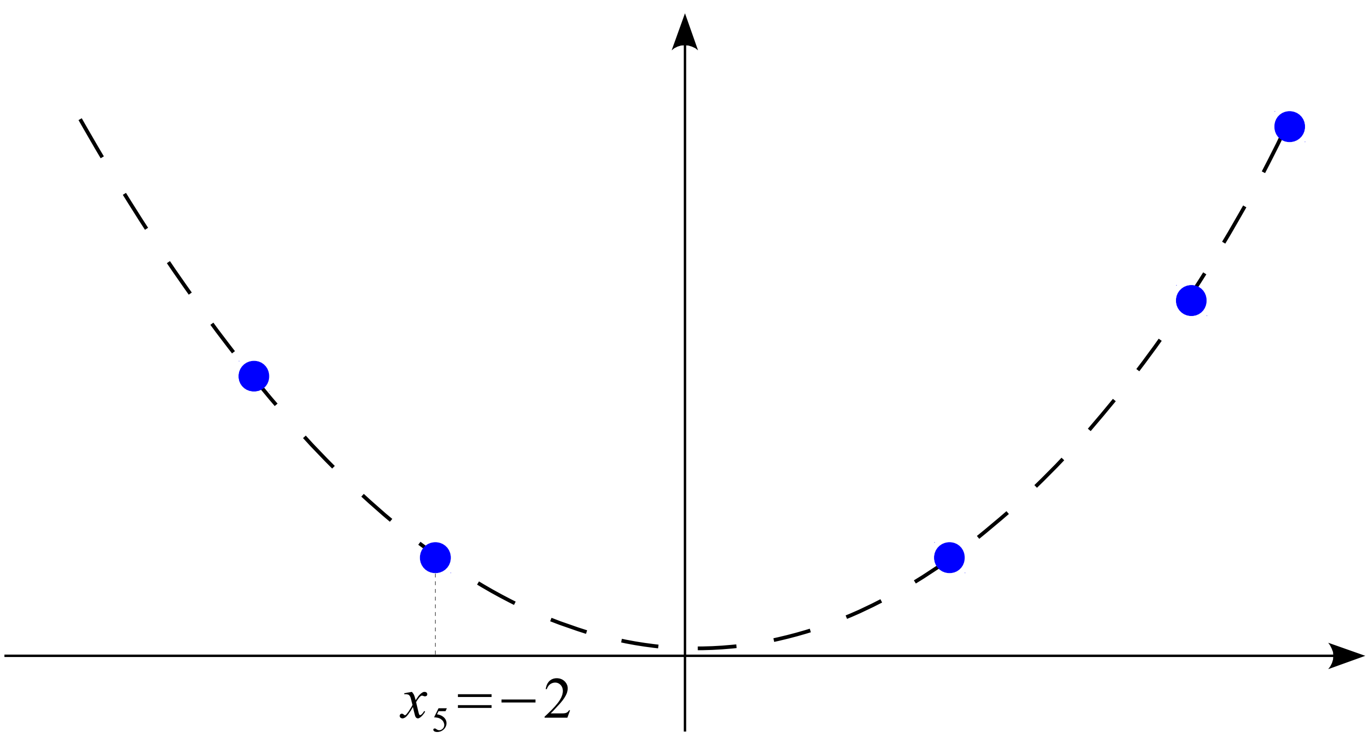

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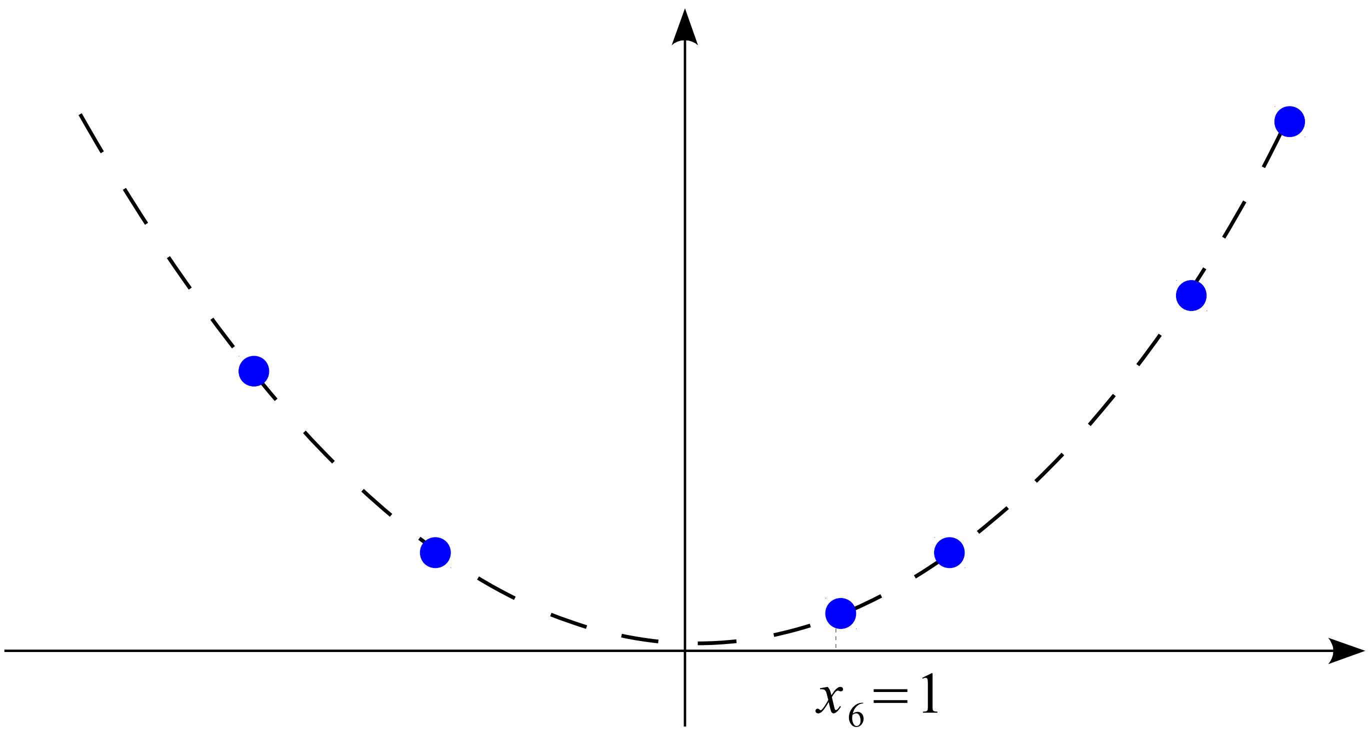

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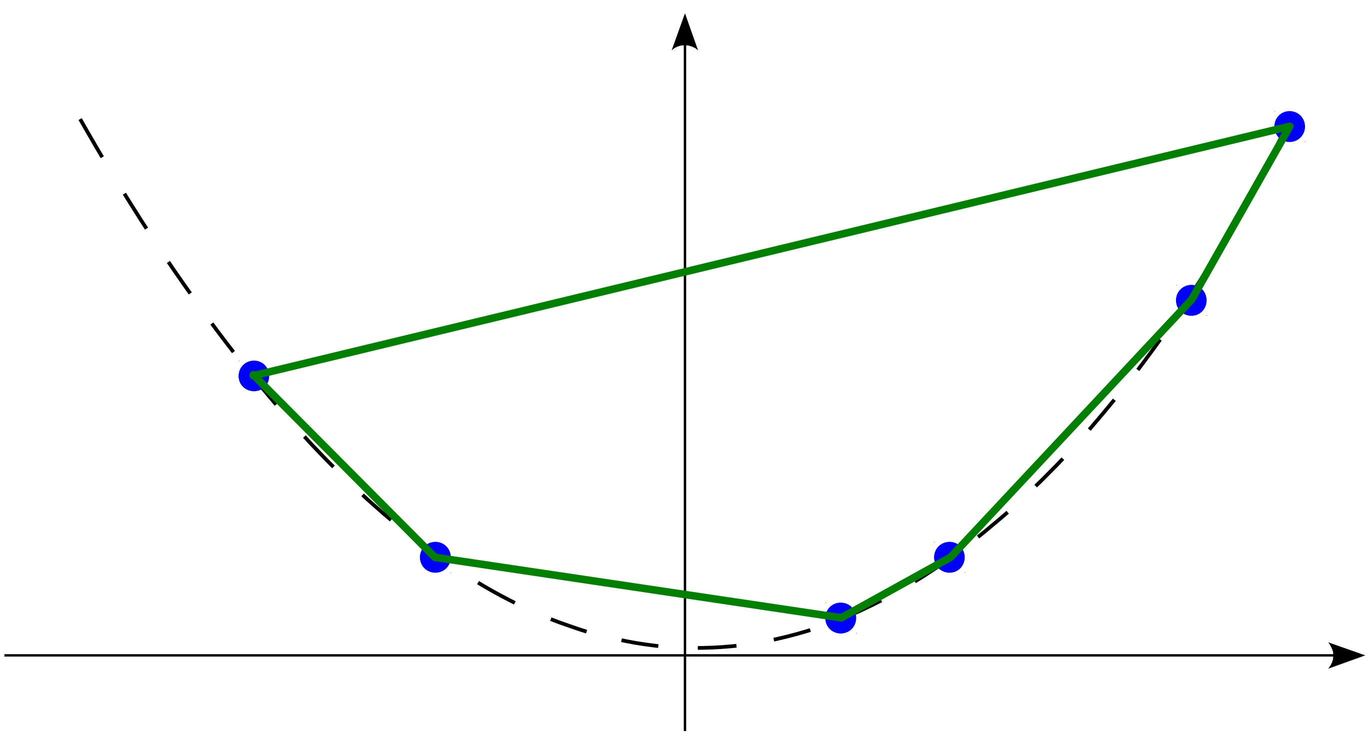

Proof by Reduction

- A key detail is that the reduction must be efficient; in this example, the reduction must be faster than $n \log(n)$.
- Otherwise, we would be proving nothing — we know sort can't be faster than $n \log(n)$. But if we do this trick, and the reduction requires an amount of time proportional to n^2, then because of the reduction part, we're not doing faster than $n \log(n)$. But we already knew that, so our argument would prove absolutely nothing!

Summary

- During today's lesson, we discussed:
  - Some additional aspects of Mathematical proofs
  - Some of the basic techniques, with emphasis on:
    - Proofs by Contradiction
    - Proofs by Reduction
  - We also introduced the notion of algorithmic reduction (more on this later on the course)