Asymptotic Analysis – Cont'd

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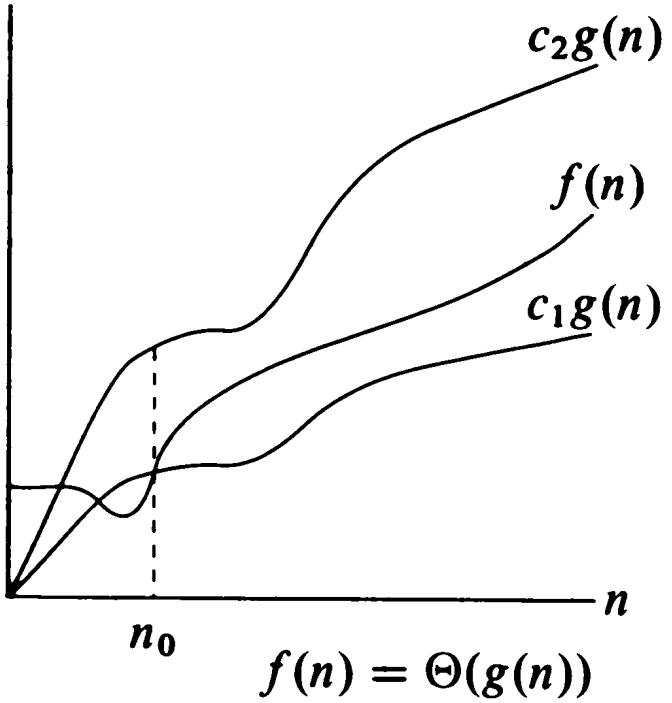

https://ece.uwaterloo.ca/~cmoreno/ece250

These slides, the course material, and course web site are based on work by Douglas W. Harder

Announcements

- We have class this Wednesday, Jan 18 at 12:30
  - That is, we have two sessions this Wednesday: at 12:30 in the make-up time slot, and at 4:30 in the regular class time slot.

Asymptotic Analysis

- Today's class:
  - We'll continue with Asymptotic analysis.
  - We'll see some additional examples for the common functions we saw last time.
  - Investigate an alternative technique, using limits to determine the appropriate Landau symbol that relates two given functions.
  - We'll also introduce two additional Landau symbols (perhaps not as useful as the three we saw, but worth knowing about)

Asymptotic Analysis

- Examples of algorithms or operations exhibiting the common functions seen last time:
  - We mentioned one example, for Θ(1) — a.k.a. constant (e.g., constant time) — being subscripted access of an element of an array; regardless the number of elements *n* in an array, accessing an element by subscript takes a constant amount of time (it maps to a single assembly-level instruction on most CPUs)

Asymptotic Analysis

- Examples of algorithms or operations exhibiting the common functions seen last time:
  - Perhaps a less obvious example for Θ(1) — an operation that may involve (perhaps at random), 1, 2, or 3 operations.
  - Why is this Θ(1)?

Asymptotic Analysis

- Examples of algorithms or operations exhibiting the common functions seen last time:
  - Perhaps a less obvious example for Θ(1) — an operation that may involve (perhaps at random), 1, 2, or 3 operations.
  - Why is this Θ(1)?
  - Perhaps more strange is calling it *constant* time, when it is not constant at all.

Asymptotic Analysis

- For the “why is Θ(1)” part — should be clear; it can be upper- and lower-bounded by constants (1 and 3, for example)

Asymptotic Analysis

- For the “why is Θ(1)” part — should be clear; it can be upper- and lower-bounded by constants (1 and 3, for example)
- As for why calling it constant time, when it is not constant? Well, you can think of it in terms of a “deadline” by which you're sure the operation has completed — and this deadline is a fixed value, regardless the number of elements.

Asymptotic Analysis

- Examples of algorithms or operations exhibiting the common functions seen last time:
  - You guys give me one example for logarithmic time ($\Theta(\log n)$)

Asymptotic Analysis

- Examples of algorithms or operations exhibiting the common functions seen last time:
  - You guys give me one example for logarithmic time ($\Theta(\log n)$)
  - Binary search on an array of *n* elements.
  - Hmmm.... array only? Not a linked list ??

Asymptotic Analysis

- Examples of algorithms or operations exhibiting the common functions seen last time:
  - How about linear time, or $\Theta(n)$?

Asymptotic Analysis

- Examples of algorithms or operations exhibiting the common functions seen last time:
  - How about linear time, or $\Theta(n)$?
  - Regular search on an array of n elements.
  - Computing the average of n values.
    - However, computing the variance would be .... ?

Asymptotic Analysis

- Examples of algorithms or operations exhibiting the common functions seen last time:
  - How about linear time, or $\Theta(n)$?
  - Regular search on an array of n elements.
  - Computing the average of n values.
    - However, computing the variance would be .... ?
      - I would hope that it was not a big surprise that it is also $\Theta(n)$

Asymptotic Analysis

- Examples of algorithms or operations exhibiting the common functions seen last time:
  - We saw an example for $n \log n$ — it was, however, $\Omega(n \log n)$, and not $\Theta(n \log n)$
  - Remember that sort can not be done faster than $n \log n$ — which is equivalent to say that sorting is $\Omega(n \log n)$.

Asymptotic Analysis

- Examples of algorithms or operations exhibiting the common functions seen last time:
  - We saw an example for $n \log n$ — it was, however, $\Omega(n \log n)$, and not $\Theta(n \log n)$
  - Remember that sort can not be done faster than $n \log n$ — which is equivalent to say that sorting is $\Omega(n \log n)$.
  - BTW — we will see several algorithms that achieve this bound; that is, that have run time $\Theta(n \log n)$

Asymptotic Analysis

- Examples of algorithms or operations exhibiting the common functions seen last time:
  - Later in the course, we'll see an example of sub-quadratic time.
  - A preview — an algorithm that recursively splits the job for size n into three recursive calls for size $n/2$.
    - Thus, if we do k recursive calls, that's 3^k operations (for each of the three recursive calls, there will be three other), but k will be lg n (right?), so that's $3^{\lg n}$ — from Q1 of the assignment, you recall that this is $n^{\lg 3} \approx n^{1.585}$

Asymptotic Analysis

- Examples of algorithms or operations exhibiting the common functions seen last time:
  - An example for quadratic time, $\Theta(n^2)$?
    - How about this for a trick question: can a sort algorithm be $\Theta(n^2)$?

Asymptotic Analysis

- Examples of algorithms or operations exhibiting the common functions seen last time:
  - An example for quadratic time, $\Theta(n^2)$?
    - How about this for a trick question: can a sort algorithm be $\Theta(n^2)$?
    - Sure — in fact, $\Theta(n^2) \subset \Omega(n \log n)$; we know that the run time of any sort algorithm must be in the set $\Omega(n \log n)$; if it is in the set $\Theta(n^2)$, then it is also in $\Omega(n \log n)$, which means that the possibility is valid.

Asymptotic Analysis

- Examples of algorithms or operations exhibiting the common functions seen last time:
  - Let's see a (very simple/intuitive) sorting algorithm that is $\Theta(n^2)$:
  - Find the lowest element in the array (that's linear time), then swap it with the first element; then, find the lowest element in the $n-1$ elements starting at the second position, and swap it with the second element; ... and so on.
  - (for an animated demo, look up *Selection Sort* on Wikipedia — the animation with numbers 0 to 9)

Asymptotic Analysis

- Examples of algorithms or operations exhibiting the common functions seen last time:
  - Let's see why it is $\Theta(n^2)$

Asymptotic Analysis

- Examples of algorithms or operations exhibiting the common functions seen last time:
  - The searching part takes n operations the first time; then, $n-1$ operations, then $n-2$ .... all the way until just one operation.

Asymptotic Analysis

- Examples of algorithms or operations exhibiting the common functions seen last time:
  - The searching part takes n operations the first time; then, $n-1$ operations, then $n-2$ .... all the way until just one operation.
  - For each of the first $n-1$ elements, you need to swap (for the last one, we don't need to do anything), which is 1 operation .... Well, ok, 3, but that's the same as 1 ... right?
    - This “abuse” of arithmetic is justified by the fact that we're analyzing the algorithm, and $3 = \Theta(1)$

Asymptotic Analysis

- Examples of algorithms or operations exhibiting the common functions seen last time:
  - Anyway, we don't really need to replace that 3 with 1 (we could simply argue that swap takes a constant amount of time, so we may consider it one operation — but let's proceed with 3; you probably already saw that it won't make any difference!)
  - We have, $n + (n-1) + (n-2) + \cdots + 3 + 2 + 1$ ... We recognize this .... (right?) ... And we have $3(n-1)$ operations for the swap — that's $n(n+1)/2 + 3(n-1)$, which is $\Theta(n^2)$

Asymptotic Analysis

- Examples of algorithms or operations exhibiting the common functions seen last time:
  - How about another trick question (maybe the question won't be too hard for you to see the answer right away, but I estimate that the answer may be very surprising)

Asymptotic Analysis

- Examples of algorithms or operations exhibiting the common functions seen last time:
  - How about another trick question (maybe the question won't be too hard for you to see the answer right away, but I estimate that the answer may be very surprising)
  - What is the run time (in asymptotic notation) of selection sort when executed only in arrays of 10 elements or less? (we don't necessarily know how many, and it may vary between runs, but we know that it is always 10 or less elements)

Asymptotic Analysis

- And speaking of .... Let's bring back the example of linear search.
- We said that searching for a value in an unordered group of values requires $\Theta(n)$ operations — what if the value is found at the first, or second location?
- In fact, on average we will not require n operations — on average, it will be found half-way ... Oh, wait! That's still $\Theta(n)$!!

Asymptotic Analysis

- Maybe we could suggest the idea that we want to describe linear search as O(n) instead of Θ(n) — big-Oh indicates upper-bound, instead of the tight bound that Θ describes.

Asymptotic Analysis

- Maybe we could suggest the idea that we want to describe linear search as O(n) instead of Θ(n) — big-Oh indicates upper-bound, instead of the tight bound that Θ describes.
- Side-note: the C++ Standard (as in, the ISO document that defines C++) typically dictates run time requirements for operations or algorithms that are part of the Standard Library; those are almost always given in big-Oh notation (why would this be a good thing?)

Asymptotic Analysis

- But where I'm really going with this is the notion of how exactly we describe the run time of a given algorithm, with three important options to consider:
  - Worst-case analysis
  - Average-case analysis
  - Best-case analysis

Asymptotic Analysis

- Worst-case analysis:
  Important when we need a guarantee about execution time.
  - For example, in Embedded Systems (in particular *real-time* embedded systems), this often plays an important role — example: if you're controlling a robot that is performing surgery, you have to calculate the movements, and once every, say, 10 ms, you absolutely need to output data to the actuators. You require the guarantee of execution time.

Asymptotic Analysis

- Average-case analysis:
  Quite often important — for most non-critical systems (e.g., games, web applications, server-side software, etc.); you want an average that allows you to guarantee a certain level of "smoothness" in the operation for a given number of concurrent requests.

Asymptotic Analysis

- Best-case analysis:
  Errmm.... Does *anyone* ever care about this one? :-)

Asymptotic Analysis

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  Errmm.... Does *anyone* ever care about this one? :-)

  (yes, the implied sarcastic tone is because the answer is, «*of course not!*» Maybe as an academic exercise it may be interesting, perhaps for the purpose of making sure that we understand the complete picture, etc.; but it typically has no practical value)

Asymptotic Analysis

- In general, if we are restricted to one of these to describe a given algorithm, we use the worst-case to describe it.
- Often enough, we use both worst-case and average-case.
  - However, very often, in asymptotic notation, worst-case and average-case are in the same class (example, linear search — both worst-case and average-case are $\Theta(n)$)

Asymptotic Analysis

- Examples of exponential run time:
  - Algorithms where we need to check all possibilities (for a suitable definition of "need to" — there are many situations where no better approach has been found; that does not mean that there aren't any)
  - One of such cases is the "subset-sum" problem; given *n* values, is there a subset of those values that adds 0?
    - An exponential time algorithm ($\Theta(2^n)$) consists of checking all combinations of each element being included or not (why is this 2^n?)

Asymptotic Analysis

- Examples of exponential run time:
  - Somewhat of a trick question: wouldn't this be $\Theta(n\,2^n)$? (checking the sum requires adding elements, and the number of elements goes up to n, and is indeed $n/2$ on average, given that we take all combinations)

Asymptotic Analysis

- Examples of exponential run time:
  - Somewhat of a trick question: wouldn't this be $\Theta(n\,2^n)$? (checking the sum requires adding elements, and the number of elements goes up to n, and is indeed $n/2$ on average, given that we take all combinations)
  - In fact, a naive implementation would indeed be $\Theta(n\,2^n)$. But the sums can be computed incrementally (when adding a new elements to the subset, don't recompute the sum — just add that value; same when removing)

Asymptotic Analysis

- Examples of exponential run time:
  - Interestingly enough, the best-case run time for this example is constant time for a YES answer and linear time for a NO answer (do you see why?)

Asymptotic Analysis

- Examples of exponential run time:
  - Interestingly enough, the best-case run time for this example is constant time for a YES answer and linear time for a NO answer (do you see why?)
  - If the first element is 0, then in just one operation the algorithm outputs YES; if all values are positive, or all are negative (which requires looking at all values before knowing), then the algorithm outputs NO.

Asymptotic Analysis

- Let's look at an alternative technique to figure out the appropriate descriptions for given functions.
- For example, consider two asymptotically non-negative functions f(*n*) and g(*n*), and consider:

$$\lim_{n \to \infty} \frac{\mathrm{f}(n)}{\mathrm{g}(n)}$$

- Say that the limit exists and is finite and non-zero — what does that tell us?

Asymptotic Analysis

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$$\lim_{n \to \infty} \mathrm{f}(n) = L \quad if$$

$$\forall\ \epsilon > 0,\ \exists\ N > 0\ \Big|\ |f(n) - L| < \epsilon\ \ \forall\ n > N$$

Asymptotic Analysis

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$$\forall\ \epsilon > 0,\ \exists\ N > 0\ \Big|\ |f(n) - L| < \epsilon\ \ \forall\ n > N$$

- Since we're saying that the limit of f(n) / g(n) exists (and is finite), we know that for all ε > 0, such *N* exists.

Asymptotic Analysis

- That is:

$$\forall\ \epsilon > 0,\ \ \exists\ N > 0\ \ \left|\ \left| \frac{\mathrm{f}(n)}{\mathrm{g}(n)} - L \right| < \epsilon\ \ \forall\ n > N \right.$$

- Since the above holds for every ε > 0, and $L > 0$ (why is $L > 0$?), then it must hold for every ε with $0 < \varepsilon < L$:

$$\left| \frac{\mathrm{f}(n)}{\mathrm{g}(n)} - L \right| < \epsilon \ \Rightarrow\ -\epsilon < \frac{\mathrm{f}(n)}{\mathrm{g}(n)} - L < \epsilon$$

Asymptotic Analysis

- Thus,

$$-\epsilon < \frac{f(n)}{g(n)} - L < \epsilon \;\Rightarrow\; L-\epsilon < \frac{f(n)}{g(n)} < L+\epsilon$$

- Since g(*n*) is non-negative (right? why?):

$$(L-\epsilon)g(n) < f(n) < (L+\epsilon)g(n) \quad \forall\ n > N$$

- But $L-\epsilon > 0$ (right?) ... So, what does this remind us of? (Hint: choose c1 = $L-\epsilon$ and c2 = L+ε)

Asymptotic Analysis

- Bottom line:

  For every asymptotically non-negative functions f(*n*) and g(*n*), we have:

$$0 < \lim_{n \to \infty} \frac{f(n)}{g(n)} < \infty \quad \Rightarrow \quad f(n) = \Theta(g(n))$$

- Important: this is a one-direction implication, and *not* an if-and-only-if — can you think of a counter-example (to show that the other direction does not necessarily hold)?

Asymptotic Analysis

- There may be cases where the limit does not even exist, and yet one function is Θ the other function.
- Example: $2+\sin(n) = \Theta(1)$ — right?
- However, the limit of $2+\sin(n)$ over 1 as $n \to \infty$ does not exist.

Asymptotic Analysis

- Similar criteria can be found for the other two Landau symbols that we saw last class:

$$\lim_{n \to \infty} \frac{\mathrm{f}(n)}{\mathrm{g}(n)} < \infty \quad \Rightarrow \quad \mathrm{f}(n) = O(\mathrm{g}(n))$$

Asymptotic Analysis

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- The difference to notice being that now the limit may be 0 — and sure, f(n) = O(g(n)) allows for f(n) to be either asymptotically proportional to g(n) or asymptotically negligible with respect to g(n) — in which case the limit would be 0.

Asymptotic Analysis

- Similarly, for Ω we have:

$$0 < \lim_{n \to \infty} \frac{f(n)}{g(n)} \quad \Rightarrow \quad f(n) = \Omega(g(n))$$

- The difference being that the limit may be ∞, corresponding to the situation where g(*n*) is asymptotically negligible with respect to f(*n*).

Asymptotic Analysis

- We can repeat the example (which will this time seem trivial by comparison) from last class, showing that $5n^2 + 2n = \Theta(n^2)$:

$$\lim_{n \to \infty} \frac{5n^2 + 2n}{n^2} = 5$$

- Directly showing that the numerator is Θ the denominator.

Asymptotic Analysis

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Asymptotic Analysis

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- We could claim that log grows slower than any power n^d, for every $d > 0$, and thus the limit is ∞.
- But we might as well use L'Hôpital's rule ...

Asymptotic Analysis

- Another example: let's verify that selection sort is, as we know it must be, $\Omega(n \log n)$:

$$\lim_{n \to \infty} \frac{n}{\log n} = \lim_{n \to \infty} \frac{1}{1/n} = \infty$$

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- Perhaps the only warning is that n is not supposed to be a continuous variable .... (is this really a problem?)

Asymptotic Analysis

- There are two additional Landau symbols that we'll want to know about (though they're not as useful — in practical situations — for describing algorithms as the three that we have seen).
- These are little-oh (o), and little-omega (ω).
- You could visualize these as the part, or the subset, from their corresponding big counterparts that excludes the big-Theta.

Asymptotic Analysis

- For example, informally speaking, big-Oh means that the function is either proportional or negligible with respect to the other one.
  - Little-oh means that the function is negligible with respect to the other one.
- In terms of limits, it means:

$$\lim_{n \to \infty} \frac{\mathrm{f}(n)}{\mathrm{g}(n)} = 0 \quad \Rightarrow \quad \mathrm{f}(n) = o(\mathrm{g}(n))$$

Asymptotic Analysis

- In terms of the formal definition, it may seem tricky/subtle:

$$o(g(n)) = \left\{ f(n) \;\middle|\; \begin{array}{c} \forall\ c>0,\ \exists\ \ N>0\ \ such\ that \\ 0 \leqslant f(n) \leqslant c\, g(n)\ \ \forall\ n \geqslant N \end{array} \right\}$$

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- See the difference with respect to big-Oh's definition? (Hint: in the above, what if instead of *c*, we use ε to denote that value?)

Asymptotic Analysis

- Similar idea for little-omega — in terms of limits:

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = \infty \quad \Rightarrow \quad f(n) = \omega(g(n))$$

Asymptotic Analysis

- Similar idea for little-omega — in terms of limits:

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- In terms of formal definition:

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Asymptotic Analysis

- Summarizing the definitions, in terms of the criteria using limits:

$$f(n) = o(g(n)) \qquad \lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$$

$$f(n) = O(g(n)) \qquad \lim_{n\to\infty} \frac{f(n)}{g(n)} < \infty$$

$$f(n) = \Theta(g(n)) \qquad 0 < \lim_{n\to\infty} \frac{f(n)}{g(n)} < \infty$$

$$f(n) = \Omega(g(n)) \qquad \lim_{n\to\infty} \frac{f(n)}{g(n)} > 0$$

$$f(n) = \omega(g(n)) \qquad \lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$$

Asymptotic Analysis

- Finally, as an analogy, to remember the meanings of these symbols; since they are used to compare functions with the idea of upper- and lower-bounds, we could definitely establish an analogy with comparing numbers --- upper-bound being associated with greater than, lower-bound associated with less than, and asymptotically proportional associated with being equal:

Asymptotic Analysis

- Thus, with our analogy, we would have:

$$\mathrm{f}(n) = o(\mathrm{g}(n)) \qquad f < g$$

$$\mathrm{f}(n) = O(\mathrm{g}(n)) \qquad f \leqslant g$$

$$\mathrm{f}(n) = \Theta(\mathrm{g}(n)) \qquad f = g$$

$$\mathrm{f}(n) = \Omega(\mathrm{g}(n)) \qquad f \geqslant g$$

$$\mathrm{f}(n) = \omega(\mathrm{g}(n)) \qquad f > g$$

Summary

- During today's class, we discussed:
  - Several examples of algorithms with run times corresponding to common functions (by name)
  - Saw an alternative criterion to determine the appropriate Landau symbol for given functions.
  - Introduced two additional Landau symbols; namely, little-oh (o) and little-omega (ω).