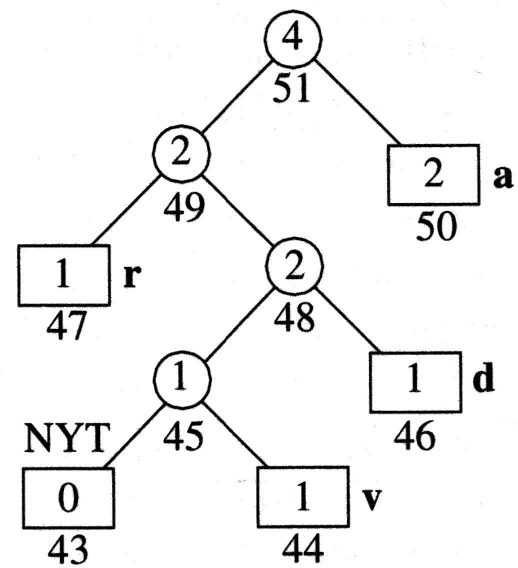
Binary Trees



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These slides, the course material, and course web site are based on work by Douglas W. Harder

Binary Trees

Standard reminder to set phones to silent/vibrate mode, please!

Binary Trees

- Today's class:
  - We'll look at binary trees — definition and some properties and related concepts.
  - Talk about its implementation.
  - Look at the notions of perfect and complete binary trees.
    - Implementing it with array storage.

Binary Trees

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  - A tree with the structure constrained such that each node has exactly two children.

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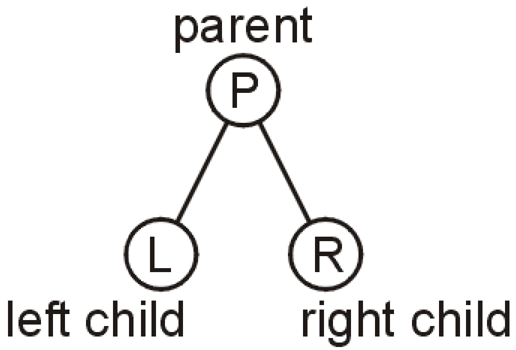
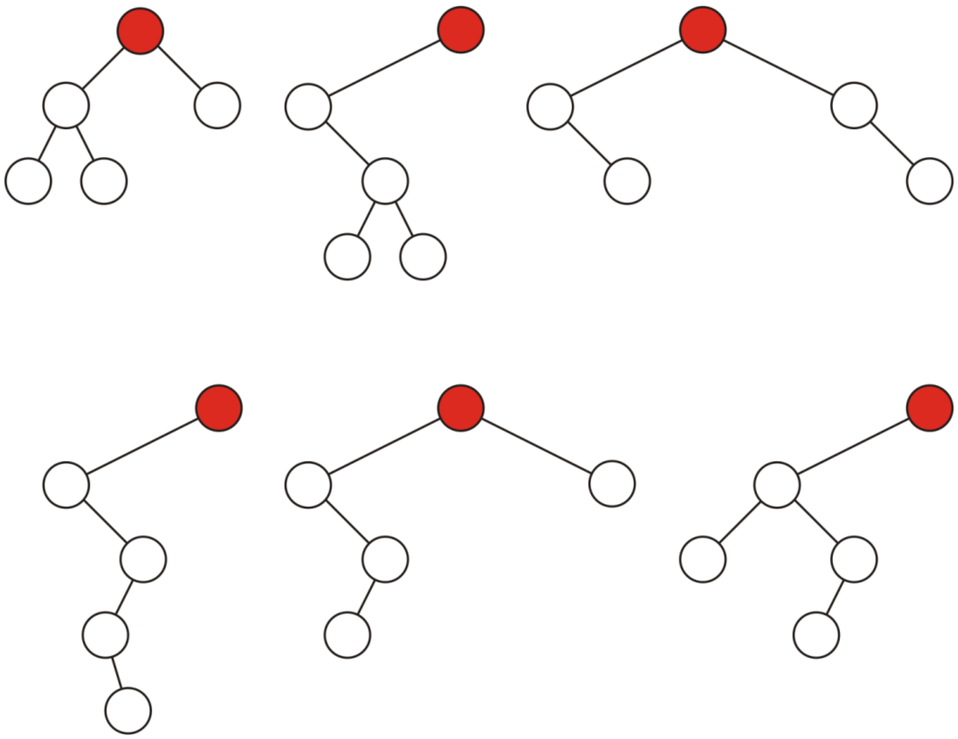
- The definition of a binary tree is quite straightforward:
  - A tree with the structure constrained such that each node has exactly two children.
    - Notice, *exactly* two children — not up to two children!
  - Each child is either empty or another binary tree.
  - Given this constraint, we can label the two children as *left* and *right* nodes or subtrees.



Binary Trees

- Examples of binary trees with five nodes:



Binary Trees

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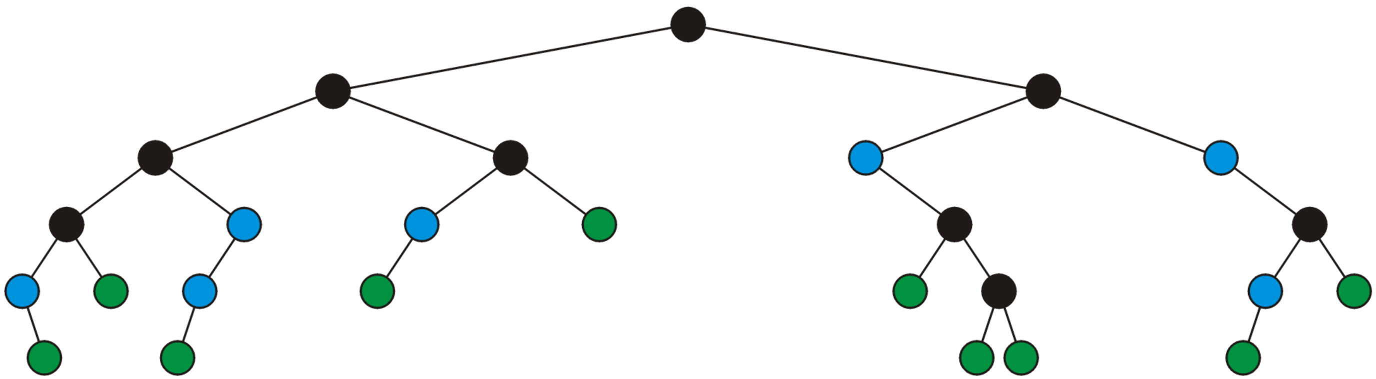

■ Full nodes ■ Leaf nodes ■ Neither

Binary Trees

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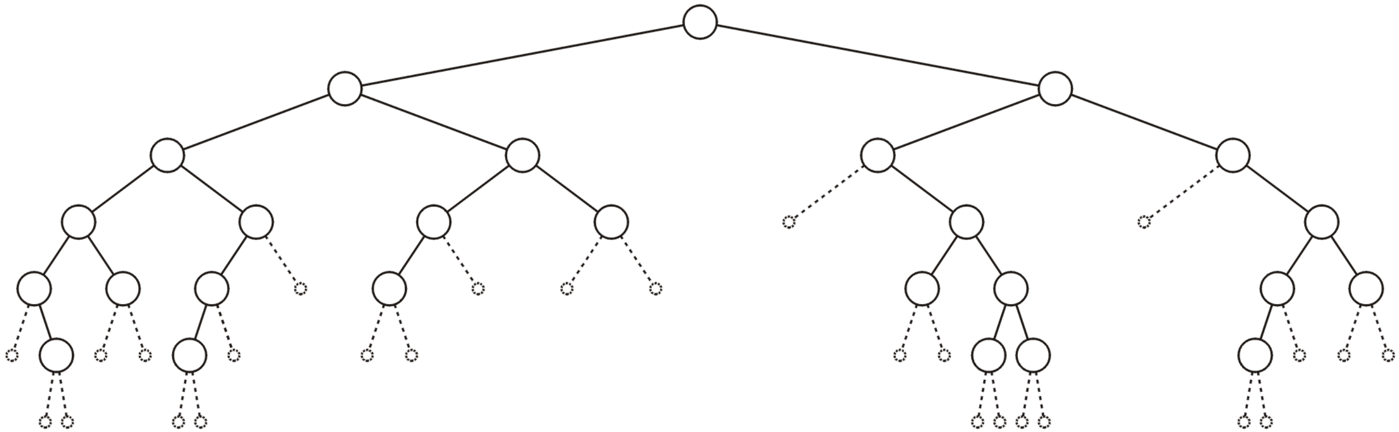
  - Graphically, the missing branches.



Binary Trees

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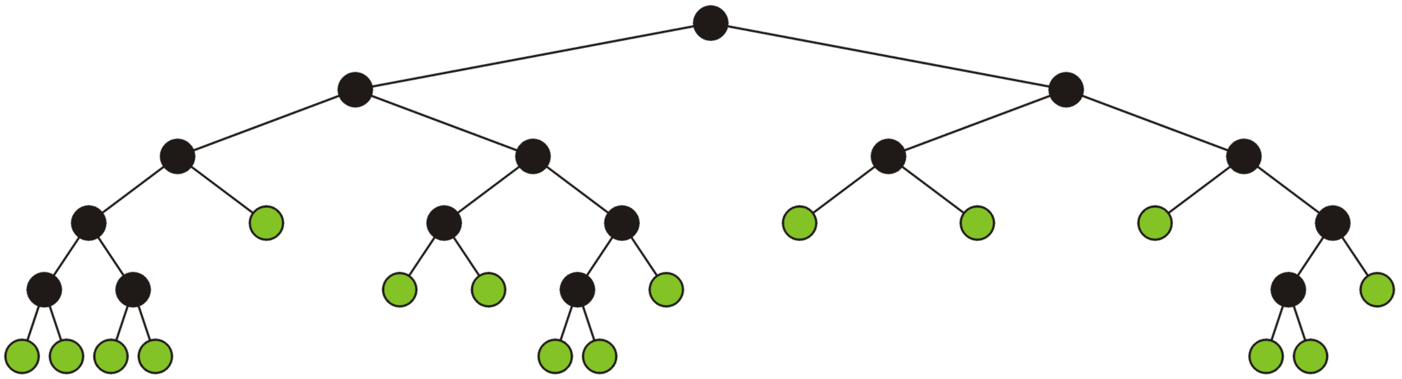

Binary Trees

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  - Clearly, since these are specific (constrained) types of trees, we could use a normal implementation of a tree, and ensure that the constraints are always applied.

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Binary Trees

- Implementing binary trees...
  - Clearly, since these are specific (constrained) types of trees, we could use a normal implementation of a tree, and ensure that the constraints are always applied.
    - Not a very interesting approach!
  - Some of the aspects in the general implementation are there to meet the general requirements (e.g., a linked list of children because we can have variable number of children).

Binary Trees

- Implementing binary trees...
  - Why use a linked list if we know that we have exactly two child nodes?

Binary Trees

- Implementing binary trees...
  - Why use a linked list if we know that we have exactly two child nodes?
    - Not only that — we want to *label* those as left and right!

Binary Trees

- Implementing binary trees...
  - A better approach is, of course, having two named pointers (as in, two data members), `left` and `right` (well, or `d_left`, `d_right`, or whatever naming convention for data members).

Binary Trees

- Implementing binary trees...
  - A better approach is, of course, having two named pointers (as in, two data members), `left` and `right` (well, or `d_left`, `d_right`, or whatever naming convention for data members).
    - If a child node is absent (i.e., an empty node or null sub-tree), we represent it with a null pointer in the corresponding child (left or right).

Binary Trees

```
template <typename Type>
class Binary_node
{
    Type d_element;
    Binary_node<Type> * d_parent;
    Binary_node<Type> * d_left;
    Binary_node<Type> * d_right;

public:
    Binary_node (const Type & obj);

    Type retrieve() const;
    Node<Type> * left() const;
    Node<Type> * right() const;

    // etc.
};
```

Binary Trees

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  - The code for traversal (e.g., recursive functions or recursive node's methods) can become a bit more "verbose" than in the case of general trees.
  - If a pointer to a child node is null, we're not allowed to dereference it.
    - And since each of the two pointers has its own name, we have to explicitly check them (as in, *individually*!)

Binary Trees

- A small caveat ....
  - The code for traversal (e.g., recursive functions or recursive node's methods) can become a bit more “verbose” than in the case of general trees.
  - If a pointer to a child node is null, we're not allowed to dereference it.
    - And since each of the two pointers has its own name, we have to explicitly check them (as in, *individually*!)
  - For example, here's what a recursive size() method could look like...

Binary Trees

```
template <typename Type>
int Binary_node<Type>::size() const
{
    int count = 1;   // this one
    if (left() != NULL)
    {
        count += left()->size();
    }

    if (right() != NULL)
    {
        count += right()->size();
    }

    return count;
}
```

Binary Trees

- In this particular example, a recursive function (*standalone* function, as opposed to a method) would be simpler with respect to that detail....

Binary Trees

```
template <typename Type>
int size (const Binary_node<Type> * node)
{
    if (node == NULL)
    {
        return 0;
    }

    return 1 + size(node->left()) + size(node->right());
}
```

Binary Trees

```
template <typename Type>
int size (const Binary_node<Type> * node)
{
    if (node == NULL)
    {
        return 0;
    }

    return 1 + size(node->left()) + size(node->right());
}
```

Key difference: we're not dereferencing the null pointer — we pass it to a function, and that function *compares* the pointer against NULL. In the other case, simply invoking **`node->size()`** when node is null invokes undefined behaviour.

Binary Trees

- Let's look at *perfect* binary trees...

Binary Trees

- A *perfect binary tree* of height *h* is a binary tree where:
  - All leaf nodes have the same depth *h*.
  - All other nodes are full.

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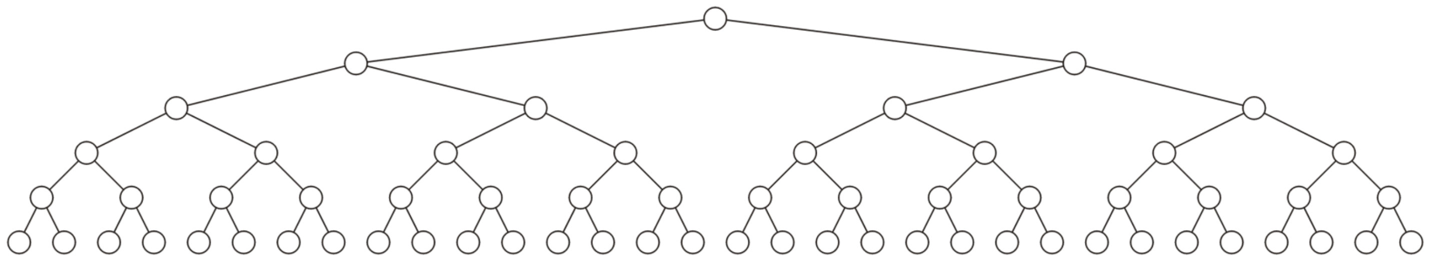
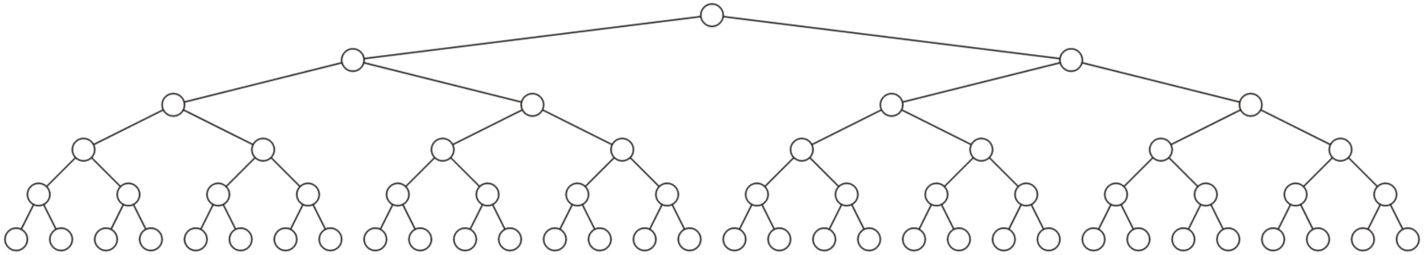
- A *perfect binary tree* of height *h* is a binary tree where:
  - All leaf nodes have the same depth *h*.
  - All other nodes are full.
  - Here's an example of a perfect binary tree:



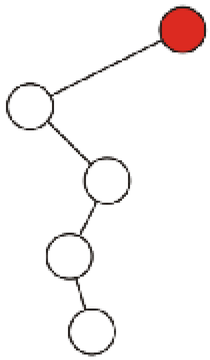
Binary Trees

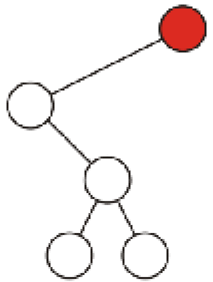
- Why do we need both conditions?
  - All leaf nodes have the same depth *h*.
  - All other nodes are full.
- Can you give counter-examples showing how each condition individually fails to describe this idea of a “maxed-out” tree?



Binary Trees

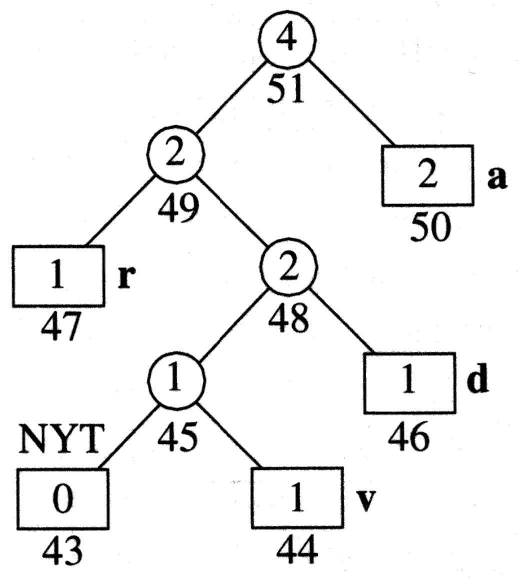
- Here's a (rather overkill) couple of examples of a tree where all leaf nodes are at the same depth:





Binary Trees

- Here's an example where all the non-leaf nodes are full:



Binary Trees

- We also have a nice recursive definition:
  - A binary tree of height 0 is perfect.
  - A binary tree with height $h > 0$ is perfect if both sub-trees are perfect binary trees of height $h-1$.

Binary Trees

- From this definition, we can prove, for example, that a perfect binary tree of height h has $2^{h+1}-1$ nodes.

Binary Trees

- From this definition, we can prove, for example, that a perfect binary tree of height *h* has $2^{h+1}-1$ nodes.
- We'll proceed by induction on *h*. So, we have to prove that: (1) The statement is true for $h = 0$; and (2) that if the statement is true for *h*, that implies that it is also true for $h+1$

Binary Trees

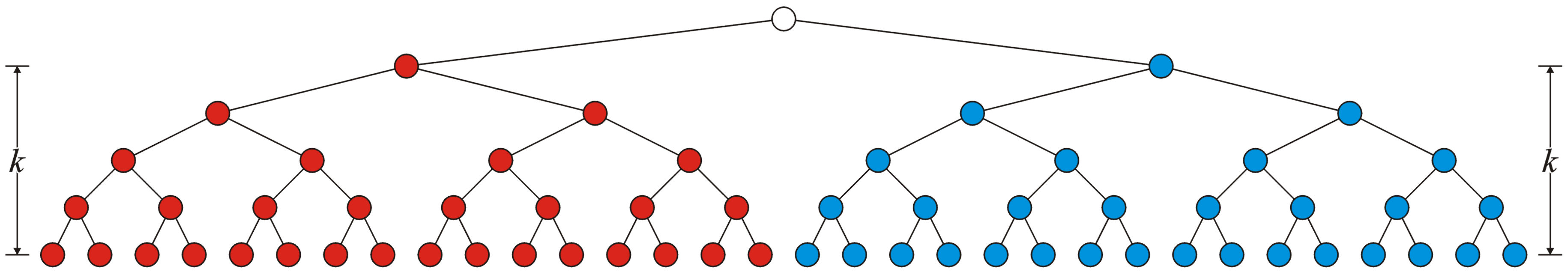
- Base case is trivial; a tree of height 0 is perfect by definition, and it is just a single (root) node. Thus, the formula matches ($2^{0+1} - 1 = 1$)

Binary Trees

- For the induction step, we assume (induction hypothesis) that the statement is true for h, and consider a tree of height $h+1$.
  - By definition, both sub-trees of a perfect tree of height $h+1$ are perfect trees of height h.
  - And by induction hypothesis, each of those perfect sub-trees have $2^{h+1}-1$ nodes.
  - Thus, we have in total the root node + twice the above number: $1 + 2(2^{h+1}-1) = 2^{h+2} - 1$

Binary Trees

- Graphically, the induction step goes like this:



Total number of nodes: $(2^{h+1} - 1) + 1 + (2^{h+1} - 1) = 2^{h+2} - 1$

Binary Trees

- As a direct consequence of this, the height of a perfect binary tree of *n* nodes is Θ(log *n*):

$$n = 2^{h+1} - 1 \;\Rightarrow\; h = \lg(n+1) - 1 \;=\; \Theta(\log n)$$

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$$n = 2^{h+1} - 1 \;\Rightarrow\; h = \lg(n+1) - 1 \;=\; \Theta(\log n)$$

- This is interesting — many operations with trees have a run time that goes with the depth of some path within the tree; if we have a perfect tree (or something *close* to it), we know that those operations run in O(log *n*).

Binary Trees

- Now, what could be *close* to a perfect binary tree?

Binary Trees

- Now, what could be *close* to a perfect binary tree?
  - One of the limitations with perfect binary trees is that the number of nodes is always $n = 2^k - 1$.
  - It would be nice to have something similar, but defined for all values of n.

Binary Trees

- Definition (informal): A *complete* binary tree is a binary tree that is filled at each depth from left to right (sort of filled in the same order as a breadth-first traversal).
  - That is, we are not allowed insertions at arbitrary positions!

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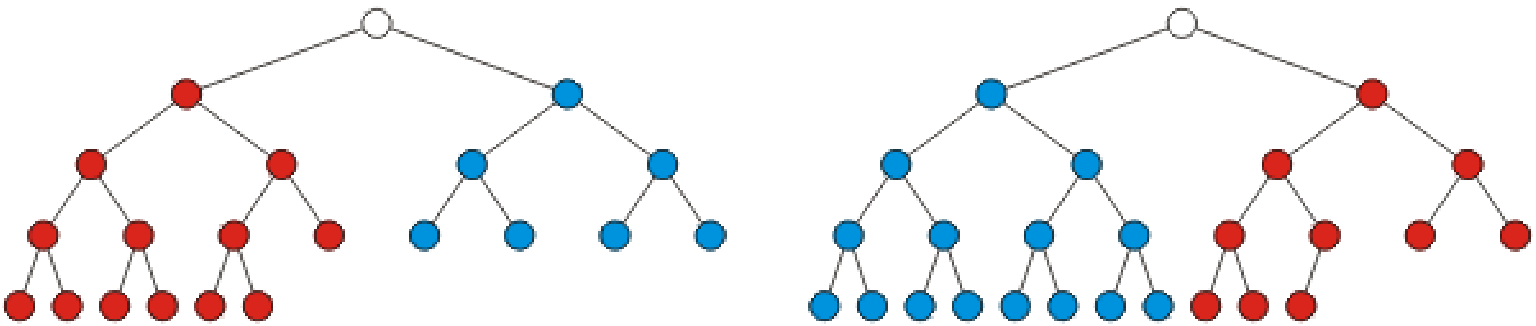
  - That is, we are not allowed insertions at arbitrary positions!
  - Also, removals are only allowed from the "last" position.
- This is could be seen like a perfect tree with the deepest level not full, but filled contiguously from left to right.

Binary Trees

- Definition – recursive:
  - A binary tree with height 0 is a *complete* binary tree.
  - A complete binary tree of height $h > 0$ is a binary tree where either:
    - The left sub-tree is a complete tree of height $h-1$ and the right sub-tree is a perfect tree of height $h-2$, or
    - The left sub-tree is a perfect tree of height $h-1$ and the right sub-tree is a complete tree of height $h-1$.

Binary Trees

- Graphically:



Binary Trees

- The *very* interesting aspect of a complete binary tree is that we can efficiently store it using an array!

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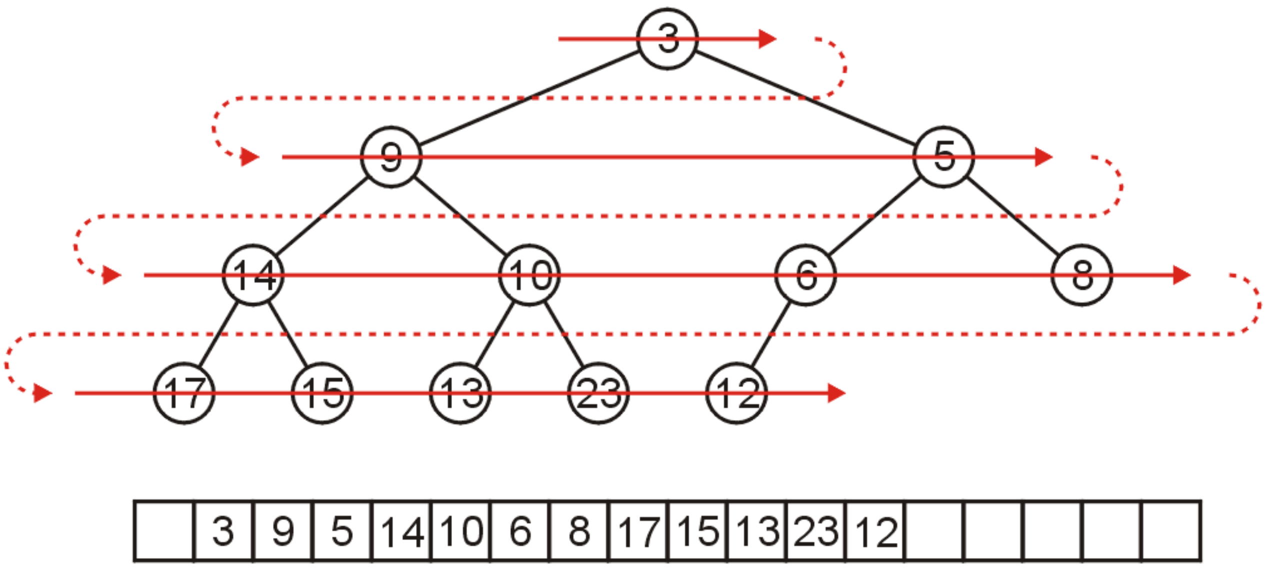
  - We traverse the tree in breadth-first order, placing the entries into the array



Binary Trees

- We notice that insertions and removals can only be done at the end of the array (not a bad thing — quite the contrary, if we think about it!)

Binary Trees

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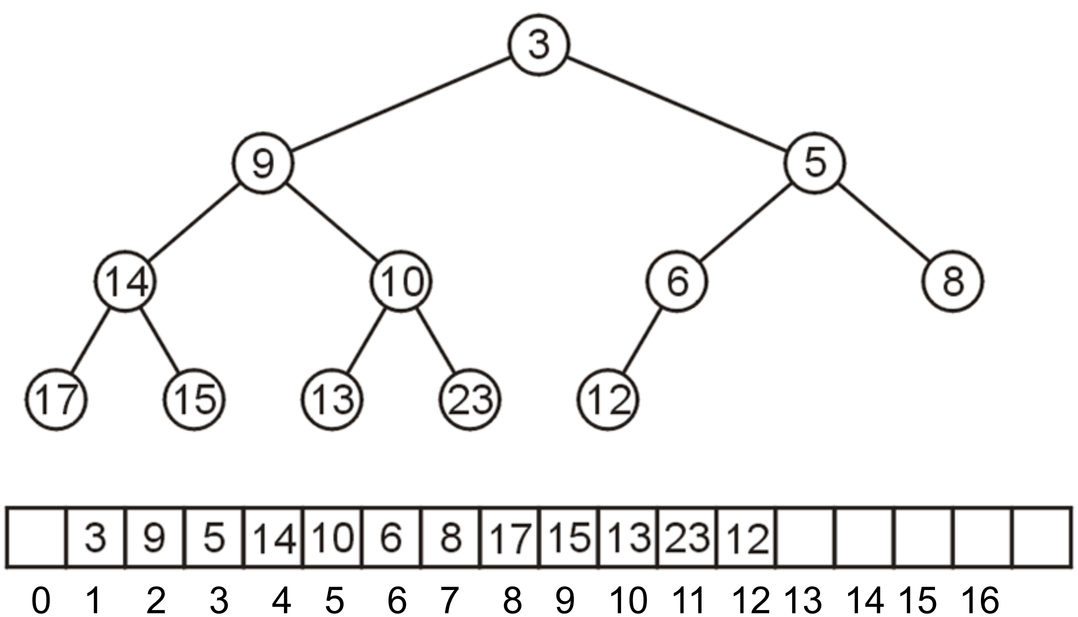
  - The problem would be: how do we efficiently access the nodes? (e.g., given a node, how do we access its children? Its parent? etc.)
  - With binary trees, the “fixed” structure of each node having exactly two children yields a nice and simple formula to relate these!
    - At each depth, there are twice as many nodes as in the previous depth!

Binary Trees

- For the formula to work, we have use 1 as the first subscript (we could look at it as we leave the first entry of the array unused).
- With this, we have:
  - The children of node at index *k* are the nodes at index 2*k* (left child) and 2*k*+1 (right child).
  - The parent of node at index k is at index $k \div 2$

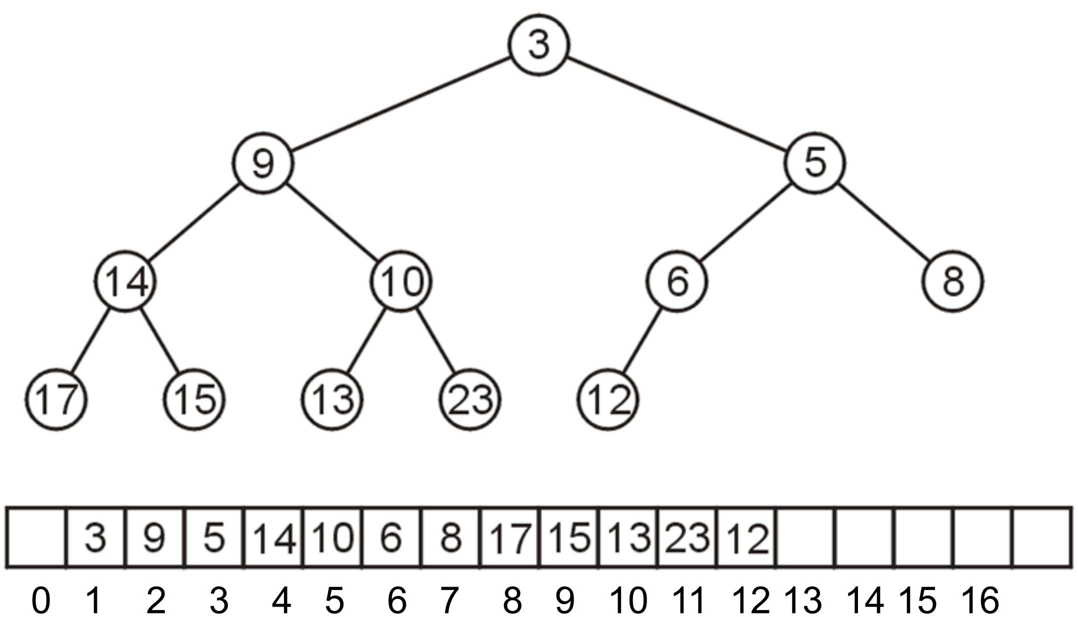
Binary Trees

- For example, node 10, at index 5, has its children, 13 and 23, at indices 10 and 11



Binary Trees

- For example, node 10, at index 5, has its children, 13 and 23, at indices 10 and 11
- Its parent, node 9, is at index 5 ÷ 2 = 2



Binary Trees

- Back to our "why is this only for complete binary trees" case ...

Binary Trees

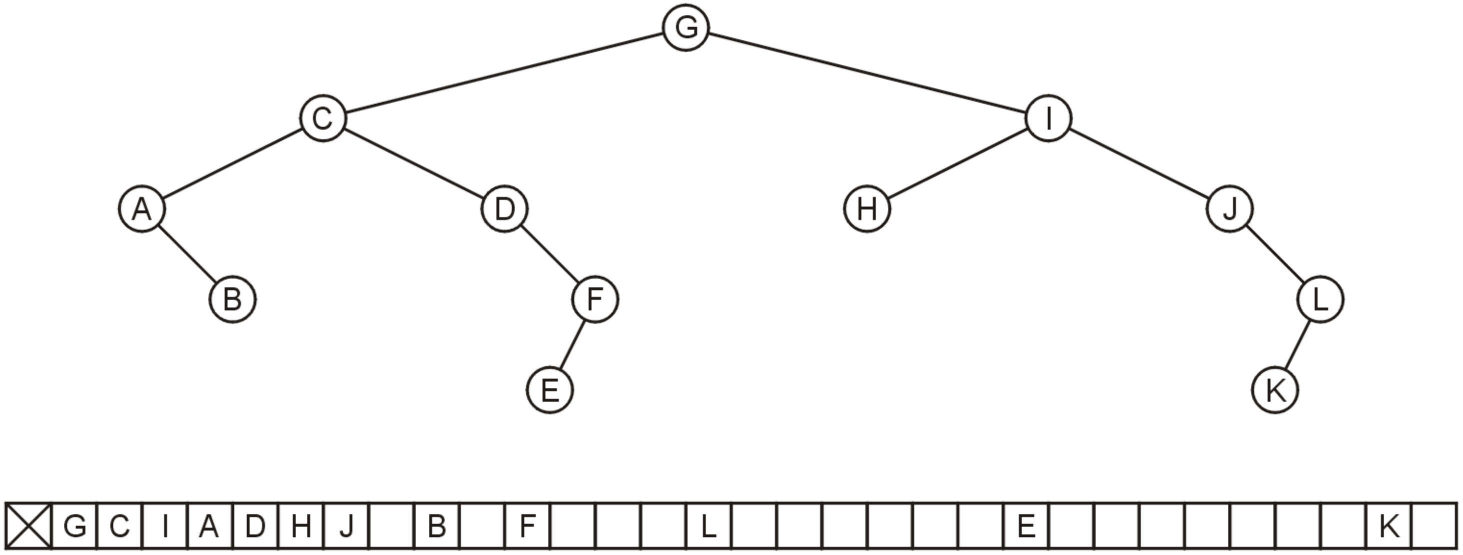
- Back to our “why is this only for complete binary trees” case ...
- Again, we could ask: why can't we do this with any binary tree? (we agree that with a general tree, efficient access to children and parent is a problem). But any binary tree does have the structure to facilitate this.

Binary Trees

- The problem with storing an arbitrary binary tree using an array is the inefficiency *in memory usage*.

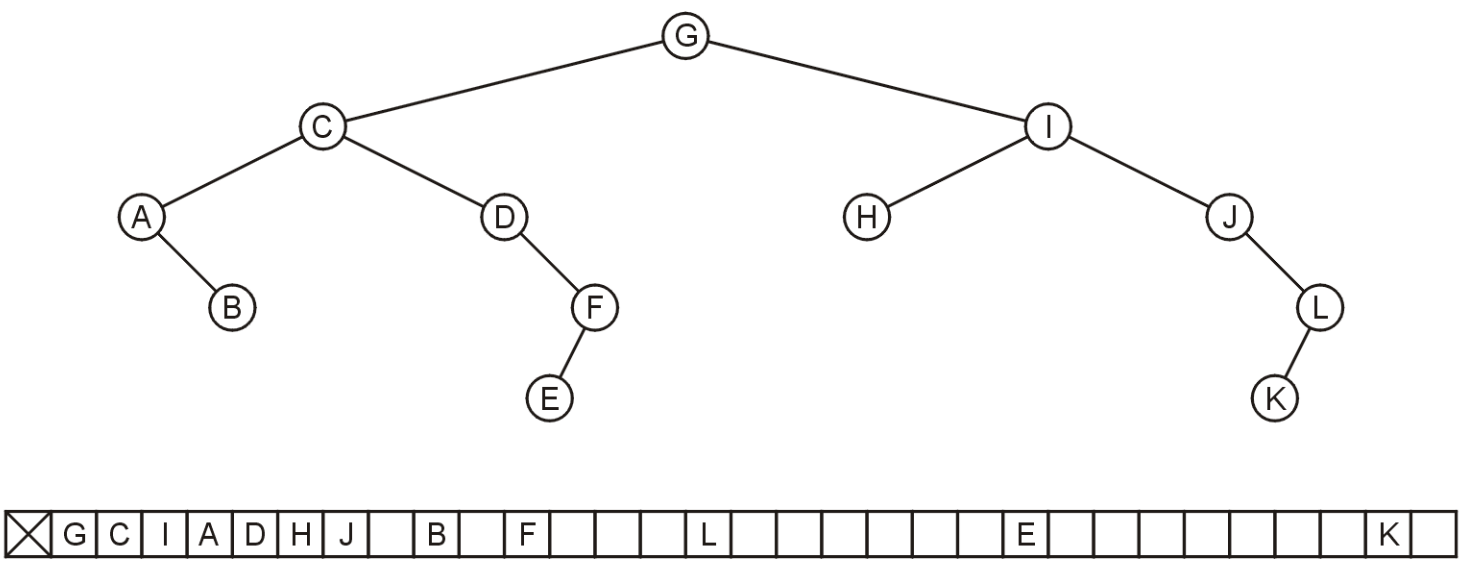
Binary Trees

- This tree has 12 nodes, and requires an array of 32 elements.



Binary Trees

- This tree has 12 nodes, and requires an array of 32 elements.
  - Adding just one extra node, as a child of node K doubles the required memory for the array!



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    (why exactly is it exponential in n?)

  - For this particular case, the number of nodes n happens to be the height of the tree, leading to $2^{n+1}-1$ nodes if it was a perfect tree (and a complete tree has at least half that many)

Summary

- During today's lesson, we discussed:
  - Binary trees
    - Definition
    - Some of its properties and related concepts
    - Discussed some aspects of their implementation
  - Perfect binary trees
  - Complete binary trees
    - Implementing them in terms of an array!