ECE-250 – Algorithms and Data Structures (Winter 2012)

Assignment 1 – Suggested solutions

1 - Show that $n^{\log_b m} = m^{\log_b n}$ for any $b > 0$, $n > 0$, $m > 0$.

Solution:

We start with the equality $n = b^{\log_b n}$. Then, we raise both sides of the equation to the $\log_b m$ th power and use a bit of algebra to obtain the result:

$$n^{\log_b m} \;=\; \left(b^{\log_b n}\right)^{\log_b m} \;=\; b^{\log_b n \cdot \log_b m} \left(=\; b^{\log_b m \cdot \log_b n}\right) \;=\; \left(b^{\log_b m}\right)^{\log_b n} \;=\; m^{\log_b n}$$

2 - (a) When writing binary numbers, you may have noticed the pattern of numbers that are written with all-ones: $3\ (= 4 - 1 = 2^2 - 1) = 11_2$, $7\ (= 8 - 1 = 2^3 - 1) = 111_2$, $15 = 1111_2$, and so on. The pattern being that $2^n - 1$ is written in binary as n ones; from the formal definition of binary notation, use the formula for the geometric sum to verify the above pattern (i.e., to show that the pattern holds for any arbitrary n).

(b) In decimal notation, n nines represent the value $10^n - 1$; using the same geometric sum formula, show that the pattern holds for any arbitrary n.

Solution:

In both cases, this is a direct application of the geometric sum, in (a) with $a = 2$, and in (b) with $a = 10$:

A number written in binary as $b_{n-1}b_{n-2}\cdots b_2b_1b_0$ represents the value $\displaystyle\sum_{k=0}^{n-1} b_k \cdot 2^k$; if the n bits are all-ones, then:

$$\sum_{k=0}^{n-1} 1 \cdot 2^k = \frac{2^n - 1}{2 - 1} \;=\; 2^n - 1$$

For base 10, a number written as n nines represents the value:

$$\sum_{k=0}^{n-1} 9 \cdot 10^k = 9 \sum_{k=0}^{n-1} 10^k = 9 \cdot \frac{10^n - 1}{10 - 1} = 9 \cdot \frac{10^n - 1}{9} \;=\; 10^n - 1$$

3 - Solve the following recurrence relation: $\mathrm{f}(n) = 2\,\mathrm{f}(n/2) + n$, with $\mathrm{f}(1) = 1$. To simplify the problem, you may assume that n is a power of 2 (that is, the relation holds for $n = 2^k$ for some $k \in \mathbb{Z}^+$)

Solution:

As suggested, we expand the sub-expressions, until reaching $f(1)$:

$$\mathrm{f}(n/2) = 2\,\mathrm{f}((n/2)/2) + (n/2) \;\Rightarrow\; \begin{aligned} \mathrm{f}(n) &= 2\,(2\,\mathrm{f}(n/4) + n/2) + n \\ &= 4\,\mathrm{f}(n/4) \,+\, 2n \\ &= 4\,(2\,\mathrm{f}(n/8) + n/4) \,+\, 2n \\ &= 8\mathrm{f}(n/8) \,+\, 3n \\ &= \cdots \end{aligned}$$

We recognize the pattern — after k times, we obtain $\mathrm{f}(n) = 2^k \mathrm{f}(n\,/\,2^k) + kn$

We reach f(1) when applied $k = \lg n$ times (such that $2^k = n$). Substituting, we obtain:

$$\mathrm{f}(n) = 2^{\lg n}\mathrm{f}(n\,/\,2^{\lg n}) + \lg n \cdot n = n\mathrm{f}(1) + \lg n \cdot n$$

Since we know $\mathrm{f}(1) = 1$, we finally get

$$\mathrm{f}(n) \;=\; n(1 + \lg n)$$

4 - Prove, by induction, that an exponential function grows faster than any integer power. That is, for any integer $n \geqslant 1$, it holds that:

$$\lim_{x\to\infty} \frac{\mathrm{e}^x}{x^n} \;=\; \infty$$

Solution:

Step 1 – Base case, $n = 1$ (applying L'Hôpital's rule):

$$\lim_{x\to\infty} \frac{\mathrm{e}^x}{x} \;=\; \lim_{x\to\infty} \frac{\mathrm{e}^x}{1} \;=\; \infty$$

Step 2 – Induction step; show that if the statement is true for n, then that implies that it is true for $n+1$; thus, assume our induction hypothesis, that the statement is true for n, and consider:

$$\lim_{x\to\infty} \frac{\mathrm{e}^x}{x^{n+1}}$$

Applying L'Hôpital's rule, we obtain:

$$\lim_{x\to\infty} \frac{\mathrm{e}^x}{x^{n+1}} \;=\; \lim_{x\to\infty} \frac{\mathrm{e}^x}{(n+1)x^n} \;=\; \frac{1}{n+1} \lim_{x\to\infty} \frac{\mathrm{e}^x}{x^n}$$

Since $n+1$, though it can be arbitrarily large, is a constant value.

By induction hypothesis, the right-hand side limit is ∞, so the limit for the case $n+1$ is also ∞, completing the induction step, and thus completing the proof.

5 - Prove, by induction, the result of the arithmetic sum — i.e., prove that $\displaystyle\sum_{k=1}^{n} k \;=\; \frac{n(n+1)}{2} \;\; \forall\, n \geqslant 1$

Solution:

Step 1: Base case, $n = 1$: the statement trivially holds; the actual sum is 1, the formula for $n = 1$ gives us $1 \cdot 2/2 = 1$.

Step 2: Induction step; our induction hypothesis is that the formula holds for n and we want to show that this implies that the formula also holds for $n + 1$.

Induction hypothesis: $\displaystyle\sum_{k=1}^{n} k \;=\; \frac{n(n+1)}{2}$

Based on that, we want to show that $\displaystyle\sum_{k=1}^{n+1} k \;=\; \frac{(n+1)(n+2)}{2}$

We know that $\displaystyle\sum_{k=1}^{n+1} k \;=\; \sum_{k=1}^{n} k + (n{+}1)$, and by induction hypothesis, we know that $\displaystyle\sum_{k=1}^{n} k \;=\; \frac{n(n+1)}{2}$.
Thus:

$$\sum_{k=1}^{n+1} k \;=\; \frac{n(n+1)}{2} + (n+1) \;=\; (n+1)\left(\frac{n}{2}+1\right) \;=\; \frac{(n+1)(n+2)}{2}$$

This completes the induction step, and thus completes the proof.

5% Bonus Marks:

The total torque T (with respect to the origin) produced by N objects with distinct masses $m_1,\, m_2,\, \cdots,\, m_N$ and aligned on the x-axis, at positions $x = 1,\, x = 2,\, \cdots,\, x = N$, is given by

$$T \;=\; \sum_{k=1}^{N} k\, m_k$$

Prove (perhaps by contradiction?) that the arrangement of the N objects that minimizes the torque is that where the objects are sorted by mass in decreasing order (i.e., the object with highest mass at position $x = 1$, the object with second highest mass at position $x = 2$, and so on).

Solution:

The basic intuition behind the solution is that if they're not sorted by descending mass, then there must be at least one pair of objects that are in ascending order — switching those two produces a total torque that is lower, contradicting the assumption that this arrangement was the one with lowest torque.

Formally, we could write it as follows:

Assume, for a contradiction, that an arrangement of objects $m_{k_1}, m_{k_2}, m_{k_3},\ \cdots\ , m_{k_n}$, not sorted by descending mass, produces the lowest possible torque, denoted T_{min}:

$$T_{\text{min}} = m_{k_1} + 2m_{k_2} + 3m_{k_3} +\ \cdots\ + n m_{k_n}$$

Because the objects are not sorted by descending mass, there are at least two objects, say m_{k_i} and m_{k_j} such that $i < j$ and $m_{k_i} < m_{k_j}$. If we swap the positions of these two objects, we obtain a new value for the torque:

$$T' = m_{k_1} + 2m_{k_2} + \ \cdots \ + im_{k_j} + \ \cdots \ + jm_{k_i} + \ \cdots \ + nm_{k_n}$$

Thus,

$$T_{\min} - T' \ = \ im_{k_i} + jm_{k_j} - im_{k_j} - jm_{k_i} \ = \ (j-i)(m_{k_j} - m_{k_i})$$

Since $i < j$ and $m_{k_i} < m_{k_j}$, the two factors in the above expression are positive, and thus:

$$T_{\min} - T' > 0 \ \Rightarrow \ T' < T_{\min}$$

Contradicting the assumption that $T_{\min}$ is the lowest possible torque.