ECE-250 – Algorithms and Data Structures (Winter 2012)

Assignment 1

Due on Friday, January 13, in class.

All questions have equal weight for the assignment grade (except for the bonus marks question).

1 - Show that $n^{\log_b m} = m^{\log_b n}$ for any $b > 0$, $n > 0$, $m > 0$.

Hint: Start with the equality $n = b^{\log_b n}$. Then, looking at the left-hand side of the equality that you need to prove should suggest what to do next.

2 - **(a)** When writing binary numbers, you may have noticed the pattern of numbers that are written with all-ones: $3\ (= 4 - 1 = 2^2 - 1) = 11_2$, $7\ (= 8 - 1 = 2^3 - 1) = 111_2$, $15 = 1111_2$, and so on. The pattern being that $2^n - 1$ is written in binary as n ones; from the formal definition of binary notation, use the formula for the geometric sum to verify the above pattern (i.e., to show that the pattern holds for any arbitrary n).

(b) In decimal notation, n nines represent the value $10^n - 1$; using the same geometric sum formula, show that the pattern holds for any arbitrary n.

3 - Solve the following recurrence relation: $\mathrm{f}(n) = 2\,\mathrm{f}(n/2) + n$, with $\mathrm{f}(1) = 1$. To simplify the problem, you may assume that n is a power of 2 (that is, the relation holds for $n = 2^k$ for some $k \in \mathbb{Z}^+$)

Hint: Try expanding the expression $\mathrm{f}(n/2)$, for which the relation also holds, and then repeatedly for the new sub-expressions:

$$\mathrm{f}(n/2) = 2\,\mathrm{f}((n/2)/2) + (n/2) \;\Rightarrow\; \begin{aligned} \mathrm{f}(n) &= 2\ (2\,\mathrm{f}(n/4) + n/2) + n \\ &= 4\,\mathrm{f}(n/4) + 2n \\ &= 4\ (2\,\mathrm{f}(n/8) + n/4) + 2n \\ &= \cdots \end{aligned}$$

4 - Prove, by induction, that an exponential function grows faster than any integer power. That is, for any integer $n \geqslant 1$, it holds that:

$$\lim_{x\to\infty} \frac{\mathrm{e}^x}{x^n} = \infty$$

Hint: Both for the base case and for the induction step, you could use L'Hôpital's rule.

5 - Prove, by induction, the result of the arithmetic sum — i.e., prove that $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \quad \forall\, n \geqslant 1$

Notice that for questions 4 and 5, you must provide proofs *by induction* — any proof through other means is not accepted.

5% Bonus Marks:

The total torque T (with respect to the origin) produced by N objects with distinct masses $m_1, m_2, \cdots, m_N$ and aligned on the x-axis, at positions $x = 1$, $x = 2$, $\cdots$, $x = N$, is given by

$$T = \sum_{k=1}^{N} k\, m_k$$

Prove (perhaps by contradiction?) that the arrangement of the N objects that minimizes the torque is that where the objects are sorted by mass in decreasing order (i.e., the object with highest mass at position $x = 1$, the object with second highest mass at position $x = 2$, and so on).