ECE 250 *Algorithms and Data Structure*
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with the subject ECE 250 Notes 1.2.
Assistances and comments will be acknowledged.

1.4 Proof by Induction

Suppose we have a formula $F(n)$ which we wish to show is true for all $n \geq n_0$. For example, we may want to show that

$$F(n) = \sum_{k=0}^{n} k = \frac{n(n+1)}{2}.$$

Usually, $n_0 = 0$ or $n_0 = 1$. To show the formula is true, we will do is the following:

1. Prove that $F(n_0)$ is true,
2. We will assume that the formula $F(n)$ is true for any $n \geq n_0$, and
3. Under the assumption that $F(n)$ is true, we will attempt to demonstrate that it follows that $F(n + 1)$ must also be true.

If we can demonstrate that $F(n)$ implies the truth of $F(n + 1)$, the inductive principle allows us to conclude that the formula is true for all $n \geq n_0$. This follows from the simple observation that, for example,

> If $F(0)$ is true, $F(1)$ is true;
> If $F(1)$ is true, $F(2)$ is true,
> If $F(2)$ is true, $F(3)$ is true, and thus $F(4)$ is true, *etc*.

Note: to show that $F(n)$ is false, it is only necessary to find one n for which it is false.

1.4.1 Formulation

Very often, $F(n)$ is a formula, such as

$$\sum_{k=0}^{n} k = \frac{n(n+1)}{2} \text{ for } n \geq 0, \; \sum_{k=1}^{n} 2k - 1 = n^2 \text{ for } n \geq 1, \text{ and } \sum_{k=0}^{n} 2^k = 2^{n+1} - 1 \text{ for } n \geq 0.$$

Alternatively, it could be a statement such as “the integer $n^3 - n$ is divisible by 3 for all $n \geq 1$.

1.4.2 Examples

We will now look at ten examples, showing the steps of the proofs at each step.

1.4.2.1 Example 1

Prove that $\sum_{k=0}^{n} k = \frac{n(n+1)}{2}$ for $n \geq 0$.

First, we observe that $\sum_{k=0}^{0} k = 0$ and that $\frac{0(0+1)}{2} = 0$, so the formula is true when $n = 0$.

Next, we will assume that $\sum_{k=0}^{n} k = \frac{n(n+1)}{2}$ is true for all $n \geq 0$.

We now want to show that $\sum_{k=0}^{n+1} k = \frac{(n+1)(n+2)}{2}$.

Looking at $\sum_{k=0}^{n+1} k$, we see we can that this is $0 + 1 + 2 + \cdots + n + (n + 1)$ which can be written as

$$\left(\sum_{k=0}^{n} k\right) + (n+1).$$

We assumed that the formula was correct for *n*, so we may substitute that value:

$$\frac{n(n+1)}{2} + (n+1).$$

Getting a common denominator and factoring out the common term $(n + 1)$, this equals

$$\frac{n(n+1) + 2(n+1)}{2} = \frac{(n+2)(n+1)}{2},$$

which is the desired result.

1.4.2.2 Example 2

Prove that the sum of the first n odd integers is n^2.

First, we observe that the first odd integer is 1, and its sum equals $1^2 = 1$.

Next, we will assume that the sum of the first n odd integers is n^2. We can write this as

$$\sum_{k=1}^{n} 2k-1 = n^2 \text{ for } n \ge 0.$$

We now want to show that $\sum_{k=1}^{n+1} 2k-1 = (n+1)^2$.

As we saw in the previous example, we note that the sum to the $(n + 1)^{\text{st}}$ integer contains the sum to the n^{th} integer, so we can thus write

$$\begin{aligned}\sum_{k=1}^{n+1} 2k-1 &= \left(\sum_{k=1}^{n} 2k-1\right) + \left(2(n+1)-1\right)\\ &= \left(\sum_{k=1}^{n} 2k-1\right) + (2n+1).\end{aligned}$$

We assumed that the formula was correct for n, so we may substitute that value:

$$\sum_{k=1}^{n+1} 2k-1 = n^2 + (2n+1).$$

The right-hand side can either be recognized to be a perfect square, or it can be simplified to

$$\begin{aligned}n^2 + (2n+1) &= (n^2+n) + (n+1)\\ &= n(n+1) + (n+1)\\ &= (n+1)(n+1)\\ &= (n+1)^2,\end{aligned}$$

which is the desired result.

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1.4.2.3 Example 3

Prove that $\sum_{k=0}^{n} 2^k = 2^{n+1} - 1$ for $n \geq 0$.

First, we observe that $\sum_{k=0}^{0} 2^k = 1$ and that $2^{0+1} - 1 = 1$, so the formula is true when $n = 0$.

Next, we will assume that $\sum_{k=0}^{n} 2^k = 2^{n+1} - 1$ is true for all $n \geq 0$.

We now want to show that $\sum_{k=0}^{n+1} 2^k = 2^{(n+1)+1} - 1 = 2^{n+2} - 1$.

As we saw in the previous example, the series to $n + 1$ contains the series to n:

$$\sum_{k=1}^{n+1} 2^k = \left(\sum_{k=1}^{n} 2^k \right) + \left(2^{n+1} \right).$$

We assumed that the formula was correct for n, so we may substitute that value:

$$\sum_{k=1}^{n+1} 2^k = \left(2^{n+1} - 1 \right) + \left(2^{n+1} \right).$$

The right-hand side can be simplified to:

$$2^{n+1} - 1 + 2^{n+1} = 2 \cdot 2^{n+1} - 1 = 2^{n+2} - 1,$$

which is the desired result.

1.4.2.4 Example 4

Prove that $\sum_{k=0}^{n}\binom{n}{k} = 2^n$ for $n \ge 0$.

First, we observe that $\sum_{k=0}^{0}\binom{n}{k} = \binom{0}{0} = 1$ and that $2^0 = 1$, so the formula is true when $n = 0$.

Next, we will assume that $\sum_{k=0}^{n}\binom{n}{k} = 2^n$ is true for all $n \ge 0$.

We now want to show that $\sum_{k=0}^{n+1}\binom{n+1}{k} = 2^{n+1}$.

Unlike the previous examples, it is more difficult to see how we can write this expression in terms of the one which we assumed, but we know that

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}.$$

Thus, we can write

$$\begin{aligned}\sum_{k=0}^{n+1}\binom{n+1}{k} &= \binom{n+1}{0} + \left[\sum_{k=1}^{n}\binom{n+1}{k}\right] + \binom{n+1}{n+1}\\ &= 1 + \left\{\sum_{k=1}^{n}\left[\binom{n}{k} + \binom{n}{k-1}\right]\right\} + 1\\ &= 1 + \left[\sum_{k=1}^{n}\binom{n}{k}\right] + \left[\sum_{k=1}^{n}\binom{n}{k-1}\right] + 1.\end{aligned}$$

The first sum is missing the first term, when $k = 0$, but $\binom{n}{0} = 1$, so we can combine that term. For the second, we can do a change of variables, replacing $\kappa = k - 1$ which requires us to add one to the limits of the summation as $k = \kappa + 1$:

$$\sum_{k=1}^{n}\binom{n}{k-1} = \sum_{\kappa=0}^{n-1}\binom{n}{\kappa}.$$

> Aside:
> This is similar to a change of variables for an integral; for example, suppose we have $\int_0^1 x\,dx$. If we make the change of variable to $y = x + 1$, we have $dy = dx$ and $x = y - 1$, yielding $\int_{-1}^{0}(y+1)\,dy$.

Now we have

$$\left[\sum_{k=0}^{n}\binom{n}{k}\right]+\left[\sum_{k=0}^{n-1}\binom{n}{k}\right]+1,$$

and we can apply the same observation: $\binom{n}{0}=1$. Substituting this completes the second summation:

$$\left[\sum_{k=0}^{n}\binom{n}{k}\right]+\left[\sum_{k=0}^{n-1}\binom{n}{k}\right]+\binom{n}{n}=\left[\sum_{k=0}^{n}\binom{n}{k}\right]+\left[\sum_{k=0}^{n}\binom{n}{k}\right]=2\left[\sum_{k=0}^{n}\binom{n}{k}\right].$$

By assumption, the summation equals 2^n and substituting this in yields $2\cdot 2^n = 2^{n+1}$, which is the desired result.

1.4.2.5 Example 5

Prove that $\sum_{k=0}^{n} r^k = \frac{1 - r^{n+1}}{1 - r}$ for $n \geq 0$.

First, we observe that $\sum_{k=0}^{0} r^k = r^0 = 1$ and that $\frac{1 - r^{0+1}}{1 - r} = \frac{1 - r^1}{1 - r} = 1$, so the formula is true when $n = 0$.

Next, we will assume that $\sum_{k=0}^{n} r^k = \frac{1 - r^{n+1}}{1 - r}$ is true for all $n \geq 0$.

We now want to show that $\sum_{k=0}^{n+1} r^k = \frac{1 - r^{(n+1)+1}}{1 - r} = \frac{1 - r^{n+2}}{1 - r}$.

As we saw in previous examples, the series to n + 1 contains the series to n:

$$\sum_{k=0}^{n+1} r^k = \left(\sum_{k=0}^{n} r^k \right) + r^{n+1}.$$

We assumed that the formula was correct for n, so we may substitute that value:

$$\frac{1 - r^{n+1}}{1 - r} + r^{n+1}.$$

Finding a common denominator, expanding the one product, and cancelling the two inner terms yields

$$\frac{1 - r^{n+1} + (1 - r) r^{n+1}}{1 - r} = \frac{1 - r^{n+1} + r^{n+1} - r^{n+2}}{1 - r} = \frac{1 - r^{n+2}}{1 - r},$$

which is the desired result.

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1.4.2.6 Example 6

This example is a little different: show that $n^3 - n$ is divisible by 3 for all integers. Induction only works for semi-infinite intervals, say, 0, 1, 2, 3, … but we could also go the other way: 0, –1, –2, –3, …; that is, assume $F(n)$ is true and use this to show that $F(n - 1)$ is true. Thus we will prove the base case when $n = 0$ and then proceed in two steps.

First, we observe that $0^3 - 0 = 0$ is a number divisible by 3.

Next, we will assume that $n^3 - n$ is divisible by 3 for any $n \geq 0$.

Now, $(n + 1)^3 - (n + 1) = n^3 + 3n^2 + 3n + 1 - n - 1 = n^3 - n + 3(n^2 + n)$. By assumption, the first term is divisible by 3 and the second term is, by definition, divisible by 3. Thus, because the sum of two terms divisible by 3 is itself divisible by 3, we have our desired result.

Going in the other direction, we will assume that $n^3 - n$ is divisible by 3 for any $n \leq 0$.

Now, $(n - 1)^3 - (n - 1) = n^3 - 3n^2 + 3n - 1 - n + 1 = n^3 - n + 3(n - n^2)$. Again, by assumption, the first term is divisible by 3 and the second term is, by definition, divisible by 3. Thus, because the sum of two terms divisible by 3 is itself divisible by 3, we have our desired result.

As we have shown that this statement is true for $n = 0$ and, based on our two assumptions, that it is true for all $n \geq 0$ and for all $n \leq 0$, it follows the statement is true for all n.

Note that we could prove this using a completely different approach. For example, any integer is of the form $3m$, $3m + 1$, or $3m + 2$ for some m. We may now observe that

$$(3m)^3 - (3m) = 3m(3m-1)(3m+1), (3m+1)^3 - (3m+1) = 3m(3m+1)(3m+1) \text{ and}$$

$$(3m+2)^3 - (3m+2) = 3(m+1)(3m+1)(3m+1),$$

respectively. Each of these is a multiple of 3.

1.4.2.7 Example 7

As a different problem, show that the derivative of x^n w.r.t. x is nx^{n-1} for $n \geq 1$ by using the chain rule and the fact that $\frac{d}{dx}x = 1$.

The assumption gives us our default case when $n = 1$.

Next, we will assume that $\frac{d}{dx}x^n = nx^{n-1}$ is true for all $n \geq 1$.

We want to show that $\frac{d}{dx}x^{n+1} = (n+1)x^n$. Using the chain rule, we see that

$$\begin{aligned}\frac{d}{dx}x^{n+1} &= \frac{d}{dx}\left(x \cdot x^n\right)\\ &= \left(\frac{d}{dx}x\right)x^n + x\frac{d}{dx}\left(x^n\right).\end{aligned}$$

In the first case, we can substitute the default case, and in the second, we substitute our assumption to get

$$\begin{aligned}\left(\frac{d}{dx}x\right)x^n + x\frac{d}{dx}\left(x^n\right) &= 1 \cdot x^n + x \cdot nx^{n-1}\\ &= x^n + nx^n = (n+1)x^n\end{aligned}$$

which is the desired result.

1.4.2.8 Example 8

Here is another interest example were we attempt to demonstrate an inequality. Show that $\ln(n!) \le n\ln(n)$ for all $n \ge 1$.

Again, we start with our base case when $n = 1$: $\ln(1!) = \ln(1) = 0$ and $1\cdot\ln(1) = \ln(1) = 0$.

Next, we will assume that $\ln(n!) \le n\ln(n)$ is true for all $n \ge 0$.

Using the rule that ln(*ab*) = ln(*a*) + ln(*b*), we may now proceed:

$$\ln((n+1)!) = \ln(n+1) + \ln(n!).$$

Using our assumption, we can rewrite the right-hand side as

$$\ln(n+1) + \ln(n!) \le \ln(n+1) + n\ln(n).$$

Now, the logarithm function is a strictly monotonically increasing function, meaning ln(*a*) < ln(*b*) whenever *a* < *b*. Therefore, ln(*n*) < ln(*n* + 1) and as *n* > 0, *n* ln(*n*) < *n* ln(*n* + 1). Therefore:

$$\begin{aligned}\ln(n+1) + n\ln(n) &< \ln(n+1) + n\ln(n+1)\\ &= (n+1)\ln(n+1)\end{aligned}$$

Thus, it follows that $\ln(n!) \le n\ln(n)$ implies that $\ln((n+1)!) < (n+1)\ln(n+1)$, and as $a < b$ implies that $a \le b$, it follows that $\ln((n+1)!) \le (n+1)\ln(n+1)$; our desired result.

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1.4.2.9 Example 9

The harmonic numbers are $H_n = \sum_{k=1}^{n} \frac{1}{k}$. In calculus, you learned that this series is divergent; that is, $\sum_{k=1}^{\infty} \frac{1}{k} = \infty$ or $\lim_{n\to\infty} H_n = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k} = \infty$. Show that this is true by showing that $H_{2^n} \geq 1 + \frac{n}{2}$ when $n \geq 0$.

Starting with the base case when $n = 0$: $H_{2^0} = H_1 = \sum_{k=1}^{1} \frac{1}{k} = 1$ and $1 + \frac{0}{2} = 1$.

Next, we will assume that $H_{2^n} \geq 1 + \frac{n}{2}$ is true for all $n \geq 0$. We will want to show that $H_{2^{n+1}} \geq 1 + \frac{n+1}{2}$

In previous examples, we saw that each subsequent series included one additional term. In this case, we have something slightly different:

$$\sum_{k=1}^{2^{n+1}} \frac{1}{k} = \left(\sum_{k=1}^{2^n} \frac{1}{k} \right) + \left(\sum_{k=2^n+1}^{2^{n+1}} \frac{1}{k} \right).$$

Using our assumption, we can rewrite the right-hand side as

$$\left(\sum_{k=1}^{2^n} \frac{1}{k} \right) + \left(\sum_{k=2^n+1}^{2^{n+1}} \frac{1}{k} \right) \geq 1 + \frac{n}{2} + \left(\sum_{k=2^n+1}^{2^{n+1}} \frac{1}{k} \right).$$

The second sum is a little more interesting: if $a_k \geq b_k$, then it follows that $\sum_{k=1}^{n} a_k \geq \sum_{k=1}^{n} b_k$, and $\frac{1}{2^{n+1}} \geq \frac{1}{k}$ for all values of $k = 2^n + 1, \ldots, 2^n$. Therefore, the right-hand side may be rewritten as

$$1 + \frac{n}{2} + \left(\sum_{k=2^n+1}^{2^{n+1}} \frac{1}{k} \right) \geq 1 + \frac{n}{2} + \left(\sum_{k=2^n+1}^{2^{n+1}} \frac{1}{2^{n+1}} \right) = 1 + \frac{n}{2} + \frac{1}{2^{n+1}} \left(\sum_{k=2^n+1}^{2^{n+1}} 1 \right).$$

The second sum has 2^n terms, so we may now write this as

$$\begin{aligned} 1 + \frac{n}{2} + \frac{1}{2^{n+1}} \left(\sum_{k=2^n+1}^{2^{n+1}} 1 \right) &= 1 + \frac{n}{2} + \frac{1}{2^{n+1}} 2^n \\ &= 1 + \frac{n}{2} + \frac{1}{2} = 1 + \frac{n+1}{2} \end{aligned},$$

which is the desired result.

Note: this isn't that far off as an approximation. Letting $n = 1998$,

$$H_{2^{1998}} \approx 1385.485283 > 1000 = 1 + \frac{1998}{2}.$$

1.4.2.10 Example 10

Finally, we will conclude with an example that requires a little more algebra but is still interesting in that it is true:

$$\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k\right)^2$$

for all $n \geq 0$. Now, looking up in any table of summations, we could quickly find that:

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4} = \left(\frac{n(n+1)}{2}\right)^2 = \left(\sum_{k=1}^{n} k\right)^2,$$

but we will show this result here using a proof by induction. In this example, we will start with the right-hand side:

$$\begin{aligned}\left(\sum_{k=1}^{n+1} k\right)^2 &= \left((n+1) + \sum_{k=1}^{n} k\right)^2 \\ &= (n+1)^2 + 2(n+1)\left(\sum_{k=1}^{n} k\right) + \left(\sum_{k=1}^{n} k\right)^2\end{aligned}$$

Substituting our assumption, and using the previous result that $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$, by collecting similar terms, we have

$$\begin{aligned}(n+1)^2 + 2(n+1)\left(\sum_{k=1}^{n} k\right) + \left(\sum_{k=1}^{n} k\right)^2 &= (n+1)^2 + 2(n+1)\frac{n(n+1)}{2} + \sum_{k=1}^{n} k^3 \\ &= (n+1)^2 + n(n+1)^2 + \sum_{k=1}^{n} k^3 \\ &= (n+1)(n+1)^2 + \sum_{k=1}^{n} k^3 \\ &= (n+1)^3 + \sum_{k=1}^{n} k^3 \\ &= \sum_{k=1}^{n+1} k^3,\end{aligned}$$

which is the desired result.

1.4.3 Where Proofs by Induction Fail

It is common to simply show examples where a proof by induction works, but what can cause the proof to fail? There are three possible steps where the proof may fail, including

1. the inductive step failing,
2. the initial inductive step is false, and
3. the proof is invalid.

We will look at four examples.

1.4.3.1 Non-example 1

The definition of the Fibonacci sequence is

$$F(n) = \begin{cases} 1 & n = 0,1 \\ F(n-1) + F(n-2) & n \geq 2 \end{cases}$$

One might observe that $F(2) = 2$ and $F(3) = 3$, and ask "Is $F(n) = n$ for $n \geq 1$?"

We would continue by observing that F(1) = 1, by definition; however, if we try to prove the inductive step by assuming $F(n) = n$ for all $n \geq 1$, we would then carry on to find that:

$$\begin{aligned} F(n+1) &= F(n) + F(n-1) \\ &= n + n - 1 \\ &= 2n - 1, \end{aligned}$$

and $2n - 1 \neq n + 1$ for virtually any $n > 2$.

1.4.3.2 Non-example 2

Consider the recursive formula

$$F(n) = \begin{cases} 1 & n = 0 \\ \sum_{k=0}^{n-1} F(k) & n \geq 1 \end{cases}$$

This is not a closed form solution, but notice that if we take the difference of two values, we get:

$$\begin{array}{rr} & F(n) = \sum_{k=0}^{n-1} F(k) \\ + & F(n-1) = \sum_{k=0}^{n-2} F(k) \\ \hline & F(n) - F(n-1) = F(n-1) \end{array}$$

Consequently, F(n) = 2F(n – 1). This might suggest quickly that $F(n) = 2^n$. Indeed, it would appear the formula is correct: F(0) = 1 by definition, and F(n) = 2F(n – 1); consequently, if we use the assumption that $F(n) = 2^n$ and substitute that into calculating $F(n + 1) = 2F(n) = 2\cdot 2^n = 2^{n+1}$. This seems to be a *bona fide* proof.

However, what happens when we calculate $F(1)$? $F(1) = F(0) = 1 \neq 2^1$ which contradicts our proof. The issue is that the sum above requires that both $F(n)$ and $F(n-1)$ are defined in terms of the sum, but if $n = 1$, then $F(0) = 1$ by definition and not by a sum. Consequently, the inductive step is valid only for $n > 1$. The correct formula would be

$$F(n) = \begin{cases} 1 & n = 0 \\ 2^{n-1} & n \geq 1 \end{cases}.$$

1.4.3.3 Non-example 3

Consider the statement that "x is a horse of a different color". Use this to prove that all horses are the same color. This would say that all horses in a set of size n are the same color. We will attempt to prove this by induction.

First, if $n = 1$, then there is only one horse in the set, and it must consequently have the same color as itself.

Let us assume for $n \geq 1$ that all horses in a set of size n have the same color.

Now, suppose we have a set of $n + 1$ horses: $\{h_1, h_2, \ldots, h_n, h_{n+1}\}$. We could break this up into to two sets, $\{h_1, h_2, \ldots, h_n\}$ and $\{h_2, \ldots, h_n, h_{n+1}\}$. Each of these is a set of size n and thus all horses in each, by assumption, are the same color. As there is an overlap between the two sets, it follows that all the horses in both sets must be the same color, and thus, all the horses in $\{h_1, h_2, \ldots, h_n, h_{n+1}\}$ have the same color.

Similar to the previous non-example, the issue occurs right at the start: if $n = 1$. In this case, $n + 1 = 2$ and {Sea Horse, Barbaro} form a set of size 2, but the two subsets that are generated, {Sea Horse} and {Barbaro} have no overlap, so the argument does not follow.

1.4.3.4 Non-example 4

Finally, prove by induction that you cannot become full eating peas.

The proof goes as follows: Eating one pea will not make you full. If you are not full having eaten n peas, you will not be full by eating one additional pea. Consequently, by induction, you cannot become full by eating peas.

The issue in this case is that "full" is not a quantitative measure. One could use a similar argument to say that no one is tall: a person who is 130 cm in height is not tall, and if a person is n cm in height is not tall, then a person who is $n + 1$ cm in height is also not tall. Consequently, no one is tall. This is also a mistake in perception: because of the size of a pea, one cannot imagine that something that small makes an impact on the "fullness" of your stomach. Additionally, food moves out of the stomach: if you were literally to eat and swallow one pea at a time, it is likely at some point, some will be moving on to the small intestines.

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1.4.4 Justification

Mathematical systems are based on axioms that are assumed to be true and theorems are derived from those axioms. The same system can be described by different sets of axioms and consequently, one set of axioms becomes theorems in the other system and vice versa.

In some descriptions, the induction principle is simply made an axiom. In other cases, it can be assumed based on the existence of other axioms or theorems. For example, mathematical induction can be derived from the following:

1. The natural numbers ($\mathbf{N} = \{0, 1, 2, 3, \ldots\}$) are linearly ordered,
2. Every natural number is either 0 or the successor ($n + 1$) of some other natural number n, and
3. The successor is by definition of the order greater than it succeeds ($n + 1 > n$).

Alternatively, suppose that $F(n)$ is false even though we have shown a proof by induction. We cannot be dealing with the base case, because we were required to prove that step. Consequently, the inductive step must apply, but then, $F(n-1) \rightarrow F(n)$. Now, if $F(n)$ is false, so must $F(n-1)$. If you don't see that, recall the truth table for implication: $a \rightarrow b$ is false if and only if a is true while b is false, and so as we proved the inductive step, because b is false, a must also be false. Thus, $F(n-1)$ is false, and so is $F(n-2)$, etc. At some point, however, we must reach the base case, and this would imply the base case is false. This contradicts the fact we proved the base case is true.

1.4.5 Strong Induction

A related principle is that of strong induction. Here we replace the assumption that $F(n)$ is true with the assumption that $F(n_0)$, $F(n_0 + 1)$, $F(n_0 + 2)$, … $F(n)$ are true.

This can be used to prove, for example, that given 3- and 7-cent coins, it is possible to make change for any amount greater than or equal to 12 cents.

The principle of strong induction could be deduced from an axiomatic system that has the induction principle.

1.4.6 Exercises

The following are some exercises you can do for youerself.

1.4.6.1 For $n \geq 0$, show that $\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$.

1.4.6.2 For $n \geq 1$, show that $\sum_{k=1}^{n} 3k^2 - 3k + 1 = n^2$.

1.4.6.3 For $n \geq 1$, show that $\sum_{k=1}^{n} \frac{1}{\sqrt{k}} \geq \sqrt{n}$.

1.4.6.4 For $n \geq 1$, show that 4 divides $3^{2n-1} + 1$.

1.4.6.5 For $n \geq 1$, show that $\left|\sum_{k=1}^{n} x_k\right| \leq \sum_{k=1}^{n} |x_k|$.

1.4.6.6 For $n \geq 2$, show that $n^2 \geq 2n$.

1.4.6.7 Given an $n \times 2$ grid, how many ways can that grid be covered with dominos (considering orientation only)? For example, consider Figure 1.

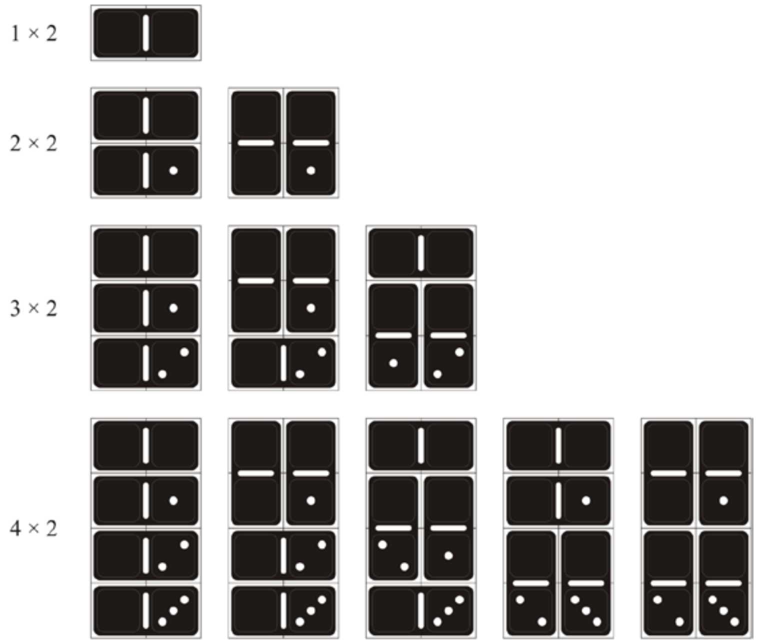

Figure 1. The different coverings for 1 × 2, 2 × 2, 3 × 2, and 4 × 2 grids.

Find a formula for d_n and prove your formula is true using induction.

1.4.6.8 Let $F(n)$ represent the n^{th} Fibonacci number where $F(0) = F(1) = 1$. Two numbers are said to be relatively prime if they share no common divisor. Prove that $F(n)$ and $F(n + 1)$ are always relatively prime.

Note: you may need the lemma that if m and n are relatively prime, then $m + n$ and $m - n$ are also relatively prime.

1.4.6.9 For $n \geq 1$, how that 133 divides $11^{n+2} + 12^{2n+1}$.

1.4.6.10 Show that every third Fibonacci number is even.

1.4.6.11 For $n \geq 1$, show that $x^n - y^n$ is always divisible by $x - y$.

1.4.6.12 For all n, show that $n^2 \geq 3n - 2$.

1.4.6.13 Come up with a recursive formula that demonstrates that lies will divide the plane into $\frac{n^2 + n + 2}{2}$ regions and show that the formula is correct using recursion. See Figure 2

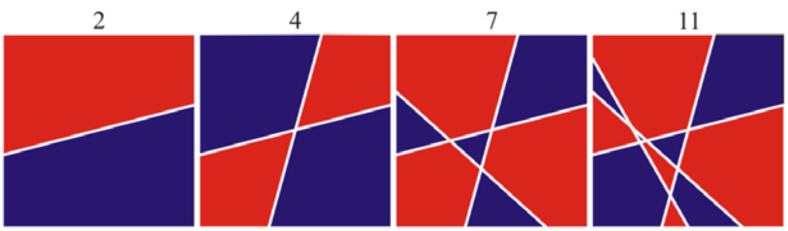

Figure 2. Dividing the plane using regions.

1.4.6.14 Demonstrate that any grid of size $2^n \times 2^n$ can, with one square deleted, be tiled with triominos. See Figure 3.

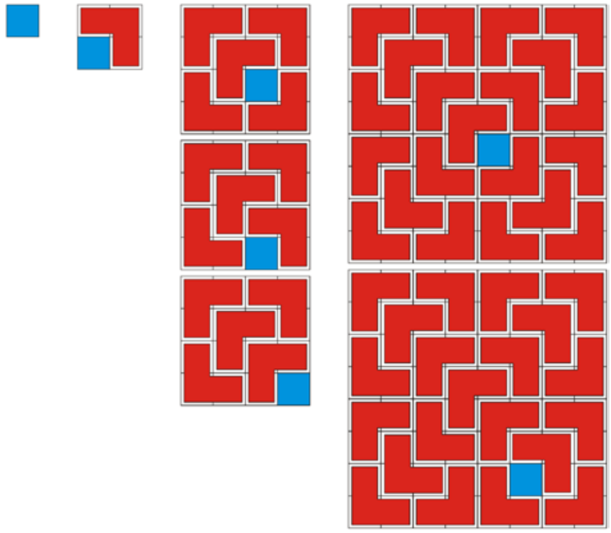

Figure 3. Tiling a square with triominos.