Approximating solutions to initial-value problems

Introduction

• In this topic, we will
  – Review initial-value problems (IVPs)
  – Discuss the differences in approaches to finding or approximating solutions to IVPs
  – Introduce cubic splines
  – Describe the upcoming lectures
Initial-value problems

- An initial-value problem (IVP) is can be:
  - The first derivative described in terms of the independent variable and the function
    \[
    y'(t) = f(t, y(t)) \quad \quad y'(t) = t y(t) + t - 1
    \]
    \[
    y(t_0) = y_0 \quad \quad y(0) = 1
    \]
  - The \(n^{th}\) derivative described in terms of the independent variable, lower derivatives and the function
    \[
    y^{(n)}(t) = f(t, y(t), y^{(1)}(t), \ldots, y^{(n-1)}(t))
    \]
    \[
    y(t_0) = y_0 \quad \quad y^{(1)}(t_0) = y^{(1)}_0 \quad \quad y^{(1)}(0) = 2 \quad \quad y^{(2)}(0) = 3 \quad \quad y^{(3)}(0) = 4
    \]

- A system of coupled IVPs, for example
  \[
  y_1'(t) = 0.02 y_1(t) - 0.1 y_1(t) y_2(t)
  \]
  \[
  y_2'(t) = -0.04 y_2(t) + 0.02 y_1(t) y_2(t)
  \]
  \[
  y_1(0) = 5233 \quad \quad y_2(0) = 323
  \]
Solutions to IVPs

- Recall your approach in calculus:
  \[ y^{(1)}(t) = -y(t) - 1 \]
  \[ y(0) = 1 \]
- In calculus, you find a single exact solution:
  \[ y(t) = 2e^{-t} - 1 \]

- What if you cannot find an exact solution?

Approximating solutions to initial value problems

Approximate solutions to IVPs

- What do we have?
  \[ y^{(1)}(t) = -ty(t) - 1 \]
  \[ y(0) = 1 \]
  - At time \( t = 0 \), the value is 1
  - The first equation says:
    - If \( t = 0 \) and \( y(0) = 1 \), then \( y^{(1)}(0) = -0 \cdot 1 - 1 = -1 \)
  - Taylor series now say that:
    \[ y(0 + h) \approx y(0) + y^{(1)}(0)h \]
    \[ = 1 + (-1)h \]
    - Thus, \( y(0.1) \approx 0.9 \)
    - If \( t = 0.1 \) and \( y(0.1) = 0.9 \), then \( y^{(1)}(0) = -0.1 \cdot 0.9 - 1 = -1.09 \)
    - Thus \( y(0.2) = y(0.1) + y^{(1)}(0.1) = 0.9 + (-1.09)0.1 = 0.791 \)
Approximating solutions to IVPs

- In this course, we will approximate the solution at specific points:
  \[(t_0, y(t_0)), (t_1, y_1), (t_2, y_2), (t_3, y_3), \ldots\]
- Thus, \(y(t_k) \approx y_k\)

This is the initial condition

Approximating at intermediate values of \(t\)

- Suppose we want to approximate the solution at some point 
  \(t_{k-1} < t < t_k\)
  - Do we find the interpolating linear polynomial between 
    \((t_{k-1}, y_{k-1})\) and \((t_k, y_k)\)?
  - Do we find the interpolating cubic polynomial between 
    \((t_{k-2}, y_{k-2}), (t_{k-1}, y_{k-1}), (t_k, y_k)\) and \((t_{k+1}, y_{k+1})\)?
Interpolating cubic polynomials

• Let’s implement this function
  – We assume \( t_k - t_{k-1} = h \)

\[
\begin{bmatrix}
-1 & 1 & -1 & 1 \\
0 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 \\
8 & 4 & 2 & 1
\end{bmatrix}
\begin{bmatrix}
a_1 \\
a_2 \\
a_3 \\
a_4
\end{bmatrix}
= 
\begin{bmatrix}
y_{k-2} \\
y_{k-1} \\
y_k \\
y_{k+1}
\end{bmatrix}
\]

double ivp_interp_4pt( double t, 
  double ts[4], 
  double ys[4] ) {
  double delta{ (t - ts[1])/(ts[2] - ts[1]) };
  assert( (0.0 <= delta) && (delta <= 1.0) );
  return (
    )*delta + ((ys[0] + ys[2])/2.0 - ys[1])
    )*delta + (-ys[3]/6.0 + ys[2] - ys[1]/2.0 - ys[0]/3.0)
    )*delta + ys[1];
}

Splines

• Recall that \( y^{(1)}(t) = f'(t, y(t)) \), so
  \( y^{(1)}(t_{k-1}) = f'(t_{k-1}, y_{k-1}) \) and \( y^{(1)}(t_k) = f'(t_k, y_k) \)
  – Can we find a cubic polynomial \( p \) that satisfies:

\[
\begin{align*}
p(t_{k-1}) &= y_{k-1} \\
p^{(1)}(t_{k-1}) &= f(t_{k-1}, y_{k-1}) \\
p(t_k) &= y_k \\
p^{(1)}(t_k) &= f(t_k, y_k)
\end{align*}
\]

\[
\begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 \\
3 & 2 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
a_1 \\
a_2 \\
a_3 \\
a_4
\end{bmatrix}
= 
\begin{bmatrix}
y_{k-1} \\
y_{k-1} \\
y_k \\
y_{k+1}
\end{bmatrix}
\]

\( \frac{\partial \mathbf{a}}{\partial t} = \mathbf{h}(t_k, y_k) \)
Splines

• This is now fun:

\[
\begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 \\
3 & 2 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
a_3 \\
a_2 \\
a_1 \\
a_0
\end{pmatrix} =
\begin{pmatrix}
y_{k-1} \\
hf(t_{k-1}, y_{k-1}) \\
y_k \\
hf(t_k, y_k)
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
a_3 \\
a_2 \\
a_1 \\
a_0
\end{pmatrix} =
\begin{pmatrix}
y_{k-1} \\
hf(t_{k-1}, y_{k-1}) \\
y_k - y_{k-1} - hf(t_{k-1}, y_{k-1}) \\
h(f(t_k, y_k) + f(t_{k-1}, y_{k-1})) + 2(y_{k-1} - y_k)
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
a_3 \\
a_2 \\
a_1 \\
a_0
\end{pmatrix} =
\begin{pmatrix}
y_{k-1} \\
hf(t_{k-1}, y_{k-1}) \\
3(y_k - y_{k-1}) - h(2f(t_{k-1}, y_{k-1}) + f(t_k, y_k)) \\
h(f(t_k, y_k) + f(t_{k-1}, y_{k-1})) + 2(y_{k-1} - y_k)
\end{pmatrix}
\]

Let’s implement this function:

```cpp
double ivp_spline_2pt( double t, double ts[2], double ys[2], double dys[2] ) { 
    double h{ ts[1] - ts[0] };  
    double delta{ (t - ts[0])/h };  
    assert( (0.0 <= delta) && (delta <= 1.0) );  
    
    return (  
        (  
            h*(dys[0] + dys[1]) + 2.0*(ys[0] - ys[1])  
        )*delta  
        + h*(2.0*dys[0] + dys[1]) + 3.0*(ys[0] - ys[1])  
    )*delta + *dys[0]  
    +*delta + ys[0];
}
```
Approximating at intermediate values of $t$

- Which is better?
  - We will take an IVP to which we know the solution and find:
    1. The linear polynomial interpolating $(0.20, y(0.20)), (0.25, y(0.25))$
    2. The cubic polynomial interpolating $(0.15, y(0.15)), (0.20, y(0.20)), (0.25, y(0.25)), (0.30, y(0.30))$  
    3. The cubic spline $(0.20, y(0.20)), (0.25, y(0.25))$
  - We will then evaluate the actual solution and these approximations at the point $t = 0.2353243$

Approximating solutions to initial-value problems

- First, let's start the 1st-order IVP:
  
  \[ y^{(1)}(t) = -y(t) \]
  \[ y(0) = 1 \]
  \[ y(t) = e^{-t} \]

  - Here, $y(0.2353243) = 0.7903145090700692$
    
    Linear interpolating polynomial:
    
    \[
    \begin{align*}
    y(t) & = 0.7905207882879153 \\
    & \quad - 0.0002062 \cdot (t - 0.20) \\
    \end{align*}
    \]

    Cubic interpolating polynomial:
    
    \[
    \begin{align*}
    y(t) & = 0.7903144140636057 \\
    & \quad + 0.0000009501 \cdot (t - 0.20)^3 \\
    \end{align*}
    \]

    Cubic spline:
    
    \[
    \begin{align*}
    y(t) & = 0.7903144140636057 \\
    & \quad + 0.00000008924 \cdot (t - 0.20)^3 \\
    \end{align*}
    \]
Approximating at intermediate values of $t$

- Next, let’s consider:
  \[ y^{(i)}(t) = \left(t - y(t) + 1\right) \left(y(t) - 1\right) \]
  \[ y(0) = 1 \]
  - Here, $y(0.2353243) = 1.022125607413852$
    - Linear interpolating polynomial: $1.022252377336976$
    - Cubic interpolating polynomial: $1.022125194141359$
    - Cubic spline: $1.022125568692043$

Also, we can do this with any continuous and differentiable function:
- Given the sine function, here we see the error of:
  - A cubic polynomial interpolating the values 0.2, 0.4, 0.6, 0.8
  - A cubic spline matching the values and derivatives at 0.4 and 0.6
  - The error of the spline is smaller by a factor of 10
Our approach

- We will begin by approximating the solution to a 1st-order IVP
  - The techniques used here will trivially generalize to allow us to:
    - A system of $n$ coupled 1st-order IVPs
      \[\frac{v^{(0)}}{RC} - \frac{i^{(0)}}{C} = \frac{v(t)}{RC} - \frac{i^{(0)}}{C} \]
      \[i^{(0)}(t) = \frac{v(t)}{L} \]
    - An $n$th-order IVP
      \[\theta^{(2)}(t) = -\frac{g}{L} \sin(\theta(t)) \]
      \[i^{(2)}(t) + \frac{R}{L} i^{(1)}(t) + \frac{1}{CL} i(t) = \frac{1}{L} v^{(0)}(t) \]
  - A system of higher-order coupled IVPs

Looking ahead

- To approximate a solution to a 1st-order IVP, we will look at:
  - Euler’s method
  - Heun’s method
  - 4th-order Runge Kutta
  - Adaptive Euler-Heun
  - Dormand-Prince method
  - Stiff ODEs and backward Euler
- We will then generalize these algorithms to approximate the solution to a system of 1st-order coupled IVPs
- We will use such an approach to approximate the solution to an $n$th-order IVP
- We will then see it is trivial to approximate the solution to a system of higher-order IVPs
Summary

- Following this topic, you now
  - Understand the various types of initial-value problems
  - Are aware of the approach we will use
  - Know about splines as opposed to interpolating polynomials
  - Are aware that we will approximate solutions to:
    - 1st-order IVPs
    - Systems of 1st-order IVPs
    - Higher-order IVPs
    - Systems of higher-order IVPs

References

Acknowledgments

None so far.

Colophon

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