Hooke-Jeeves method for finding extrema in n dimensions

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Introduction

• In this topic, we will
  – Describe the idea of exploring surrounding points
  – Use this exploration to indicate a direction to move
  – Describe the Hooke-Jeeves method
  – Discuss using this exploratory move/pattern move approach for solving problems in general
  – Look at an implementation
  – Look at an example
Definitions

• Recall that in $\mathbb{R}^n$, the canonical basis is represented by the vectors $e_1, \ldots, e_n$
  – For $e_k$, all entries are zero except for the $k^{th}$ entry which is one
  – For example, in $\mathbb{R}^4$, the canonical basis is

$$
\begin{align*}
e_1 &= \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, & e_2 &= \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, & e_3 &= \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, & e_4 &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}
\end{align*}
$$

The Hooke-Jeeves method

• Given a real-valued function of an $n$-dimensional vector variable, we will start with an initial guess $u_0$ and an initial step size $h$
  – The idea is we will make a local search to find a direction of greatest decrease, and then continue in that direction as far as possible
The Hooke-Jeeves method

The algorithm is as follows:

- Given an approximation $u_k$ and a current step size $h$
  - Let $\Delta u_k \leftarrow 0$
  - For $j$ going from 1 to $n$,
    - Evaluate $f_{-1} \leftarrow f(u_k + \Delta u_k - he_j)$
    - Evaluate $f_0 \leftarrow f(u_k + \Delta u_k)$
    - Evaluate $f_1 \leftarrow f(u_k + \Delta u_k + he_j)$
    - If $f_{-1} < f_0, f_1$, set $\Delta u_k \leftarrow \Delta u_k - he_j$
    - If $f_1 < f_0, f_{-1}$, set $\Delta u_k \leftarrow \Delta u_k + he_j$
  - If $\Delta u_k = 0$, if $h$ is sufficient small, we are finished, otherwise, reduce $h$ and return to the first step
  - Otherwise, evaluate $f(u_k + m\Delta u_k)$ for successively larger integer values of $m$ until $f(u_k + m\Delta u_k) < f(u_k + (m + 1)\Delta u_k)$
    - Set $u_{k+1} \leftarrow u_k + m\Delta u_k$ and return to the first step

To summarize the strategy:

- Explore the points around $u_k$ and find the $\Delta u_k$ that offers the best move towards the minimum
  - These are called exploratory moves
- If no better point was found, either we are finished, or we try again in a smaller neighborhood
- If a better point is found, continue moving in the direction indicated by this $\Delta u_k$ until we find a minimum in that direction
  - These are called pattern moves
Problem-solving techniques

- This problem-solving strategy can be used for other searches:
  - In a process of exploration, determine a local improvement
    - If an improvement is found, use this pattern to move towards a better solution
    - If no improvement is found,
      - Either declare the current approximation to be acceptable,
      - Or try again with different searching criteria

```
std::pair<vector, double>
hooke_jeeves( double f( vector u ), vector u,
              double h,
              double eps_step, double eps_abs,
              unsigned int max_iterations ) { 
    unsigned int dim{ u.dim() };
    double min{ f( u ) };

    for ( unsigned k{0}; k < max_iterations; ++k ) { 
        // Exploratory moves
        // Check conditions
        // Pattern moves
    }

    return std::make_pair( vector{ dim, 0.0 }, NAN );
}
```
Implementation

// Exploratory moves
vector u0( u );
double min0{ min };
vector du{ vector{ dim, 0.0 } };  // The n-dimensional zero vector

for ( unsigned int j{0}; j < dim; ++j ) {
    du( j ) = -h;
    double fn{ f( u + du ) };
    du( j ) = h;
    double fp{ f( u + du ) };

    if ( (fp < fn) && (fp < min) ) {
        min = fp;
    } else if ( fn < min ) {
        du( j ) = -h;
        min = fn;
    } else {
        du( j ) = 0.0;
    }
}

u += du;

Implementation

// Check conditions
if ( norm( du ) == 0.0 ) {
    if ( h < eps_step ) {
        return std::make_pair( u, min );
    } else {
        h /= 2.0;
        continue;
    }
}

Implementation

// Pattern moves
// - We stored the initial values in u0 and min0

while ( k < max_iterations ) {
    double fm{ f( u + du )};
    ++k;
    if ( fm < min ) {
        u += du;
        min = fm;
    } else {
        break;
    }
}

if ( (k < max_iterations) && (norm( u - u0 ) < eps_step) && ((min0 - min) < eps_abs) ) {
    return std::make_pair( u, min );
}

Example

• The example on the Wikipedia page is most appropriate
  – Created by Guillaume Jacquenot
Summary

• Following this topic, you now
  – Understand the Hooke-Jeeves method for finding a minimum
  – Are aware of this exploratory/pattern approach to solving problems
  – Have seen an implementation
  – Have seen an example

References

Acknowledgments

None so far.

Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using Consolas.

Examples may be formulated and checked using Maple by Maplesoft, Inc.

The photographs of flowers and a monarch butterfly appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see https://www.rbg.ca/ for more information.
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