Complex numbers for numerical analysis

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April 2, 2025

Not unexpectedly, a number of students forget most of what they learned with respect to complex numbers following a period of time between learning the material and using that material. This is a review of the concepts you should have seen about complex numbers that you will require in this course on numerical analysis.

Please note, despite being given these notes, topics that use complex numbers and arithmetic will never-the-less be summarized and described in class. Thus, while you should read these now, it is not expected that you have mastered all this material for the course lectures.

We define $j = \sqrt{-1}$, so $j^2 = -1$. Additionally, we note that $j^3 = -j$ and $j^4 = 1$, so $j^5 = j^{1001} = j$.

Given a variable x, we can define a linear polynomial $4 + 5x$. Similarly, given $\sqrt{2}$, we have the real number $4 + 5\sqrt{2}$. In the same way, we can define a complex number $4 + 5j$.

If $z = 4 + 5j$, then the "real component" of z is 4, and the "imaginary component" is the real coefficient 5. These are written as $\Re e(z)$ and $\Im m(z)$, respectively. The imaginary component is a real number.

Two complex numbers w and z are equal, written $w = z$, if and only if both the real and imaginary components are equal.

The zero linear polynomial is $0 + 0x$. The value $0 + 0\sqrt{2} = 0$. The "zero" for a complex number is $0 + 0j$.

All complex numbers with a real component greater than zero are said to form the "open right-hand complex plane", while all complex numbers with a real component less than zero are said to form the "open left-hand complex plane." All complex numbers with an imaginary part equal to zero are said to be "real" and all complex numbers with a real part equal to zero are said to be "imaginary."

We will look at complex arithmetic and then the roots of polynomials.

1 Complex arithmetic

We will review complex addition, complex multiplication, division by a real number, the complex conjugate, complex division, and some properties of complex numbers.

1.1 Complex addition

Given a linear polynomial $7 - 4x$, we can add to this $5 + 3x$ to get the polynomial $12 - x$. Similarly, given two real numbers $7 - 4\sqrt{2}$ and $5 + 3\sqrt{2}$, we can add them to get $12 - \sqrt{2}$. Finally, given two complex numbers $7 - 4j$ and $5 + 3j$, adding these two we get $12 - j$.

Thus, $(\alpha + \beta j) + (\gamma + \delta j) = (\alpha + \gamma) + (\beta + \delta)j$. For example, $(3 + 4j) + (-2 + 7j) = 1 + 11j$.

Note that $(\alpha + \beta j) + (0 + 0j) = \alpha + \beta j$, and $w + z = w$ if and only if $z = 0 + 0j$. Also note that $w + z = z + w$, and that if you have the sum of n complex numbers, like the real numbers, it doesn't matter what order you add them in.

As you may suspect, subtraction is defined similarly: $(\alpha + \beta j) - (\gamma + \delta j) = (\alpha - \gamma) + (\beta - \delta)j$. For example, $(3 + 4j) - (-2 + 7j) = 5 - 3j$.

1.2 Complex multiplication

Given the linear polynomials $-2 + 5x$ and $4 + 7x$, we can multiply these two using FOIL (first, outside, inside and last) to get the

$$(-2 + 5x)(4 + 7x) = -8 - 14x + 20x + 35x^2 = -8 + 6x + 35x^2.$$

Similarly, you can multiply $-2 + 5\sqrt{2}$ and $4 + 7\sqrt{2}$ to get

$$(-2 + 5\sqrt{2})(4 + 7\sqrt{2}) = -8 - 14\sqrt{2} + 20\sqrt{2} + 35\sqrt{2}^2 = -8 + 6\sqrt{2} + 35 \cdot 2 = 62 + 6\sqrt{2}.$$

Finally, given the two complex numbers $-2+5j$ and $4+7j$, we can multiply these two to get

$$(-2+5j)(4+7j) = -8 - 14j + 20j + 35j^2 = -8 + 6j + 35j^2,$$

but $j^2 = -1$, so this equals $-8 + 6j - 35 = -43 + 6j$.

Thus,

$$(\alpha + \beta j)(\gamma + \delta j) = (\alpha\gamma - \beta\delta) + (\alpha\delta + \beta\gamma)j.$$

For example, $(3+4j)(-2+7j) = -6 + 21j - 8j + 28j^2 = -34 + 13j$.

Note that $(\alpha + \beta j)(0 + 0j) = 0 + 0j$ and that $(\alpha + \beta j)(1 + 0j) = \alpha + \beta j$. Also, $wz = 0$ if and only if $w = 0$, $z = 0$ or both are zero. Similarly $wz = w$ if and only if either $w = 0 + 0j$, $z = 1 + 0j$, or both. Also note that $wz = zw$, and that if you have the product of n complex numbers, like the real numbers, it doesn't matter what order you multiply them in.

1.3 Dividing a complex number by a real number

Given a polynomial $-5 - 9x$, we can divide this by a real number by simply dividing both coefficients, so $\frac{-5-9x}{-2.5} = 2 + 3.6x$. Similarly, we can divide the radical $-5 - 9\sqrt{2}$ by that same real number to get $\frac{-5-9\sqrt{2}}{-2.5} = 2 + 3.6\sqrt{2}$. We can do the same for a complex number $-5 - 9j$ and divide it by -2.5 to get $\frac{-5-9j}{-2.5} = 2 + 3.6j$. Clearly, we cannot divide by zero.

Thus, for $\gamma \neq 0$, $\frac{\alpha+\beta j}{\gamma} = \frac{\alpha}{\gamma} + \frac{\beta}{\gamma} j$. For example, $\frac{8-15j}{-6} = -\frac{4}{3} + \frac{5}{2} j$.

1.4 Complex division

Given two polynomials $5 + 7x$ and $-4 + 3x$, we generally cannot simplify the ratio $\frac{5+7x}{-4+3x}$ unless there is a common factor; however, given the radicals $5 + 7\sqrt{2}$ and $-4 + 3\sqrt{2}$, we can simplify this by multiplying by the "radical conjugate" of the denominator over itself:

$$\frac{5+7\sqrt{2}}{-4+3\sqrt{2}} = \frac{5+7\sqrt{2}}{-4+3\sqrt{2}} \cdot 1 = \frac{5+7\sqrt{2}}{-4+3\sqrt{2}} \cdot \frac{-4-3\sqrt{2}}{-4-3\sqrt{2}}$$

You may note that the numerator expands to $-62 - 43\sqrt{2}$ and the denominator simplifies to -2, so we can use the previous property to simply the expression to:

$$\frac{5+7\sqrt{2}}{-4+3\sqrt{2}} = (-62 - 43\sqrt{2}) \cdot \frac{1}{-2} = 31 + 21.5\sqrt{2}.$$

If $\alpha + \beta\sqrt{2}$ is such that α and β are rational numbers, then $(\alpha + \beta\sqrt{2})(\alpha - \beta\sqrt{2})$ will be a rational number $\alpha^2 - 2\beta^2$.

Definition: The "complex conjugate" of the complex number $z = \alpha + \beta j$, denoted by z^*, is the complex number $z^* = \alpha - \beta j$. Thus, the complex number of $-4 + 3j$ is $(-4 + 3j)^* = -4 - 3j$. One important property of the complex conjugate is that a complex number multiplied by its complex conjugate must be a non-negative real number:

$$zz^* = (\alpha + \beta j)(\alpha - \beta j) = \alpha^2 + \beta^2,$$

and $zz^* = 0$ if and only if $z = 0 + 0j$.

Like we did with radicals, we can do the same for complex numbers: given the two complex numbers $5 + 7j$ divided by $-4 + 3j$, by multiplying by the complex conjugate of the denominator over itself:

$$\frac{5+7j}{-4+3j} = \frac{5+7j}{-4+3j} \cdot 1 = \frac{5+7j}{-4+3j} \cdot \frac{-4-3j}{-4-3j}$$

You may note that the numerator expands to $1 - 43\sqrt{2}$ and the denominator simplifies to 25, so complex division is now reduced to complex multiplication and the division of a complex number by a positive real number, in this case, $(-4 + 3j)(-4 - 3j) = 4^2 + 3^2 = 25$:

$$\frac{5+7j}{-4+3j} = (1 - 43j) \cdot \frac{1}{25} = 0.04 - 1.72j.$$

You will observe that if $z = \alpha + \beta j$, then $zz^* = \alpha^2 + \beta^2$.

Thus,

$$\frac{\alpha+\beta j}{\gamma+\delta j} = \frac{(\alpha+\beta j)(\gamma-\delta j)}{\gamma^2+\delta^2} = \frac{\alpha\gamma+\beta\delta}{\gamma^2+\delta^2} + \frac{\beta\gamma-\alpha\delta}{\gamma^2+\delta^2} j.$$

However, do not memorize this formula: just remember that $\frac{w}{z} = \frac{wz^*}{zz^*}$ and then work out the details.

Like real numbers, you can cancel terms out, so if w and z are complex numbers, then $\frac{w^3 z^2}{wz^4} = \frac{w^2}{z^2} = \left(\frac{w}{z}\right)^2$. Similarly, like the real numbers, the only number you cannot divide by is $0 + 0j$: for every other complex number w, the ratio $\frac{z}{w}$ is well defined.

1.5 The inverse or reciprocal of a complex number

As a consequence, the reciprocal of z is

$$z^{-1} = \frac{1}{z} = \frac{z^*}{zz^*},$$

so for example, the reciprocal of $3+4j$ is $\frac{3-4j}{3^2+4^2} = \frac{3-4j}{25} = 0.12 - 0.16j$. You can multiply these together to get that $(3+4j)(0.12-0.16j) = 0.36 - 0.48j + 0.48j - 0.64j^2 = 1 + 0j$.

1.6 The length or absolute value of a complex number

The "length" or "absolute value" of a complex number $z = \alpha + \beta j$ is the distance to the origin if the point (α, β) was plotted on the Cartesian plane, or in other words, $|z| = \sqrt{\alpha^2 + \beta^2}$. For example, if $z = 4 - 3j$, then $|z| = \sqrt{4^2 + (-3)^2} = \sqrt{25} = 5$ and if $w = -3 + 1.25j$, then $|w| = \sqrt{(-3)^2 + 1.25^2} = \sqrt{10.5625} = 3.25$. You will notice that $|z| \geq 0$ and $|z| = 0$ if and only if $z = 0 + 0j$.

Note that $|z|^2 = zz^*$ as $zz^* = \alpha^2 + \beta^2 = |z|^2$ and. Also, $\frac{w}{z} = \frac{wz^*}{|z|^2}$ and $\frac{1}{z} = \frac{z^*}{|z|^2}$.

Note that $|wz| = |w||z|$, just like the real numbers, and that $|z^{-1}| = \frac{1}{|z|}$ for a non-zero complex number z. The second is interesting, because on the left-hand side, you are calculating the complex reciprocal first, and only then calculating the absolute value. In the second, you calculate the absolute value of z first and then find the reciprocal of the result.

All complex numbers that have $|z| = 1$ form a circle in the complex plane, and this is called the "unit circle." All complex numbers such that $|z| \leq 1$ form what is called the "unit disk". All complex numbers such that $|z| < 1$ form what is called the "open unit disk," meaning that it does not include the boundary.

Just like two real numbers x and y are "close" to each other if $|x - y| < \epsilon$, two complex numbers w and z may be said to be "close" if $|w - z| < \epsilon$ where $|w - z|$ is the absolute value of the difference between the two complex numbers.

1.7 Properties of the complex conjugate

If w and z are complex numbers, then $z^{**} = z$, $(w+z)^* = w^* + z^*$ and $(wz)^* = w^*z^*$. Also, $z = z^*$ if and only if z is real, $z = -z^*$ if and only if z is imaginary, and $z^* = z^{-1}$ if and only if z lies on the unit circle.

1.8 Properties of complex numbers

Note that the complex numbers $0+0j$ and $1+0j$ have all the same properties as 0 and 1 do for real numbers:

1. $(0+0j)(\alpha+\beta j) = 0+0j$ for all complex numbers $z = \alpha + \beta j$.
2. $(1+0j)(\alpha+\beta j) = \alpha+\beta j$ for all complex numbers $z = \alpha + \beta j$.
3. For every complex number $z = \alpha + \beta j$, there is an "additive inverse" $-z = -\alpha - \beta j$ such that $z + (-z) = 0 + 0j$. The additive inverse may be found by multiplying a complex number by $-1 + 0j$.
4. For every non-zero complex numbers $z = \alpha + \beta j$, there is a "multiplicative inverse" $z^{-1} = \frac{\alpha - \beta j}{\alpha^2 + \beta^2}$ such that $zz^{-1} = 1 + 0j$.

Because of this, we often just write $0 + 0j$ as 0 and $1 + 0j$ as 1. In fact, we usually write $\alpha + 0j$ as α.

2 The square root and the quadratic formula

You are aware that for two non-negative numbers x and y, $\sqrt{xy} = \sqrt{x}\sqrt{y}$. Because we are now discussing $\sqrt{-1}$, we must emphasize that this is not true in general: For example, $1 = \sqrt{1} = \sqrt{(-1)(-1)}$, but if we claim that $\sqrt{(-1)(-1)} = \sqrt{-1}\sqrt{-1}$, then $1 = \sqrt{-1}\sqrt{-1} = j^2 = -1$, which is false. There is, however, one true statement: if z is any complex number and $x \geq 0$, then $\sqrt{xz} = \sqrt{x}\sqrt{z}$.

We can use this in finding the roots of the quadratic polynomial $x^2 + 8x + 41$:

$$\begin{aligned} \frac{-8 \pm \sqrt{8^2 - 4 \cdot 1 \cdot 41}}{2 \cdot 1} &= \frac{-8 \pm \sqrt{64 - 164}}{2} = \frac{-8 \pm \sqrt{-100}}{2} \\ &= \frac{-8 \pm \sqrt{100}\sqrt{-1}}{2} = \frac{-8 \pm 10j}{2} = -\frac{8}{2} \pm \frac{10j}{2} = -4 \pm 5j. \end{aligned}$$

3 Geometric series

You may recall that $1 + \gamma + \gamma^2 + \gamma^3 + \cdots = \sum_{k=0}^{\infty} \gamma^k = \frac{1}{1-\gamma}$ for real numbers so long as $|\gamma| < 1$. For example, $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2$. The same is true for complex numbers: If you calculate $1 + (0.4 + 0.2j) + (0.4 + 0.2j)^2 + \cdots$, you will find this converges to

$$\frac{1}{1-(0.4+0.2j)} = \frac{1}{0.6-0.2j} = \frac{0.6+0.2j}{0.36+0.04} = \frac{0.6+0.2j}{0.4} = 1.5 + 0.5j.$$

You can try this out, as

$$\sum_{k=0}^{15} (0.4+0.2j)^k = 1.499999540084736 + 0.499995978498048j,$$

exactly and is clearly very close to $1.5 + 0.5j$.

4 Roots of a polynomial and the fundamental theorem of algebra

Recall that you learned polynomial division in secondary school. Thus, if you divided a non-constant polynomial $p(x)$ of degree n by $x - x_0$, this would result in a quotient polynomial $q(x)$ of degree $n-1$ plus the remainder equal to $p(x_0)$ so that $p(x) = (x - x_0)q(x) + p(x_0)$. For example,

$$x^5 - 3x^3 + 2x + 1 = (x+2)(x^4 - 2x^3 + x^2 - 2x + 6) - 11,$$

where $(-2)^5 - 3(-2)^3 + 2(-2) + 1 = -32 + 24 - 4 + 1 = -11$.

For example, if $p(x) = x^4 + 3x^3 - 9x^2 - 23x - 12$, then dividing this by $x - 2$ yields $q(x) = x^3 + 5x^2 + x - 21$ with a remainder of -54, and therefore $p(x) = q(x)(x-2) - 54$. If you expand $(x-2)(x^3+5x^2+x-21)-54$, you will indeed get back the original polynomial $p(x)$:

$$\begin{aligned}(x-2)(x^3+5x^2+x-21)-54 &= (x^4+5x^3+x^2-21x) - (2x^3+10x^2+2x-42) - 54\\ &= (x^4+3x^3-9x^2-23x+42) - 54\\ &= x^4+3x^3-9x^2-23x-12\end{aligned}$$

Therefore, a given real or complex number r and a polynomial $p(x)$, r is a root of the polynomial if and only if dividing the polynomial by $x - r$ has a remainder of zero. For example, given the quartic polynomial $p(x)$ defined above, we observe that $p(x) = (x-3)(x^3+6x^2+9x+4)+0$, $p(x) = (x+4)(x^3-x^2-5x-3)+0$ and $p(x) = (x+1)(x^3+2x^2-11x-12)+0$, and therefore 3, -4 and -1 are all roots of the polynomial $p(x)$.

A polynomial $p(x)$ has a "multiple root" at r if the quotient polynomial $q(x)$ also has a root at r:

1. A polynomial has a root at r of multiplicity equal to 1 if $p(r) = 0$, so $p(x) = (x-r)q(x) + 0$ but $q(r) \neq 0$.
2. A polynomial has a root at r of multiplicity m if $p(x) = (x-r)q(x)+0$ (so $p(r) = 0$) and $q(x)$ has a root at r of multiplicity $m-1$.

For example, continuing with the example above:

- $p(x) = (x-3)(x^3+6x^2+9x+4)+0$ but $3^3 + 6\cdot 3^2 + 9\cdot 3 + 4 = 112$, so $p(x)$ has a root of multiplicity one at 3.
- $p(x) = (x+4)(x^3-x^2-5x-3)+0$ but $(-4)^3 - (-4)^2 - 5(-4) - 3 = -63$, so $p(x)$ has another root of multiplicity one at -4.
- $p(x) = (x+1)(x^3+2x^2-11x-12)+0$ and $x^3+2x^2-11x-12 = (x+1)(x^2+x-12)+0$ but $(-1)^2 + (-1) - 12 = -12$, so $p(x)$ has a root of multiplicity two at -1.

Theorem 1. If $p(x)$ is a polynomial with real coefficients of degree n, then $p(x)$ has at most n real roots if you count multiplicity.

For example, consider these cubic polynomials:

- The polynomial $x^3 - 12x$ has three roots of multiplicity 1 at $x = 0$ and $x = \pm 2\sqrt{3}$.
- The polynomial $x^3 - 12x + 16$ has a root of multiplicity 1 at $x = -4$ and a root of multiplicity 2 at $x = 2$.
- The polynomial $x^3 - 12x + 65$ has a root of multiplicity 1 at $x = -5$.

However, to give a few more example:

- The polynomial $x^2 - 4$ has two roots of multiplicity 1 at $x = \pm 2$.
- The polynomial x^2 has a root of multiplicity 2 at $x = 0$.
- The polynomial $x^2 + 1$ has no real roots.

Theorem 2. The fundamental theorem of algebra says that a polynomial of degree n with complex coefficients has exactly n complex roots if you count multiplicity.

As every real number r is the complex number $r + 0j$, it follows that every polynomial of degree n with real coefficients has exactly n complex roots (some of them possibly real) if you count multiplicity.

For example, the polynomial $x^3 + x^2 - 7x - 15$ has only one real root at $x = 3$, and therefore the two other roots must be non-real complex numbers.

There is a special property of the roots of a polynomial with real coefficients:

Theorem 3. If a polynomial $p(x)$ has real coefficients and z is a non-real root of $p(x)$ with multiplicity m, then so is z^*, the complex conjugate of z.

A complex number and its complex conjugate are said to constitute a "complex conjugate pair."

In the above example, because the polynomial $x^3 + x^2 - 7x - 15$ has real coefficients, its two complex roots must form a complex conjugate pair. Specifically,

$$x^3 + x^2 - 7x - 15 = (x - 3)(x + 2 - j)(x + 2 + j),$$

so the non-real complex roots are $-2 + j$ and $-2 - j$.

For example, the polynomial

$$\begin{aligned} p(x) = x^{12} &+ 5x^{11} - 27x^{10} + 11x^9 + 1000x^8 - 1338x^7 - 12890x^6 \\ &+ 8186x^5 + 27357x^4 + 20405x^3 + 97525x^2 + 10875x + 56250 \end{aligned}$$

must have twelve roots if you count multiplicity, and if any non-real complex number is a root, then so is its complex conjugate. In this case, this polynomial has a complex conjugate pair of roots of multiplicity two at $x = \pm j$, a complex conjugate pair of roots of multiplicity one at $x = 3 \pm 4j$, a real root of multiplicity one at $x = -2$, a real root of multiplicity two at $x = 3$ and a real root of multiplicity three at $x = -5$, for a total of twelve roots. Therefore,

$$p(x) = (x - j)^2(x + j)^2(x - 3 + 4j)(x - 3 - 4j)(x + 2)(x - 3)^2(x + 5)^3.$$

You are not expected to be able to know how to find all twelve roots, but we could approximate at least the roots by using Newton's method.

5 Newton's method

Given a polynomial $p(x)$ with at least one real root, you can find a real root using this algorithm:

1. Estimate a root with a value x_0.
2. Calculate $x_{k+1} \leftarrow x_k - \frac{p(x_k)}{p^{(1)}(x_k)}$ for $k = 0, 1, 2, 3, \ldots$ until $|x_{k+1} - x_k| < 10^{-12}$.
   Here, we use the $\leftarrow$ symbol to emphasize that we assign to x_{k+1} the result of the calculation on the right-hand side.
3. If this does not converge, the polynomial may not have a real root, so try again with a different x_0, but if you already have tried again, we are likely finished.
4. If it does converge, divide the polynomial by $p(x)$ by $x - xk + 1$ to get a polynomial $q(x)$ of one lower degree with the root we found removed:
   (a) If the resulting polynomial is of degree 0, we are done.
   (b) Otherwise, go back to Step 1 with the new polynomial $q(x)$.

For example, given the above polynomial, if we estimate a root with $x_0 = -3$, we iterate to get the sequence of points:

$$\begin{aligned} x_1 &= -2.2089249492900608519 \\ x_2 &= -2.0376884045188339208 \\ x_3 &= -2.0016434601040040205 \\ x_4 &= -2.0000033488809440001 \\ x_5 &= -2.0000000000139502783 \\ x_6 &= -2.0000000000000000000 \\ x_7 &= -2.0000000000000000000 \end{aligned}$$

Note that $|x_6 - x_5| = 1.39502783 \times 10^{-11}$, so we had to iterate one more time.

6 Euler's formula

Euler's formula relates the exponential function, the trigonometric functions and j through the formula

$$e^{yj} = \cos(y) + \sin(y)j$$

If you substitute $y = \pi$ into this formula, you get

$$e^{\pi j} = \cos(\pi) + \sin(\pi)j = -1 + 0 \cdot j = -1.$$

You may wonder how this is true, and for this, we turn to Taylor series:

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + \cdots.$$

If we substitute $x = \beta j$, assuming β is real, into this formula, we get:

$$e^{\beta j} = 1 + \beta j + \frac{(\beta j)^2}{2} + \frac{(\beta j)^3}{6} + \frac{(\beta j)^4}{24} + \frac{(\beta j)^5}{120} + \frac{(\beta j)^6}{720} + \frac{(\beta j)^7}{5040} + \cdots.$$

Now, $(\beta j)^n = \beta^n j^n$:

$$e^{\beta j} = 1 + \beta j + \frac{\beta^2 j^2}{2} + \frac{\beta^3 j^3}{6} + \frac{\beta^4 j^4}{24} + \frac{\beta^5 j^5}{120} + \frac{\beta^6 j^6}{720} + \frac{\beta^7 j^7}{5040} + \cdots.$$

However, $j^2 = -1$, $j^3 = -j$, $j^4 = 1$, $j^5 = j$, and so on, so we may simplify this:

$$\begin{aligned} e^{\beta j} &= 1 + \beta j + \frac{\beta^2 \cdot (-1)}{2} + \frac{\beta^3 \cdot (-j)}{6} + \frac{\beta^4 \cdot 1}{24} + \frac{\beta^5 \cdot j}{120} + \frac{\beta^6 \cdot (-1)}{720} + \frac{\beta^7 \cdot (-j)}{5040} + \cdots \\ &= 1 + \beta j - \frac{\beta^2}{2} - \frac{\beta^3}{6} j + \frac{\beta^4}{24} + \frac{\beta^5}{120} j - \frac{\beta^6}{720} - \frac{\beta^7}{5040} j + \cdots \end{aligned}$$

We observe that terms with even powers of β involve only real numbers, while terms with odd powers of β are real numbers multiplied by j. This allows us to separate the sum into two parts:

$$e^{\beta j} = \left(1 - \frac{\beta^2}{2} + \frac{\beta^4}{24} - \frac{\beta^6}{720} + \cdots\right) + \left(\beta - \frac{\beta^3}{6} + \frac{\beta^5}{120} - \frac{\beta^7}{5040} + \cdots\right) j$$

You will notice that the first is the Taylor series for cosine, and the second is for sine, and thus, we have:

$$e^{\beta j} = \cos(\beta) + \sin(\beta) j$$

You should note that $e^{\beta j}$ is a point on the unit circle with an angle of β radians relative to the real axis.

Now, because $e^{x+y} = e^x e^y$, it follows that:

$$e^{\alpha + \beta j} = e^\alpha e^{\beta j} = e^\alpha \left(\cos(\beta) + \sin(\beta) j\right).$$

You will note that if $\beta = 0$ in this formula, this simplifies to $e^\alpha(1 + 0j) = e^\alpha$.

To give a simple example, note that if $z = 0.1 - 0.2j$, the claim is that

$$\begin{aligned} e^{0.1-0.2j} &= e^{0.1}(\cos(-0.2) + \sin(-0.2)j) \\ &= 1.105170918075648 \cdot (0.980066577841242 - 0.198669330795061j) \\ &= 1.083141079608063 - 0.219563566708252i \end{aligned}$$

While it is a little tedious, you could calculate:

$$\begin{aligned} &1 + (0.1 - 0.2j) + \frac{(0.1-0.2j)^2}{2} + \frac{(0.1-0.2j)^3}{6} + \frac{(0.1-0.2j)^4}{24} + \frac{(0.1-0.2j)^5}{120} + \frac{(0.1-0.2j)^6}{720} \\ &= 1.0831410791\overline{6} - 0.21956356\overline{1}j, \end{aligned}$$

which seems to be converging to the calculation above.

7 Acknowledgments

Many, many thanks to Zarra Sarker for finding various mistakes and typos, as well as making valuable suggestions to make the material clearer for a second-year engineering student.