Home Assignment 1

ECE 203 – Probability and Statistics 1
Due: September 27, 2022

Exercise 1 (20 points)

> If two fair dice are rolled, what is the conditional probability that the first one lands on 6 given that the sum of the dice is i? Compute for all values of i between 2 and 12.

Solution: $P(6 \mid \text{sum of } i) = 0$ for $i = 2, 3, \ldots, 6$, while

$$
\begin{aligned}
P(6 \mid \text{sum of } 7) &= P(\{6,1\})/6/6 = 1/6 \\
P(6 \mid \text{sum of } 8) &= P(\{6,2\})/5/6 = 1/5 \\
P(6 \mid \text{sum of } 9) &= P(\{6,3\})/4/6 = 1/4 \\
P(6 \mid \text{sum of } 10) &= P(\{6,4\})/3/6 = 1/3 \\
P(6 \mid \text{sum of } 11) &= P(\{6,5\})/2/6 = 1/2 \\
P(6 \mid \text{sum of } 12) &= 1
\end{aligned}
$$

Exercise 2 (20 points)

> Consider 3 urns. Urn A contains 2 white and 4 red balls, urn B contains 8 white and 4 red balls, and urn C contains 1 white and 3 red balls. If 1 ball is selected from each urn, what is the probability that the ball chosen from urn A was white given that exactly 2 white balls were selected?

Solution:

$$P(A = w \mid 2w) = \frac{P(A = w, 2w)}{P(2w)} =$$

$$= \frac{P(A = w, B = w, C \neq w) + P(A = w, B \neq w, C = w)}{P(A = w, B = w, C \neq w) + P(A = w, B \neq w, C = w) + P(A \neq w, B = w, C = w)} =$$

$$= \frac{\frac{2}{6}\frac{8}{12}\frac{3}{4} + \frac{2}{6}\frac{4}{12}\frac{1}{4}}{\frac{2}{6}\frac{8}{12}\frac{3}{4} + \frac{2}{6}\frac{4}{12}\frac{1}{4} + \frac{4}{6}\frac{8}{12}\frac{1}{4}} = 7/11.$$

Exercise 3 (20 points)

> Suppose that an ordinary deck of 52 cards (which contains 4 aces) is randomly divided into 4 hands of 13 cards each. We are interested in determining p, the probability that each hand has an ace. Let E_i be the event that the i-th hand has exactly one ace. Determine $p = P(E_1E_2E_3E_4)$by using the multiplication rule.

Solution: $p = P(E_1E_2E_3E_4) = p(E_1)p(E_2|E_1)p(E_3|E_1E_2)p(E_4|E_1E_2E_3)$, where

$$\begin{aligned} P(E_1) &= \binom{4}{1}\binom{48}{12}\Big/\binom{52}{13} \\ P(E_2 \mid E_1) &= \binom{3}{1}\binom{36}{12}\Big/\binom{39}{13} \\ P(E_3 \mid E_1E_2) &= \binom{2}{1}\binom{24}{12}\Big/\binom{26}{13} \\ P(E_4 \mid E_1E_2E_3) &= 1 \end{aligned}$$

Exercise 4 (20 points)

> A parallel system functions whenever at least one of its components works. Consider a parallel system of n components, and suppose that each component works independently with probability $1/2$. Find the conditional probability that component 1 works given that the system is functioning.

Solution: Let W and F be the events that component 1 works and that the system functions. Then, we have

$$P(W \mid F) = \frac{P(WF)}{P(F)} = \frac{P(W)}{1 - P(F^c)} = \frac{1/2}{1 - (1/2)^{n-1}}$$

Exercise 5 (20 points)

> There are 3 coins in a box. One is a two-headed coin, another is a fair coin, and the third is a biased coin that comes up heads 75 percent of the time. When one of the 3 coins is selected at random and flipped, it shows heads. What is the probability that it was the two-headed coin?

Solution:

$$P(2 \text{ headed} \mid \text{heads}) = \frac{1 \cdot 1/3}{1 \cdot 1/3 + 1/2 \cdot 1/3 + 3/4 \cdot 1/3}$$

Note that, for all the coins, the prior probabilities are equal to $1/3$.