Home Assignment 2

ECE 203 – Probability and Statistics 1
Due: June 24, 2025

Exercise 1 (20 points)

> In a class, there are 4 freshman boys, 6 freshman girls, and 6 sophomore boys. How many sophomore girls must be present if sex and class are to be independent when a student is selected at random?

Solution: Let the number of sophomore girls be x. We have

$$P[Boy, F] = \frac{4}{16+x}, \quad P[Boy] = \frac{10}{16+x}, \quad P[F] = \frac{10}{16+x}$$

where F stands for "freshman". By the assumption of independence,

$$4 = \frac{10 \cdot 10}{16+x},$$

which suggests that $x = 9$. A direct check now shows that 9 sophomore girls (which the above shows is necessary) is also sufficient for independence of sex and class.

Exercise 2 (20 points)

> Five distinct numbers are randomly distributed to players numbered 1 through 5. Whenever two players compare their numbers, the one with the higher one is declared the winner. Initially, players 1 and 2 compare their numbers; the winner then compares her number with that of player 3, and so on. Let X denote the number of times player 1 is a winner. Find $P\{X = i\}$, $i = 0, 1, 2, 3, 4$.

Solution: We have

$$P[X = 0] = P[\text{1 loses to 2}] = 1/2$$

$$P[X = 1] = P[\text{of 1, 2, 3: 3 has largest, then 1, then 2}] = (1/3)(1/2) = 1/6$$

$$P[X = 2] = P[\text{of 1, 2, 3, 4: 4 has largest and 1 has next largest}] = (1/4)(1/3) = 1/12$$

$$P[X = 3] = P[\text{of 1, 2, 3, 4, 5: 5 has largest then 1}] = (1/5)(1/4) = 1/20$$

$$P[X = 4] = P[\text{1 has largest}] = 1/5$$

Exercise 3 (20 points)

> Suppose that two teams play a series of games that ends when one of them has won i games. Suppose that each game played is, independently, won by team A with probability p. Find the expected number of games that are played when (a) $i = 2$ and (b) $i = 3$. Also, show in both cases that this number is maximized when $p = 1/2$.

Solution: Let N denote the number of games played.
(a) The expectation of N is given by

$$\mathbb{E}(N) = 2[p^2 + (1-p)^2] + 3[2p(1-p)] = 2 + 2p(1-p)$$

The final equality could also have been obtained by using that $N = 2 + I$, where I is 0 if two games are played and 1 if three are played. Differentiation yields

$$\frac{d\mathbb{E}[N]}{dp} = 2 - 4p.$$

The minimum occurs when $2 - 4p = 0$ or $p = 1/2$.
(b) In this case, we have

$$\mathbb{E}[N] = 3[p^3+(1-p)^3]+4[3p^2(1-p)p+3p(1-p)^2(1-p)]+5[6p^2(1-p)^2] = 6p^4-12p^3+3p^2+3p+3$$

Differentiation yields

$$\frac{d\mathbb{E}[N]}{dp} = 24p^3 - 36p^2 + 6p + 3$$

Its value at $p = 1/2$ is easily seen to be 0.

Exercise 4 (20 points)

> Two coins are to be flipped. The first coin will land on heads with probability 0.6, the second with probability 0.7. Assume that the results of the flips are independent, and let X equal the total number of heads that result.
> (a) Find $P\{X = 1\}$.
> (b) Determine $E[X]$.

Solution: (a) We have

$$P(X = 1) = P(H_1 \cap T_2) + P(T_1 \cap H_2) = P(H_1) \cdot P(T_2) + P(T_1) \cdot P(H_2)$$

$$= 0.6 \cdot 0.3 + 0.4 \cdot 0.7 = 0.18 + 0.28 = \boxed{0.46}$$

(b) We also have

$$\mathbb{E}[X] = \mathbb{E}[X_1] + \mathbb{E}[X_2] = P(H_1) + P(H_2) = 0.6 + 0.7 = \boxed{1.3}$$

Exercise 5 (20 points)

> A box contains 5 red and 5 blue marbles. Two marbles are withdrawn randomly. If they are the same colour, then you win 1.10 dollar; if they are different colours, then you win -1 dollar. (That is, you lose 1 dollar.) Calculate:
> (a) the expected value of the amount you win;
> (b) the variance of the amount you win.

Solution: First, we compute

$$P[X = 1.10] = 4/9 = 1 - P[X = -1]$$

Then,

$$\mathbb{E}[X] = (1.1)4/9 - 5/9 = -0.6/9 \approx -.067$$

and

$$\mathrm{Var}(X) = (1.1)^2(4/9) + 5/9 - (0.6/9)^2 \approx 1.089.$$