Home Assignment 3

ECE 203 – Probability and Statistics 1
Due: July 7, 2025

Exercise 1 (20 points)

> Suppose that two teams play a series of games that ends when one of them has won i games. Suppose that each game played is, independently, won by team A with probability p. Find the expected number of games that are played when: (a) $i = 2$ and (b) $i = 3$. Also, show in both cases that this number is maximized when $p = 0.5$.

Solution: Let N denote the number of games played.
(a) The expectation of N is given by

$$\mathbb{E}(N) = 2[p^2 + (1-p)^2] + 3[2p(1-p)] = 2 + 2p(1-p)$$

The final equality could also have been obtained by using that $N = 2 + I$, where I is 0 if two games are played and 1 if three are played. Differentiation yields

$$\frac{d\mathbb{E}[N]}{dp} = 2 - 4p.$$

The minimum occurs when $2 - 4p = 0$ or $p = 1/2$.
(b) In this case, we have

$$\mathbb{E}[N] = 3[p^3+(1-p)^3]+4[3p^2(1-p)p+3p(1-p)^2(1-p)]+5[6p^2(1-p)^2] = 6p^4-12p^3+3p^2+3p+3$$

Differentiation yields

$$\frac{d\mathbb{E}[N]}{dp} = 24p^3 - 36p^2 + 6p + 3$$

Its value at $p = 1/2$ is easily seen to be 0.

Exercise 2 (20 points)

The number of times that a person contracts a cold in a given year is a Poisson random variable with parameter $\lambda = 5$. Suppose that a new wonder drug has just been marketed that reduces the Poisson parameter to $\lambda = 3$ for 75 percent of the population. For the other 25 percent of the population, the drug has no appreciable effect on colds. If an individual tries the drug for a year and has 2 colds in that time, how likely is it that the drug is beneficial for him or her?

Solution: We have

$$P[\text{beneficial} \mid 2] = \frac{P[2 \mid \text{beneficial}](3/4)}{P[2 \mid \text{beneficial}](3/4) + P[2 \mid \text{not beneficial}](1/4)} =$$

$$= \left(e^{-3}\frac{3^2}{2}\frac{3}{4}\right) \Big/ \left(e^{-3}\frac{3^2}{2}\frac{3}{4} + \left(e^{-5}\frac{5^2}{2}\frac{1}{4}\right)\right)$$

Exercise 3 (20 points)

Suppose that 10 balls are put into 5 boxes, with each ball independently being put in box i with probability p_i, $\sum_{i=1}^{5} p_i = 1$.

(a) Find the expected number of boxes that do not have any balls.

(b) Find the expected number of boxes that have exactly 1 ball.

Solution: (a) Define an indicator random variable

$$I_i = \begin{cases} 1 & \text{if box } i \text{ has no balls} \\ 0 & \text{otherwise} \end{cases}$$

Then the total number of empty boxes is:

$$X = \sum_{i=1}^{5} I_i$$

and thus

$$\mathbb{E}[X] = \sum_{i=1}^{5} \mathbb{E}[I_i]$$

Since each ball independently avoids box i with probability $1 - p_i$, the probability that all 10 balls avoid box i is:

$$\mathbb{E}[I_i] = P(\text{box } i \text{ is empty}) = (1 - p_i)^{10}$$

So:

$$\mathbb{E}[X] = \sum_{i=1}^{5}(1-p_i)^{10}.$$

(b) Define another indicator variable

$$J_i = \begin{cases} 1 & \text{if box } i \text{ has exactly one ball} \\ 0 & \text{otherwise} \end{cases}$$

Then:

$$Y = \sum_{i=1}^{5} J_i, \quad \mathbb{E}[Y] = \sum_{i=1}^{5} \mathbb{E}[J_i]$$

To compute $\mathbb{E}[J_i]$, note that for box i to have exactly one ball:

- Choose 1 out of the 10 balls to go to box i: $\binom{10}{1}$
- That ball goes to box i with probability p_i
- The remaining 9 balls avoid box i: each with probability $1 - p_i$

Consequently,

$$\mathbb{E}[J_i] = \binom{10}{1} p_i(1-p_i)^9 = 10p_i(1-p_i)^9$$

Thus,

$$\mathbb{E}[Y] = \sum_{i=1}^{5} 10p_i(1-p_i)^9.$$

Exercise 4 (20 points)

(a) A fire station is to be located along a road of length A, $A < \infty$. If fires occur at points uniformly chosen on $(0, A)$, where should the station be located so as to minimize the expected distance from the fire? That is, choose a so as to minimize $E[|X - a|]$, when X is uniformly distributed over $(0, A)$.

(b) Now suppose that the road is of infinite length - stretching from point 0 outward to ∞. If the distance of a fire from point 0 is exponentially distributed with rate λ, where should the fire station now be located? That is, we want to minimize $E[|X - a|]$, where X is now exponential with rate λ.

Exercise 5 (20 points)

> If Y is uniformly distributed over $(0, 5)$, what is the probability that the roots of the equation $4x^2 + 4xY + Y + 2 = 0$ are both real?

Solution: For both roots to be real the discriminant $(4Y)^2 - 44(Y + 2)$ must be greater than 0, which suggests that $Y^2 \geq Y + 2$. In the interval $0 < Y < 5$, this is equivalent to $Y \geq 2$. Thus,

$$P[Y^2 \geq Y + 2] = P[Y \geq 2] = 3/5.$$