Home Assignment 4

ECE 203 – Probability and Statistics 1
Due: July 21, 2025

Exercise 1 (20 points)

> There are two types of batteries in a bin. When in use, type i batteries last (in hours) an exponentially distributed time with rate λ_i, $i = 1, 2$. A battery that is randomly chosen from the bin will be a type i battery with probability p_i, with $\sum_{i=1}^{2} p_i = 1$. If a randomly chosen battery is still operating after t hours of use, what is the probability that it will still be operating after an additional s hours?

Solution: Let T be the lifetime of the randomly chosen battery. We are asked to compute:

$$P(T > t + s \mid T > t)$$

Using the definition of conditional probability:

$$P(T > t + s \mid T > t) = \frac{P(T > t + s)}{P(T > t)}$$

Since T is a mixture of two exponential distributions:

$$P(T > t) = p_1 e^{-\lambda_1 t} + p_2 e^{-\lambda_2 t}$$

$$P(T > t + s) = p_1 e^{-\lambda_1 (t+s)} + p_2 e^{-\lambda_2 (t+s)}$$

Thus:

$$P(T > t + s \mid T > t) = \frac{p_1 e^{-\lambda_1 (t+s)} + p_2 e^{-\lambda_2 (t+s)}}{p_1 e^{-\lambda_1 t} + p_2 e^{-\lambda_2 t}}$$

Now factor exponentials:

$$= \frac{p_1 e^{-\lambda_1 t} e^{-\lambda_1 s} + p_2 e^{-\lambda_2 t} e^{-\lambda_2 s}}{p_1 e^{-\lambda_1 t} + p_2 e^{-\lambda_2 t}}$$

Exercise 2 (20 points)

The joint probability density function of X and Y is given by $f(x, y) = (6/7)(x^2 + xy/2)$, where $0 < x < 1$ and $0 < y < 2$.
(a) Verify that this is indeed a joint density function.
(b) Compute the density function of X.
(c) Find $P[X > Y]$.
(d) Find $P[Y > 1/2 | X < 1/2]$.
(e) Find $E[X]$.
(f) Find $E[Y]$.

Solution:
(a) It is easy to verify that $\int_0^2 \int_0^1 ((6/7)(x^2 + xy/2))dxdy = 1$.
(b) $f_X(x) = \int_0^2 ((6/7)(x^2 + xy/2))dy = (6/7)(2x^2 + x)$.
(c) $P[X > Y] = \int_0^1 \int_0^x ((6/7)(x^2 + xy/2))dydx = 15/56$.
(d) $P[Y > 1/2 | X < 1/2] = \frac{P[Y>1/2, X<1/2]}{P[X<1/2]} = \frac{\int_{1/2}^2 \int_0^{1/2} ((6/7)(x^2+xy/2))dxdy}{\int_0^{1/2} (6/7)(2x^2+x)dx}$.
(e) $\mathbb{E}[X] = (6/7) \int_0^1 x(2x^2 + x)dx$.
(f) Similar to (b) and (e).

Exercise 3 (20 points)

Two points are selected randomly on a line of length L so as to be on opposite sides of the midpoint of the line. In other words, the two points X and Y are independent random variables such that $X \sim U(0, L/2)$ and $Y \sim U(L/2, L)$. Find the probability that the distance between the two points is greater than $L/3$.

Solution:

$$P[Y - X > 1/3] = \iint_{y-x>1/3} (4/L^2)dxdy =$$

$$= \frac{4}{L^2}\left[\int_0^{L/6}\int_{L/2}^{L} dydx + \int_{L/6}^{L/2}\int_{x+L/3}^{L} dydx\right] = \frac{4}{L^2}\left[\frac{L^2}{12} + \frac{5L^2}{24} - \frac{7L^2}{72}\right] = 7/9.$$

Exercise 4 (20 points)

If W is uniform on $(0, 2\pi)$ and Z, independent of W, is exponential with rate 1, show that X and Y defined by $X = \sqrt{2Z}\cos W$ and $Y = \sqrt{2Z}\sin W$ are independent standard normal random variables.

Solution: The joint density of W and Z is given by

$$f_{W,Z}(w, z) = \frac{1}{2\pi}e^{-z}, \quad z \geq 0.$$

Note that

$$Z = \frac{X^2 + Y^2}{2}, \quad W = \tan^{-1}(Y/X).$$

The determinant of the Jacobian is given by

$$J = \begin{vmatrix} \frac{2}{2\sqrt{2Z}}\cos W & -\sqrt{2Z}\sin W \\ \frac{2}{2\sqrt{2Z}}\sin W & \sqrt{2Z}\cos W \end{vmatrix} = \cos^2 W + \sin^2 W = 1.$$

Therefore, we have

$$f_{X,Y}(x, y) = \frac{1}{2\pi}e^{-(X^2+Y^2)/2} = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}\frac{1}{\sqrt{2\pi}}e^{-y^2/2}.$$

Exercise 5 (20 points)

If X_1 and X_2 are independent exponential random variables, each having parameter λ, find the joint density function of $Y_1 = X_1 + X_2$ and $Y_2 = e^{X_1}$.

Solution: From the definitions

$$X_1 = \ln Y_2, \quad X_2 = Y_1 - \ln Y_2$$

Now, we compute the Jacobian matrix of the transformation from (Y_1, Y_2) to (X_1, X_2):

$$J = \begin{bmatrix} \frac{\partial X_1}{\partial Y_1} & \frac{\partial X_1}{\partial Y_2} \\ \frac{\partial X_2}{\partial Y_1} & \frac{\partial X_2}{\partial Y_2} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{Y_2} \\ 1 & -\frac{1}{Y_2} \end{bmatrix}$$

$$\Rightarrow |\det J| = \left| -\frac{1}{Y_2} \right| = \frac{1}{Y_2}$$

Since X_1 and X_2 are independent exponential random variables with rate λ:

$$f_{X_1,X_2}(x_1, x_2) = \lambda^2 e^{-\lambda(x_1+x_2)}$$

Substitution yields

$$x_1 = \ln Y_2, \quad x_2 = Y_1 - \ln Y_2 \quad \Rightarrow \quad x_1 + x_2 = Y_1$$

So, with

$$f_{X_1,X_2}(x_1, x_2) = \lambda^2 e^{-\lambda Y_1},$$

we have

$$f_{Y_1,Y_2}(y_1, y_2) = f_{X_1,X_2}(\ln y_2, y_1 - \ln y_2) \cdot |\det J| = \lambda^2 e^{-\lambda y_1} \cdot \frac{1}{y_2}$$

Finally,

$$\boxed{f_{Y_1,Y_2}(y_1, y_2) = \frac{\lambda^2}{y_2} e^{-\lambda y_1}, \quad \text{for } y_2 > 1,\ y_1 > \ln y_2}$$