Home Assignment 5

ECE 203 – Probability and Statistics 1
Due: July 30, 2025

Exercise 1 (20 points)

> A player throws a fair die and simultaneously flips a fair coin. If the coin lands heads, then the player wins twice the value that appears on the die. If it is tails, then the winnings equal to one-half of the die value. Determine the expected winnings of the player.

Solution: Let $X = 1$ if the coin toss lands heads, and let it equal 0 otherwise. Also, let Y denote the value that shows up on the die. Then, with $p(i, j) = P[X = i, Y = j]$

$$\mathbb{E}[\text{return}] = \sum_{j=1}^{6} 2jp(1, j) + \sum_{j=1}^{6} \frac{j}{2} p(0, j) = \frac{1}{12}(42 + 10.5) = 52.5/12.$$

Exercise 2 (20 points)

> Let Z be a standard normal random variable, and, for a fixed x, set $X = Z$, if $Z > x$, and $X = 0$, otherwise. Show that $E[X] = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$.

Solution:

$$\mathbb{E}[X] = \int_{y>x} y \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

Exercise 3 (20 points)

> Let X be the number of 1's and Y the number of 2's that occur in n rolls of a fair die. Compute $\text{Cov}(X, Y)$.

Solution: Let X_i be equal to 1 if roll i lands on 1, and let it be 0 otherwise. Also, let Y_i be equal to 1 if roll i lands on 2, and let it be 0 otherwise. Then

$$\mathrm{Cov}(X_i, Y_j) = \mathbb{E}[X_i Y_j] - \mathbb{E}[X_i]\mathbb{E}[Y_j] = \begin{cases} -1/36, & i = j \\ 0, & i \neq j \end{cases}$$

Consequently,

$$\mathrm{Cov}\left(\sum_i X_i, \sum_j Y_j\right) = \sum_i \sum_j \mathrm{Cov}(X_i, Y_j) = -\frac{n}{36}.$$

Exercise 4 (20 points)

> The joint density function of X and Y is given by $f(x, y) = (1/y)e^{-(y+x/y)}$, for $x > 0, y > 0$. Find $E[X]$, $E[Y]$, and show that $\mathrm{Cov}(X, Y) = 1$.

Solution: We have

$$f_Y(y) = e^{-y} \int \frac{1}{y} e^{-x/y} dx = e^{-y}.$$

In addition, the conditional distribution of X given that $Y = y$ is exponential with mean y. Hence, $\mathbb{E}[Y] = 1$ and $\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X \mid Y]] = \mathbb{E}[Y] = 1$.

Since $\mathbb{E}[XY] = \mathbb{E}[\mathbb{E}[XY \mid Y]] = \mathbb{E}[Y\mathbb{E}[X \mid Y]] = \mathbb{E}[Y^2] = 2$ (this follows from the fact that Y is exponential with mean 1). Therefore, $\mathrm{Cov}(X, Y) = 2 - 1 = 1$.

Exercise 5 (20 points)

> Consider a gambler who, at each gamble, either wins or loses their bet with respective probabilities p and $1 - p$. A popular gambling system, known as the Kelley strategy, is to always bet the fraction $2p - 1$ of your current fortune when $p > 1/2$. Compute the expected fortune after n gambles of a gambler who starts with x units and employs the Kelley strategy.

Solution: Let F_n denote the fortune after n gambles.

$$\mathbb{E}[F_n] = \mathbb{E}[\mathbb{E}[F_n \mid F_{n-1}]] = \mathbb{E}\left(2(2p-1)F_{n-1}p + F_{n-1} - (2p-1)F_{n-1}\right) =$$

$$= (1 + (2p-1)^2)\mathbb{E}[F_{n-1}] = (1 + (2p-1)^2)^2\mathbb{E}[F_{n-2}] = \ldots$$

$$\ldots = (1 + (2p-1)^2)^n\mathbb{E}[F_0] = (1 + (2p-1)^2)^n x.$$