ECE 203 – Section 1
Introduction to Probability

- Axioms
- Sample spaces and events
- Set operations
- Sample spaces with equally likely outcomes.

The slides have been prepared based on the lecture notes of Prof. Patrick Mitran.

Axioms (or Laws) of Probability

- We start with the definition of a *random experiment* as follows.
- Random experiments do not have predictable outcomes.
- The set of all possible outcomes is called the *sample space*, and denoted S.

Example 1: If we toss two 2 coins, then $S = \{(h, h), (h, t), (t, h), (t, t)\}$.

Example 2: If we toss two 6-sided dice, then

$$S = \{(i, j) \mid i = 1, 2, \ldots, 6,\ j = 1, 2, \ldots, 6\}$$

Example 3: In a race with 5 horses, the possible order of finishing are

$$S = \{\text{all } 5! \text{ orders of } (a, b, c, d, e)\}$$

Example 4: In roulette, $S = \{00, 0, 1, \ldots, 36\}$.

Example 5: In an experiment measuring the lifetime of a solid-state drive,

$$S = \{x \in \mathbf{R} \mid x \geq 0\}$$

Example 6: Romeo and Juliette have a date. Each will arrive with a delay that is between 0 and 1 hour:

$$S = \{(x, y) \in \mathbf{R}^2 \mid 0 \leq x \leq 1, 0 \leq y \leq 1\}$$

Example 7: A standard 52-card deck has 4 suits (Clubs [$\clubsuit$], Diamonds [$\diamondsuit$], Hearts [$\heartsuit$] and Spades [$\spadesuit$]).

Each suit has a cards numbered from 1 to 13, with 1 also called Ace, 11 also called Jack, 12 also called Queen, and 13 also called King.

Let S be all possible orderings of the deck. Then $|S| = 52! \approx 8 \times 10^{67}$.

Definition: A subset $E \subset S$ is called an *event*.

Example 8: In the example with two coins,

$$E = \{(h, h), (t, t)\}$$

is the event that both coins come up identical.

Example 9: In the example with two dice,

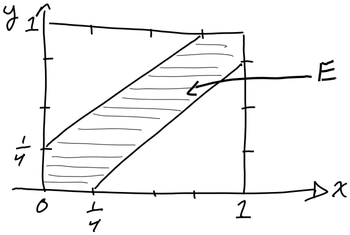
$$E = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$$

is the event that the dice add up to 7.

Example 10: In roulette, even $= \{2, 4, 6, \dots, 36\}$ is called an *even outcome* and odd $= \{1, 3, 5, \dots, 35\}$ is called an *odd outcome*.

Example 11: The event that Romeo and Juliette arrive within 1/4 hour of each other is:

$$E = \{(x, y) \in S \mid |x - y| \leq 1/4\}$$



Examples (cont.)

For 2 events E and F:

- $E \cup F$ is the event that either E **or** F occurs

$$E \cup F = \{x \in S \mid x \in E \text{ or } x \in F\}$$

- $E \cap F$ is the event that both E **and** F occur (we also write EF)

$$E \cap F = \{x \in S \mid x \in E \text{ and } x \in F\}$$

- If $EF = \emptyset$, then E and F are said to be *mutually exclusive* or *disjoint*.

- E^c is the event that E does not occur (we also write $\bar{E}$)

$$E^c = \{x \in S \mid x \notin E\}$$

- Given F and $E_1, E_2, \ldots, E_n$, if
  1. $E_i E_j = \emptyset$ for $i \neq j$ (i.e., $E_1, E_2, \ldots E_n$ are disjoint)
  2. $F = \cup_{i=1}^n E_i$

  then $E_1, E_2, \ldots, E_n$ are said to *partition* F.

Laws and Properties of Sets

- **Commutative Laws**

$$E \cup F = F \cup E \quad \& \quad EF = FE$$

- **Associative Laws**

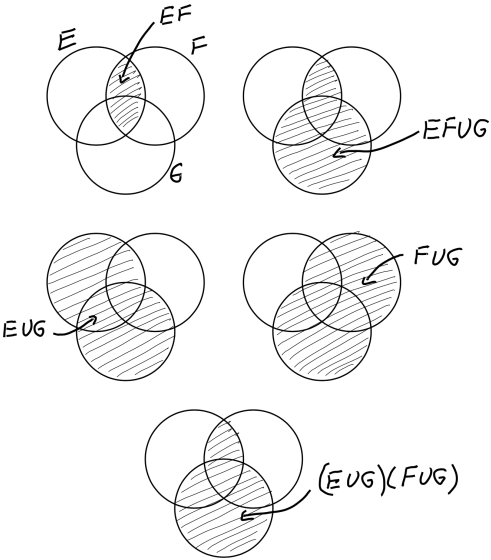
$$(E \cup F) \cup G = E \cup (F \cup G) \quad \& \quad (EF)G = E(FG)$$

- **Distributive Laws**

$$(E \cup F)G = EG \cup FG \quad \& \quad EF \cup G = (E \cup G)(F \cup G)$$

Venn Diagram

Venn diagram interpretation of $EF \cup G = (E \cup G)(F \cup G)$.



DeMorgan's Laws have the following analytic form.

$$\left(\bigcup_{i=1}^{n} E_i\right)^c = \bigcap_{i=1}^{n} E_i^c \quad \text{(1st law)}$$

$$\left(\bigcap_{i=1}^{n} E_i\right)^c = \bigcup_{i=1}^{n} E_i^c \quad \text{(2nd law)}$$

Proof of 1st law:

- **Step 1**: We will show $(\cup_i E_i)^c \subset \cap_i E_i^c$
- Let $x \in (\cup_i E_i)^c$
- Then $x \notin \cup_i E_i$
- Then, for each i, $x \notin E_i$
- Then, for each i, $x \in E_i^c$
- Then, $x \in \cap_i E_i^c$

- **Step 2**: We will show $\cap_i E_i^c \subset (\cup_i E_i)^c$
- Let $x \in \cap_i E_i^c$
- Then, for each i, $x \in E_i^c$
- Then, for each i, $x \notin E_i$
- Then, $x \notin E_1 \cup E_2 \cup \cdots \cup E_n$
- Then, $x \in \underbrace{(E_1 \cup E_2 \cup \cdots \cup E_n)^c}_{(\cup_i E_i)^c}$

- We wish to assign to each event E a probability, denoted $P(E)$ (or $P[E]$).
- How do we determine it?
- **Frequent approach:** Let $n(E)$ be number of occurrences of E in n repeated experiments. Then define

$$P[E] = \lim_{n \to \infty} \frac{n(E)}{n}$$

- Does this limit exist? In what sense?

- **Modern approach:** Instead, assume that certain rules (axioms) must hold. Specifically:

[A1] $0 \leq P[E] \leq 1$

[A2] $P[S] = 1$

[A3] If $E_1, E_2, \ldots$ are disjoint (i.e., mutually exclusive), then

$$P[E_1 \cup E_2 \cup \ldots] = \sum_i P[E_i]$$

- **Corollary:** $P[\emptyset] = 0$.
- To see that, Let $E_1 = S, E_2 = \emptyset, E_3 = \emptyset, \ldots$. Then $E_1, E_2, E_3, \ldots$ are disjoint. Hence,

$$\begin{aligned} P[E_1 \cup E_2 \cup E_3 \cup \cdots] &= P[E_1] + P[E_2] + P[E_3] + \cdots \\ &= P[S] + P[\emptyset] + P[\emptyset] + \cdots \\ &= 1 + P[\emptyset] + P[\emptyset] + \cdots \end{aligned}$$

- But this sum must be ≤ 1, so $P[\emptyset] = 0$.

- **Corollary:** Say $E_1, E_2, \ldots, E_n$ are disjoint. Then

$$P[\cup_{i=1}^{n} E_i] = \sum_{i=1}^{n} P[E_i]$$

- To see that, take $\emptyset = E_{n+1} = E_{n+2} = \cdots$. Then

$$P[\cup_{i=1}^{n} E_i] = P[\cup_{i=1}^{\infty} E_i] = \sum_{i=1}^{\infty} P[E_i] =$$

$$= \sum_{i=1}^{n} P[E_i] + \sum_{i=n+1}^{\infty} P[E_i] = \sum_{i=1}^{n} P[E_i]$$

- **Example:** Roulette has 38 possible outcomes. If each is equally likely, then
$$P[00] = P[0] = P[1] \cdots = P[36]$$

- But
$$1 = P[\{00, 0, 1, \cdots, 36\}] = P[00] + P[0] + P[1] + \cdots + P[36]$$

- Hence,
$$P[00] = P[0] = P[1] \cdots = P[36] = 1/38$$

- So,
$$\begin{aligned} P[\text{even}] &= P[\{2, 4, \ldots, 36\}] \\ &= P[2] + P[4] + \cdots + P[36] \\ &= 18/38 = 9/19 \end{aligned}$$

Consequences of Axioms (cont.)

- **Corollary:** $P[E^c] = 1 - P[E]$
- To see that we notice that E and E^c are disjoint, and $E \cup E^c = S$.
- Therefore,
$$1 = P[S] = P[E \cup E^c] = P[E] + P[E^c]$$
- So $P[E^c] = 1 - P[E]$. $\blacksquare$

Consequences of Axioms (cont.)

- **Corollary:** If $E \subset F$ then $P[E] \leq P[F]$.
- Why? Since $E \subset F$, then $F = E \cup E^c F$, while E and $E^c F$ are disjoint.
- Then
$$P[F] = P[E] + \underbrace{P[E^c F]}_{\geq 0}$$
- Therefore $P[F] \geq P[E]$. $\blacksquare$

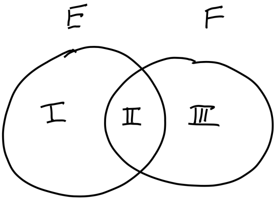
- **Example:** In roulette, $\mathsf{odd} \subset \mathsf{even}^c$, so
$$\underbrace{P[\mathsf{odd}]}_{9/19} \leq \underbrace{P[\mathsf{even}^c]}_{10/19}$$

Consequences of Axioms (cont.)

- **Corollary:** $P[E \cup F] = P[E] + P[F] - P[E \cap F]$
- To see that, we notice the following.

$$\left.\begin{aligned} I &= EF^c \\ II &= EF \\ III &= E^cF \end{aligned}\right\} \text{disjoint}$$

$$\begin{aligned} E &= I \cup II \\ F &= II \cup III \\ E \cup F &= I \cup II \cup III \end{aligned}$$



- Hence

$$\begin{aligned} P[E] + P[F] &= P[I \cup II] + P[II \cup III] \\ &= P[I] + P[II] + P[II] + P[III] \\ &= P[I] + P[II] + P[III] + P[II] \\ &= P[E \cup F] + P[EF]. \quad \blacksquare \end{aligned}$$

Example

- After 5 years, a car may need
  1. new brakes with probability 0.5
  2. new tires with probability 0.4
  3. both with probability 0.3
- What is probability it needs neither?
- To answer this question, let $B = \{\text{needs brakes}\}, T = \{\text{needs tires}\}$.
- Then $P[B] = 0.5$, $P[T] = 0.4$, and $P[BT] = 0.3$.
- Consequently,

$$
\begin{aligned}
P[\text{needs neither}] = P[B^c T^c] = P[(B \cup T)^c] = \\
= 1 - P[B \cup T] = 1 - (P[B] + P[T] - P[BT]) = \\
= 1 - (0.5 + 0.4 - 0.3) = 0.4
\end{aligned}
$$

- Can we generalize the $P[E \cup F]$ idea of the corollary? Yes!

$$
\begin{aligned}
P[E \cup F \cup G] &= \\
&= P[(E \cup F) \cup G] = \\
&= P[(E \cup F)] + P[G] - P[(E \cup F)G] = \\
&= P[E] + P[F] - P[EF] + P[G] - P[EG \cup FG] = \\
&= P[E] + P[F] + P[G] - P[EF] - (P[EG] + P[FG] - P[EGFG]) \\
&= P[E] + P[F] + P[G] - P[EF] - P[EG] - P[FG] + P[EFG]
\end{aligned}
$$

$$
\begin{aligned}
&P[E_1 \cup E_2 \cup \ldots \cup E_n] \\
&\quad = P[E_1] + P[E_2] + \cdots P[E_n] && \text{include all events} \\
&\qquad - \sum_{i_1 < i_2} P[E_{i_1} E_{i_2}] && \text{exclude intersections of pairs} \\
&\qquad + \sum_{i_1 < i_2 < i_3} P[E_{i_1} E_{i_2} E_{i_3}] && \text{include triple intersections} \\
&\qquad\quad \vdots && \vdots \\
&\qquad + (-1)^{r+1} \sum_{i_1 < \cdots < i_r} P[E_{i_1} E_{i_2} \cdots E_{i_r}] && \text{(in/ex)clude } r\text{-way intersections} \\
&\qquad\quad \vdots && \vdots \\
&\qquad + (-1)^{n+1} P[E_1 E_2 \cdots E_n] && \text{(in/ex)clude } n\text{-way intersection}
\end{aligned}
$$

Sample Spaces with Equally Likely Outcomes

- Say $S = \{1, 2, \ldots N\}$. Then

$$1 = P[S] = P[1] + P[2] + \cdots + P[N]$$

- Now suppose in addition that each individual outcome is equally likely:

$$P[1] = P[2] = \cdots = P[N]$$

- Combining the above equations, we obtain

$$P[1] = P[2] = \cdots = P[N] = 1/N$$

- Now, for any subset $E \subset S$, we have $E = \bigcup_{i \in E} \{i\}$ and so

$$\boxed{P[E] = \sum_{i \in E} P[i] = \sum_{i \in E} 1/N = |E|/N = |E|/|S|}$$

Examples

- There are $\binom{52}{5} = \frac{52!}{(52-5)!5!} = 259860$ ways to pick 5 cards out of 52.
- If we shuffle the deck well, these are all equally likely.
- The probability that 4 of the 5 cards have the same # is:

$$\frac{\#\text{ of ways to get 4 cards with same }\#}{\#\text{ of ways to get 5 cards}} = \frac{13 \times (52-4)}{\binom{52}{5}}$$

Examples (cont.)

- If two dice are rolled, what is the probability that the sum is 7?
- Assume that there are 36 equally likely outcomes, in which case

$$\{\text{sum} = 7\} = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$$

- Therefore, $P = 6/36$.

Examples (cont.)

- An urn has 7 white balls and 5 black balls.
- If we draw 3 balls at random, what is the probability that 1 is white and 2 are black?
- If we put a unique mark on each ball, there are $12 \times 11 \times 10 = 1320$ possible outcomes.
- Consider the following three cases:

Case 1: 1st ball is white; there are $7 \times 5 \times (5-1) = 140$ ways.
Case 2: 2nd ball is white; there are $5 \times 7 \times (5-1) = 140$ ways.
Case 3: 3rd ball is white; there are $5 \times (5-1) \times 7 = 140$ ways.

- Therefore $P = \frac{3 \times 140}{1320} = 7/22$.

Example: Matching Problem

- Each of n persons at a party throws their hat into the centre of a room and picks a hat at random.
- What is the probability that no person selects their own hat?
- There are $n \times (n-1) \times \cdots \times 1 = n!$ possible hat assignments.
- Let $E_i = \{\text{person } i \text{ selects hat } \# \, i\}$. Recall that

$$
\begin{aligned}
P[E_1 \cup E_2 \cup \cdots \cup E_n] = P[E_1] + P[E_2] + \cdots P[E_n] \\
- \sum_{i_1 < i_2} P[E_{i_1} E_{i_2}] \\
\vdots \\
+ (-1)^{m+1} \sum_{i_1 < \cdots < i_m} P[E_{i_1} E_{i_2} \cdots E_{i_m}] \\
\vdots \\
+ (-1)^{n+1} P[E_1 E_2 \cdots E_n]
\end{aligned}
$$

Example: Matching Problem (cont.)

- Now, $E_{i_1} E_{i_2} \cdots E_{i_m}$ means persons $i_1, i_2, \ldots, i_m$ have their own hat.
- This leaves $(n-m)$ people with an unknown hat arrangement. There are $(n-m)!$ ways to arrange these. Thus

$$P[E_{i_1} E_{i_2} \cdots E_{i_m}] = \frac{(n-m)!}{n!}$$

- Also, $\sum_{i_1 < \cdots < i_m} P[E_{i_1} E_{i_2} \cdots E_{i_m}]$ has $\binom{n}{m}$ terms in the sum. So,

$$\sum_{i_1 < \cdots < i_m} P[E_{i_1} E_{i_2} \cdots E_{i_m}] = \binom{n}{m} \frac{(n-m)!}{n!} =$$

$$= \frac{n!}{(n-m)!m!} \frac{(n-m)!}{n!} = \frac{1}{m!}$$

Example: Matching Problem (cont.)

- Consequently,

$$P[E_1 \cup E_2 \cup \cdots \cup E_n] = \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{(-1)^{n+1}}{n!}$$

- Finally,

$$P[E_1^c E_2^c \cdots E_n^c] = 1 - P[E_1 \cup E_2 \cup \cdots \cup E_n] =$$
$$= 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}$$

- This is a truncation of the Taylor series for e^{-1}. When n is large, this is ≈ 0.369.