ECE 203 – Section 4
Continuous random variables

- Probability density function
- Mean and variance of continuous random variables
- Expectation of functions of continuous random variables
- Representative distributions and memoryless random variables
- Transformation of random variables

The slides have been prepared based on the lecture notes of Prof. Patrick Mitran.

Continuous Random Variables

- We saw random variables where the set of all possible outcomes was discrete.

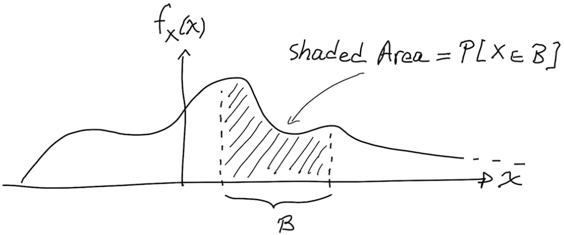
- In some cases, a random variable can take a continuum of values (e.g., time at which a train arrives, voltage across a resistor, etc)

> - **Definition:** We say that X is a *continuous random variable*, if there is a non-negative function $f_X(x)$ such that
>
> $$P[X \in B] = \int_B f_X(x)\, dx$$
>
> $f_X(x)$ is called **probability density function** (pdf).

- This is similar to mass density: if $\rho(x)$ is known, the *density of mass* in kg/m^3 at every point $x \in \mathbb{R}^3$, the mass inside any volume V is:

$$m(V) = \text{mass in volume } V = \iiint_V \rho(x)\, dx$$

- $f_X(x)$ is similar to the mass densoty, except it measures the *density of probability*, not mass.



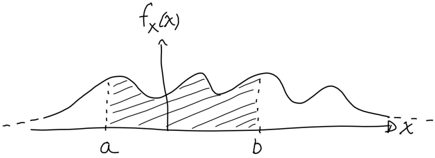
- Since X must take some value, we have:

$$\boxed{1 = P[X \in (-\infty, \infty)] = \int_{-\infty}^{\infty} f_X(x)\, dx}$$

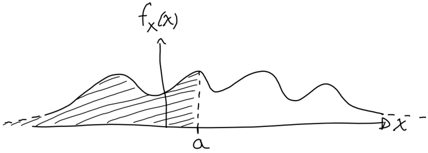
- **Note:** Say X has units of kg. Since dx has units of kg, $f_X(x)$ has units of kg^{-1}.

- Once we know $f_X(x)$, all probability statements about X can be answered.

  1. $P[X \in [a,b]] = \int_a^b f_X(x)dx$



  2. $P[X = a] = P[X \in [a,a]] = \int_a^a f_X(x)dx = 0$
  3. $F_X(a) = P[X \le a] = P[X \in (-\infty, a]] = \int_{-\infty}^a f_X(x)dx$



  4. $f_X(a) = dF_X(a)/da$

- Say X has pdf given by

$$f_X(x) = \begin{cases} C\left(4x - 2x^2\right) & 0 < x < 2 \\ 0 & \text{else} \end{cases}$$

for some constant C. Find C and $P[X > 1]$.

- *Solution:* Since the pdf must integrate to 1, then

$$1 = \int_{-\infty}^{\infty} f_X(x)dx = C\int_0^2 \left(4x - 2x^2\right) dx = C\left[2x^2 - \frac{2}{3}x^3\right]_0^2 = C \cdot \frac{8}{3}$$

and, therefore, $C = 3/8$. Also,

$$P[X > 1] = \int_1^{\infty} f_X(x)\, dx = C\int_1^2 \left(4x - 2x^2\right) dx = C\left[2x^2 - \frac{2}{3}x^3\right]_1^2 =$$

$$= C \cdot \left(\frac{8}{3} - \frac{4}{3}\right) = \frac{3}{8} \cdot \frac{4}{3} = \frac{1}{2}$$

- The lifetime of a drive in months is a random variable with pdf

$$f_X(x) = \begin{cases} \lambda\, e^{-x/100} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

for some constant λ. What is the probability that it functions for between 50 and 150 months?

- *Solution:* The pdf of X must integrate to 1, which suggests that

$$1 = \int_{-\infty}^{\infty} f_X(x)dx = \lambda \int_0^{\infty} e^{-x/100} dx = \lambda \left[-100\, e^{-x/100} \right]_0^{\infty} = \lambda(0-(-100))$$

and, therefore, $\lambda = 0.01$. Consequently,

$$P[50 < X < 150] = \int_{50}^{150} f_X(x)dx = \int_{50}^{150} \frac{1}{100} e^{-x/100} dx =$$

$$= e^{-1/2} - e^{-3/2} \approx 0.383$$

Yet another example

- Let X have pdf $f_X(x)$, and $Y = 2X$. What is $f_Y(y)$?
- *Solution:*
$$F_Y(a) = P[Y \le a] = P[2X \le a] = P[X \le \frac{a}{2}] = F_X\left(\frac{a}{2}\right)$$
and
$$f_Y(a) = \frac{d}{da} F_X\left(\frac{a}{2}\right) = f_X\left(\frac{a}{2}\right) \cdot \frac{1}{2}$$
- Note that
$$\int_{-\infty}^{\infty} f_Y(u)\, du = \int_{-\infty}^{\infty} \frac{1}{2} f_X\left(\frac{u}{2}\right)\, du = \int_{-\infty}^{\infty} f_X(v)\, dv = 1$$
where we used $v = u/2$, with $dv = du/2$.

- For a continuous random variable X, we define its *expectation* as

$$E[X] = \int_{-\infty}^{\infty} x f_X(x) dx$$

- For instance, if $f_X(x)$ is given by

$$f_X(x) = \begin{cases} 2x & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

then

$$E[X] = \int_{-\infty}^{\infty} x f_X(x) dx = \int_0^1 2x^2 dx = \frac{2}{3}$$

Example

- Let X have pdf

$$f_X(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

  Find $E[e^X]$.

- *Solution:* Let $Y = e^X$. Lets determine $f_Y(y)$ by first determining the CDF $F_Y(y)$. Since X ranges from 0 to 1, $Y = e^X$ ranges from 1 to e. So, for $1 \le y \le e$, we have

$$F_Y(y) = P[Y \le y] = P[e^X \le y] = P[X \le \ln y] = \int_0^{\ln y} f_X(x)\,dx = \ln y$$

- Consequently, $f_Y(y) = dF_Y(y)/dy = 1/y$, for $1 \le y \le e$.

- Note that Y cannot take values outside this interval, so outside this interval $f_Y(y) = 0$. Hence,

$$E[Y] = \int_{-\infty}^{\infty} y f_Y(y)\,dy = \int_1^e y \cdot \frac{1}{y}\,dy = e - 1$$

Expectation of Continuous RV (cont.)

- For a continuous random variable X, we define

$$\boxed{E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx}$$

Thus, in the previous example,

$$E[e^X] = \int_{-\infty}^{\infty} e^x f_X(x)\, dx = \int_0^1 e^x\, dx = e - 1$$

- If X is a non-negative random variable, then

$$\boxed{E[X] = \int_0^{\infty} P[X > x] dx}$$

Indeed, we have

$$\int_0^{\infty} P[Y > y] dy = \int_0^{\infty} \left[\int_y^{\infty} f_Y(u) du \right] dy = \int_0^{\infty} \int_y^{\infty} f_Y(u) du dy =$$

$$= \int_0^{\infty} \int_0^{u} f_Y(u) dy du = \int_0^{\infty} \left[\int_0^{u} dy \right] f_Y(u) du = \int_0^{\infty} u f_Y(u) du = E[Y]$$

- For a continuous random variable X, we have

$$\boxed{E[aX + b] = aE[X] + b}$$

Indeed,

$$E[aX + b] = \int_{-\infty}^{\infty} (ax + b) f_X(x) dx = a \int_{-\infty}^{\infty} x f_X(x) dx +$$

$$+ b \int_{-\infty}^{\infty} f_X(x) dx = aE[X] + b$$

- For the variance $\sigma^2 = Var(X)$ of X, one has

$$\boxed{Var[aX + b] = a^2 Var[X]}$$

with

$$\sigma^2 = Var[X] = E[(X - E[X])^2] = E[X^2] - (E[X])^2$$

- Find $Var[X]$ if

$$f_X(x) = \begin{cases} 2x & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

- *Solution:* We have already seen that, in this case, $\mu = 2/3$. Thus,

$$Var[X] = E\left[(X - 2/3)^2\right] = E\left[X^2 - \frac{4}{3}X + \frac{4}{9}\right] =$$

$$= \int_0^1 \left(x^2 - \frac{4}{3}x + \frac{4}{9}\right) f_X(x)\, dx = \int_0^1 \left(x^2 - \frac{4}{3}x + \frac{4}{9}\right) 2x\, dx =$$

$$= \int_0^1 \left(2x^3 - \frac{8}{3}x^2 + \frac{8}{9}x\right) dx = \left[\frac{1}{2}x^4 - \frac{8}{9}x^3 + \frac{4}{9}x^2\right]_0^1 =$$

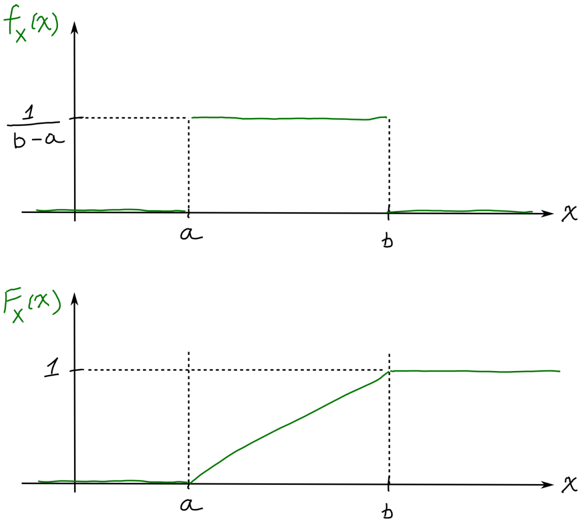
$$= \frac{1}{2} - \frac{8}{9} + \frac{4}{9} = \frac{1}{18}$$

- **Uniform random variables:** We say X is *uniform* on the interval (a, b), denoted $X \sim U(a, b)$, if

$$f_X(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \text{else} \end{cases}$$

- In this case, the CDF of X is given by

$$F_X(x) = \begin{cases} 0 & x \leq a \\ \frac{x}{b-a} - \frac{a}{b-a} & a \leq x \leq b \\ 1 & b \leq x \end{cases}$$



Note that if X has units of kg, then a and b have units of kg, and $1/(b-a)$ has units kg^{-1}.

Example

- Let $Y \sim U(0, 10)$. Find $P[Y < 3]$ and $P[6 < Y < 12]$.
- *Solution:* We are given

$$f_Y(y) = \begin{cases} \frac{1}{10} & 0 < y < 10 \\ 0 & \text{else} \end{cases}$$

- Consequently,

$$P[Y < 3] = \int_{-\infty}^{3} f_Y(y)dy = \int_0^3 \frac{1}{10}dy = \frac{3}{10}$$

and

$$P[6 < Y < 12] = \int_6^{12} f_Y(y)dy = \int_6^{10} \frac{1}{10}dy = \frac{4}{10}$$

- Let $X \sim U(a,b)$. Find $E[X]$ and $Var[X]$.

- *Solution:* We are given

$$f_X(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \text{else} \end{cases}$$

- Consequently,

$$E[X] = \int_{-\infty}^{\infty} x f_X(x) dx = \int_a^b \frac{x}{b-a} dx = \frac{1}{2} \frac{b^2 - a^2}{b-a} = \frac{a+b}{2}$$

and

$$Var[X] = E[X^2] - (E[X])^2 = \int_{-\infty}^{\infty} x^2 f_X(x) dx - (E[X])^2 =$$

$$= \int_a^b \frac{x^2}{b-a} dx - \left(\frac{a+b}{2}\right)^2 = \frac{1}{3} \frac{b^3 - a^3}{b-a} - \left(\frac{a+b}{2}\right)^2 =$$

$$= \frac{1}{3}(b^2 + ab + a^2) - \left(\frac{a+b}{2}\right)^2 = \frac{1}{12}(b-a)^2$$

- Buses arrive at a stop at 7:00, 7:15 and 7:30. If a person arrives between 7:00 and 7:30 uniformly, what is probability that they wait less than 5 minutes?
- *Solution:* Let X be # minutes past 7:00 that person arrives at stop. Then $X \sim U(0, 30)$.
- Then we have

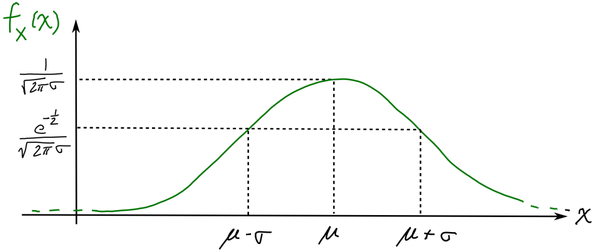
$$P[\text{wait less than 5 min}] = P[\{10 < X < 15\} \cup \{25 < X < 30\}] =$$

$$= \int_{10}^{15} \frac{1}{30} dx + \int_{25}^{30} \frac{1}{30} dx = 1/3$$

- X is said to be **normal** (or *normally distributed*) with parameters μ and σ^2 if

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

This is denoted $X \sim \mathcal{N}(\mu, \sigma^2)$.



- The area under $f_X(x)$ can be verified to be equal to 1.

- Note that, since X has units kg, then μ has units of kg as well. Thus, $(x-\mu)^2/2\sigma^2$ must be unit-less for the exponential to make sense.

Proposition: If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $Y = aX + b$ is $\mathcal{N}(a\mu + b, a^2\sigma^2)$.

- To see that, assume $a > 0$ ($a < 0$ is similar).

$$F_Y(u) = P[Y \leq u] = P[aX + b \leq u] =$$

$$= P[X \leq (u - b)/a] = F_X\left(\frac{u-b}{a}\right)$$

- Consequently,

$$f_Y(u) = \frac{d}{du}F_Y(u) = \frac{d}{du}F_X\left(\frac{u-b}{a}\right) = f_X\left(\frac{u-b}{a}\right) \cdot \frac{1}{a} =$$

$$= \frac{1}{\sqrt{2\pi}a\sigma}\exp\left(-\frac{\left(\frac{u-b}{a} - \mu\right)^2}{2\sigma^2}\right) = \frac{1}{\sqrt{2\pi}a\sigma}\exp\left(-\frac{(u - b - a\mu)^2}{2(a\sigma)^2}\right)$$

Therefore, $Y \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$.

Example

- Let $X \sim \mathcal{N}(\mu, \sigma^2)$. Find the distribution of $Z = (X - \mu)/\sigma$.

- *Solution:*
$$Z = \frac{X - \mu}{\sigma} = \frac{1}{\sigma}X - \frac{\mu}{\sigma}$$
Substituting $a = 1/\sigma$ and $b = -\mu/\sigma$ into the previous proposition, we obtain
$$Z \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$$
with $a\mu + b = 0$ and $a^2\sigma^2 = 1$. Thus, $Z \sim \mathcal{N}(0, 1)$.

- **Definition:** A random variable that is $\mathcal{N}(0, 1)$ is called a **standard normal** or **standard Gaussian** random variable.

Example

- Let $X \sim \mathcal{N}(\mu, \sigma^2)$. Find $E[X]$ and $Var[X]$.
- *Solution:* Let $Z = (X - \mu)/\sigma$. Then $Z \sim \mathcal{N}(0, 1)$ and $X = \sigma Z + \mu$.
- Consequently, we have

$$E[Z] = \int_{-\infty}^{\infty} z f_Z(z) dz = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z e^{-z^2/2} dz = \frac{-1}{\sqrt{2\pi}} \left. e^{-z^2/2} \right|_{-\infty}^{\infty} = 0$$

and

$$Var[Z] = E[Z^2] - (E[Z])^2 = E[Z^2] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2 e^{-z^2/2} dz =$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{z}_{u} \cdot \underbrace{z e^{-z^2/2} dz}_{dv} = \frac{1}{\sqrt{2\pi}} \left[uv|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} v du \right] =$$

$$= \frac{1}{\sqrt{2\pi}} \left[-z e^{-z^2/2} \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} -e^{-z^2/2} dz \right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2} dz = 1$$

- Thus,

$$E[X] = E[\sigma Z + \mu] = \sigma E[Z] + \mu = \mu$$
$$Var[X] = Var[\sigma Z + \mu] = \sigma^2 Var[Z] = \sigma^2$$

CDF of Normal Random Variables

- For a $\mathcal{N}(0,1)$ distribution, its CDF is defined as

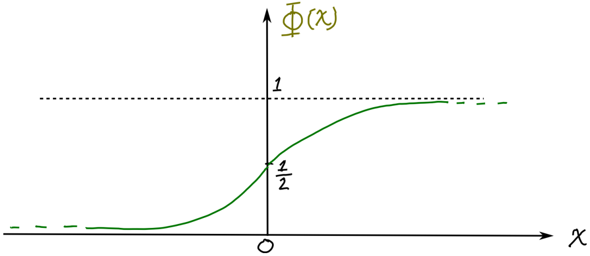
$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du$$

- It is also customary to defined its *Q-function* as

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-u^2/2} du$$

- Note that

$$\boxed{\Phi(-x) = 1 - \Phi(x) = Q(x)}$$



- There is also the so-called *error function* that is defined as

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-v^2} dv = \frac{\sqrt{2}}{\sqrt{\pi}} \int_0^{\sqrt{2}x} e^{-u^2/2} du = 2\left[\frac{1}{\sqrt{2\pi}} \int_0^{\sqrt{2}x} e^{-u^2/2} du\right] =$$

$$= 2\left[-\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{-u^2/2} du + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\sqrt{2}x} e^{-u^2/2} du\right] =$$

$$= 2\left[-\frac{1}{2} + \Phi(\sqrt{2}x)\right] = 2\Phi(\sqrt{2}x) - 1$$

Table of $\Phi(x)$

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.50000	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.52790	0.53188	0.53586
0.1	0.53983	0.54380	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.62930	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.65910	0.66276	0.66640	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.70540	0.70884	0.71226	0.71566	0.71904	0.72240
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.75490
0.7	0.75804	0.76115	0.76424	0.76730	0.77035	0.77337	0.77637	0.77935	0.78230	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.86650	0.86864	0.87076	0.87286	0.87493	0.87698	0.87900	0.88100	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.90320	0.90490	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.92220	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.94520	0.94630	0.94738	0.94845	0.94950	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.96080	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.97320	0.97381	0.97441	0.97500	0.97558	0.97615	0.97670
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.98030	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.98300	0.98341	0.98382	0.98422	0.98461	0.98500	0.98537	0.98574
2.2	0.98610	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.98840	0.98870	0.98899
2.3	0.98928	0.98956	0.98983	0.99010	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.99180	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.99430	0.99446	0.99461	0.99477	0.99492	0.99506	0.99520
2.6	0.99534	0.99547	0.99560	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.99720	0.99728	0.99736
2.8	0.99744	0.99752	0.99760	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.99900
3.1	0.99903	0.99906	0.99910	0.99913	0.99916	0.99918	0.99921	0.99924	0.99926	0.99929
3.2	0.99931	0.99934	0.99936	0.99938	0.99940	0.99942	0.99944	0.99946	0.99948	0.99950
3.3	0.99952	0.99953	0.99955	0.99957	0.99958	0.99960	0.99961	0.99962	0.99964	0.99965
3.4	0.99966	0.99968	0.99969	0.99970	0.99971	0.99972	0.99973	0.99974	0.99975	0.99976

Use of Standardization

- What if we want the CDF of a Gaussian random variable other than $\mathcal{N}(0,1)$?
- These can be expressed in terms of $\Phi(\cdot)$.
- Indeed, we already know that if $X \sim \mathcal{N}(\mu, \sigma^2)$, then

$$Z = \frac{X-\mu}{\sigma}$$

  is $\mathcal{N}(0,1)$.

- Consequently,

$$F_X(a) = P[X \leq a] = P\left[\frac{X-\mu}{\sigma} \leq \frac{a-\mu}{\sigma}\right] =$$

$$= P\left[Z \leq \frac{a-\mu}{\sigma}\right] = \Phi\left(\frac{a-\mu}{\sigma}\right)$$

Example

- Let $X \sim \mathcal{N}(3, 9)$. Compute $P[2 < X < 5]$.
- *Solution:*

$$P[2 < X < 5] = P\left[\frac{2-3}{3} < \frac{X-3}{3} < \frac{5-3}{3}\right] = P\left[-\frac{1}{3} < Z < \frac{2}{3}\right] =$$

$$= \Phi\left(\frac{2}{3}\right) - \Phi\left(-\frac{1}{3}\right) = \Phi\left(\frac{2}{3}\right) - \left[1 - \Phi\left(\frac{1}{3}\right)\right] \approx 0.37807$$

(The nearest values from the table give 0.37787.)

Another example

- If test grades in a course follow an $\mathcal{N}(\mu, \sigma^2)$ distribution, what is the probability that a random student is at least one σ above the mean μ?
- *Solution:* Let $X \sim \mathcal{N}(\mu, \sigma^2)$. Then

$$P[X > \mu + \sigma] = P\left[\frac{X - \mu}{\sigma} > \frac{\mu + \sigma - \mu}{\sigma}\right] =$$

$$= P\left[Z > 1\right] = 1 - \Phi(1) \approx 0.15866$$

- Although not asked, in addition we have

$$P[X < \mu - \sigma] \approx 0.15866$$

$$P[\mu < X < \mu + \sigma] \approx 0.34134$$

$$P[X > \mu + 2\sigma] \approx 0.02275$$

Yet another example

- In finance, the Value At Risk (VAR) of an investment is the value v such that there is only a 1% chance that the investment will lose more than $v > 0$.

- The gain from an investment is $X \sim \mathcal{N}(\mu, \sigma^2)$, what is the VAR?

- *Solution:* We are looking $v > 0$ such that $P[X < -v] = 0.01$. Then there is a 1% chance we will lose more than v.

$$0.01 = P[X < -v] = P\left[\frac{X-\mu}{\sigma} < \frac{-v-\mu}{\sigma}\right] = P\left[Z < \frac{-v-\mu}{\sigma}\right]$$

- Note that if $Z \sim \mathcal{N}(0,1)$ and $a > 0$ is such that $P[Z < -a] = 0.01$, then $P[Z > a] = 0.01$ or, equivalently, $P[Z \leq a] = 0.99 = \Phi(a)$. So, $a = \Phi^{-1}(0.99) = 2.33$.

- Finally, setting $(-v-\mu)/\sigma$ to $-a = -2.33$, we get

$$v = 2.33\sigma - \mu$$

- **Definition:** A random variable X with pdf

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{else} \end{cases}$$

is called **exponential** with parameter $\lambda > 0$ and denoted $X \sim \mathrm{Exp}(\lambda)$.

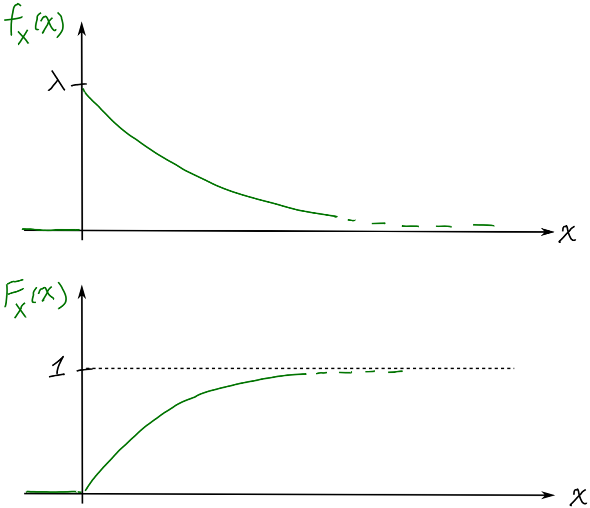
- Note that if X has units of kg, then λx must be unit-less for $\exp(-\lambda x)$ to make sense, suggesting that λ has units kg^{-1}.

- The CDF of an exponential X is given by

$$F_X(a) = \int_{-\infty}^{a} f_X(u)du = \begin{cases} 1 - e^{-\lambda a} & a \geq 0 \\ 0 & a < 0 \end{cases}$$

with $F_X(\infty) = 1$.

Exponential distribution



- What are $E[X]$ and $Var[X]$ of $X \sim \text{Exp}(\lambda)$?

$$E[X^n] = \int_{-\infty}^{\infty} x^n f_X(x)dx = \int_0^{\infty} x^n \lambda e^{-\lambda x} dx =$$

$$= \left[uv|_0^{\infty} - \int_0^{\infty} vdu \right] = \left[-x^n e^{-\lambda x}\Big|_0^{\infty} - \int_0^{\infty} -e^{-\lambda x} n x^{n-1} dx \right] =$$

$$= \frac{n}{\lambda} \int_0^{\infty} x^{n-1} \lambda e^{-\lambda x} dx = \frac{n}{\lambda} E[X^{n-1}]$$

- Accordingly, since $E[X^0] = E[1] = 1$, we have

$$E[X] = \frac{1}{\lambda} E[X^0] = \frac{1}{\lambda}$$

$$E[X^2] = \frac{2}{\lambda} E[X^1] = \frac{2}{\lambda^2}$$

- Hence

$$Var[X] = E[X^2] - (E[X])^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$$

- Say the length of time someone uses an ATM machine is an exponential random variable with $\lambda = 1/3\ \text{min}^{-1}$.
- If someone arrives at the ATM just before you, what is the probability that you wait more than 3 min?

$$P[X > 3] = 1 - F_X(3) = \exp(-3\lambda) = \exp(-1) \approx 0.36788$$

- And what is the probability that you wait between 3 and 6 min?

$$P[3 < X < 6] = F_X(6) - F_X(3) = \exp(-1) - \exp(-2) \approx 0.23254$$

- **Definition:** A non-negative random variable X is called **memoryless** if for all $s > 0$ and all $t > 0$

$$P[X > s + t \mid X > t] = P[X > s]$$

- *In words:* The probability of waiting s seconds more given you have already waited t seconds is the same as waiting s seconds from the start. In other words, no matter how long you have waited, time to wait still has the same distribution.

- Does $\mathrm{Exp}(\lambda)$ have the memoryless property? Let $X \sim \mathrm{Exp}(\lambda)$. Then

$$P[X > s + t \mid X > t] = \frac{P[X > s + t, X > t]}{P[X > t]} = \frac{P[X > s + t]}{P[X > t]} =$$

$$= \frac{e^{-\lambda(s+t)}}{e^{-\lambda t}} = e^{-\lambda s} = P[X > s]$$

- So, yes – $\mathrm{Exp}(\lambda)$ has the memoryless property.

Example

- Two people are being served by two tellers at a bank. A third person arrives, and must wait for one of the two tellers. If service times are exponential with the same parameter λ, what is the probability that the third person is the last to leave?

- *Solution:* The third person starts being served as soon as one of the initial two is finished.

  Once this happens, the time to go for remaining person and the third person has the same distribution $\mathrm{Exp}(\lambda)$ due to memoryless property.

  By symmetry, each has a probability $1/2$ of finishing last.

Another example

- A car battery has a lifetime that is exponentially distributed with the mean of $10,000$ km.

- What is the probability of completing a 5000 km trip without replacing the battery?

- *Solution:* Let $X \sim \text{Exp}(\lambda)$ with $\lambda = 10^4$. Also, let d be the number of km that battery has been operating for so far.

  Since battery has operated for d km so far, we have

$$P[X > 5000 + d \mid X > d] = P[X > 5000] = 1 - F_X(5000) =$$

$$= \exp(-5000/10000) \approx 0.607$$

- Had the distribution *not* been exponential, then we'd have had

$$P[X > d + 5000 \mid X > d] = \frac{P[X > d + 5000,\ X > d]}{P[X > d]} =$$

$$= \frac{P[X > d + 5000]}{P[X > d]} = \frac{1 - F_X(d + 5000)}{1 - F_X(d)}$$

Distribution of a function of a random variable

- Given a random variable X and $Y = g(X)$, we want to find the distribution of Y.

- According to our two-step approach, we first need to calculate

$$F_Y(y) = P[g(X) \leq y]$$

and then differentiate to compute

$$f_Y(y) = \frac{d}{dy} F_Y(y)$$

Example

- Let $X \sim U(0,1)$ and $Y = \sqrt{X}$. Find $F_Y(y)$ and $f_Y(y)$.

- *Solution:* For $y \geq 0$, we have

$$F_Y(y) = P[Y \leq y] = P[\sqrt{X} \leq y] = P[X \leq y^2] = F_X(y^2) =$$

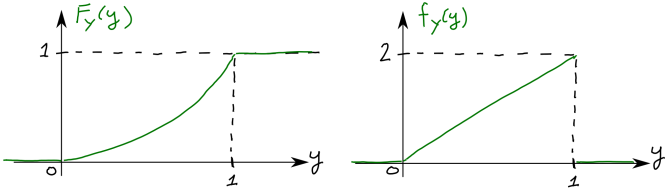
$$= \begin{cases} y^2 & \text{for } 0 \leq y \leq 1 \\ 1 & \text{for } 1 < y \end{cases}$$

- Since Y cannot be negative, $F_Y(y) = 0$ for $y < 0$. Therefore,

$$F_Y(y) = \begin{cases} 0 & \text{for } y < 0 \\ y^2 & \text{for } 0 \leq y \leq 1 \\ 1 & \text{for } 1 < y \end{cases}$$

- Finally, differentiating w. r. t. y, we get

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \begin{cases} 0 & \text{for } y < 0 \\ 2y & \text{for } 0 \le y \le 1 \\ 0 & \text{for } 1 < y \end{cases}$$



Another example

- Let $Y = X^2$. What is $f_Y(y)$ in terms of $f_X(x)$?

- *Solution:* For $y \geq 0$, we have

$$F_Y(y) = P[Y \leq y] = P[X^2 \leq y] = P[-\sqrt{y} \leq X \leq \sqrt{y}] =$$

$$= F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

- Consequently,

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} F_X(\sqrt{y}) - \frac{d}{dy} F_X(-\sqrt{y}) =$$

$$= f_X(\sqrt{y}) \frac{1}{2\sqrt{y}} - f_X(-\sqrt{y}) \frac{-1}{2\sqrt{y}} = \frac{1}{2\sqrt{y}} \left(f_X(\sqrt{y}) + f_X(-\sqrt{y}) \right)$$

- For $y < 0$, we have

$$F_Y(y) = P[X^2 \leq y] = 0$$

and, therefore, $f_Y(y) = 0$

More examples

- Let $Y = |X|$. What is $f_Y(y)$ in terms of $f_X(x)$?
- *Solution:* For $y \geq 0$, we have

$$F_Y(y) = P[Y \leq y] = P[|X| \leq y] = P[-y \leq X \leq y] =$$

$$= F_X(y) - F_X(-y)$$

- Hence,

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} F_X(y) - \frac{d}{dy} F_X(-y) =$$

$$= f_X(y) + f_X(-y)$$

- For $y < 0$, $P[|X| \leq y] = 0$, and so $f_Y(y) = 0$.

Final (important) example

- Let $Y = aX + b$. What is $f_Y(y)$ in terms of $f_X(x)$?
- *Solution:* If $a > 0$, we have

$$F_Y(y) = P[Y \le y] = P[aX + b \le y] = P\left[X \le \frac{y-b}{a}\right] = F_X\left(\frac{y-b}{a}\right)$$

and hence

$$f_Y(y) = \frac{d}{dy}F_Y(y) = \frac{d}{dy}F_X\left(\frac{y-b}{a}\right) = \frac{1}{a}f_X\left(\frac{y-b}{a}\right)$$

- On the other hand, if $a < 0$, we have

$$F_Y(y) = P[Y \le y] = P[aX+b \le y] = P\left[X \ge \frac{y-b}{a}\right] = 1-P\left[X < \frac{y-b}{a}\right]$$

$$= 1 - P\left[X \le \frac{y-b}{a}\right] = 1 - F_X\left(\frac{y-b}{a}\right)$$

and hence

$$f_Y(y) = \frac{d}{dy}F_Y(y) = \frac{d}{dy}\left(1 - F_X\left(\frac{y-b}{a}\right)\right) = -\frac{1}{a}f_X\left(\frac{y-b}{a}\right)$$

- Since a was negative in the second case, both cases can be combined into a single expression:

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

which makes sense, since the probability density cannot be negative!

- To summarize: if $X \sim f_X(x)$, then $Y = aX + b$ is distributed with

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

- **Theorem:** Let X be a continuous random variable with pdf $f_X(x)$. Let $g(x)$ be differentiable and either strictly increasing or strictly decreasing. Then $Y = g(X)$ has pdf

$$f_Y(y) = \begin{cases} f_X\left(g^{-1}(y)\right)\left|\frac{d}{dy}g^{-1}(y)\right|, & \text{if } y = g(x) \text{ for some } x \\ 0, & \text{else} \end{cases}$$

- To understand how it works, consider the case that $g(x)$ is strictly increasing. Say $y = g(x)$ for some x. Then

$$F_Y(y) = P[g(X) \leq y] = P[X \leq g^{-1}(y)] = F_X(g^{-1}(y))$$

and hence

$$f_Y(y) = \frac{d}{dy}F_Y(y) = \frac{d}{dy}F_X(g^{-1}(y)) = f_X(g^{-1}(y))\frac{d}{dy}g^{-1}(y)$$

- If there is no x such that $y = g(x)$, then either:
  1. y is less than all possible values of $g(x)$
  2. y is greater than all possible values of $g(x)$
- Then, $P[g(X) \leq y]$ is either 0 or 1.
- Either way, $f_Y(y) = 0$.