ECE 203 – Section 5
Jointly distributed random variables

- Joint cumulative distribution functions and joint probability densities
- Marginalization of discrete and continuous distributions
- Independence of continuous random variables

The slides have been prepared based on the lecture notes of Prof. Patrick Mitran.

- So far, we only considered the distribution of a single random variable.
- Sometimes we want the probability of a statement involving two or more random variables.
- For example:

1. $P[X < 3, Y > 7]$
2. $P[X < Y]$
3. $P[X^2 + Y^2 < 10]$
4. $P[XY = 3]$

For this, we need the **joint cumulative distribution function**, or **joint CDF**

$$F_{XY}(a, b) = P[X \leq a, Y \leq b]$$

which allows finding *all* probability statements involving X and Y.

Example

- For $a_1 < a_2$ and $b_1 < b_2$,

$$P[a_1 < X \le a_2, b_1 < Y \le b_2] =$$
$$= F_{XY}(a_2, b_2) + F_{XY}(a_1, b_1) - F_{XY}(a_1, b_2) - F_{XY}(a_2, b_1)$$

- To see that, we first note that

$$\underbrace{P[X \le a_2, Y \le b]}_{F_{XY}(a_2,b)} = \underbrace{P[X \le a_1, Y \le b]}_{F_{XY}(a_1,b)} + P[a_1 < X \le a_2, Y \le b]$$

and, therefore,

$$P[a_1 < X \le a_2, Y \le b] = F_{XY}(a_2, b) - F_{XY}(a_1, b)$$

Examples (cont.)

- Next, we notice that

$$P[a_1 < X \leq a_2, Y \leq b_2] =$$

$$= P[a_1 < X \leq a_2, Y \leq b_1] + P[a_1 < X \leq a_2, b_1 < Y \leq b_2]$$

  and, hence,

$$P[a_1 < X \leq a_2, b_1 < Y \leq b_2] =$$

$$= P[a_1 < X \leq a_2, Y \leq b_2] - P[a_1 < X \leq a_2, Y \leq b_1]$$

- Combining the above two results yields

$$P[a_1 < X \leq a_2,\ b_1 < Y \leq b_2] =$$

$$= F_{XY}(a_2, b_2) - F_{XY}(a_1, b_2) - (\ F_{XY}(a_2, b_1) - F_{XY}(a_1, b_1)\ ) =$$

$$= F_{XY}(a_2, b_2) + F_{XY}(a_1, b_1) - F_{XY}(a_1, b_2) - F_{XY}(a_2, b_1)$$

Discrete case

- Say X and Y both be discrete, with
  - X takes values in $\mathcal{X} = \{x_1, x_2, \ldots\}$,
  - Y takes values in $\mathcal{Y} = \{y_1, y_2, \ldots\}$.

- We define the **joint probability mass function**, or **joint pmf** as

$$p_{XY}(x, y) = P[X = x, Y = y]$$

- Consequently,

$$p_X(x) = P[X = x] = P[\cup_j \{X = x, Y = y_j\}] =$$

$$= \sum_j P[X = x, Y = y_j] = \sum_j p_{XY}(x, y_j)$$

- Likewise,

$$p_Y(y) = \sum_i p_{XY}(x_i, y)$$

- When computing $p_X(x)$ from $p_{XY}(x, y)$, we call this computing the **marginal** pmf for X, and the process is called **marginalization**.

Notes

- f we list $p_{XY}(x_i, y_j)$ in a table on a piece of paper, then the sum over j is summing each column of the table, and writing each sum at the bottom of the column, in the margin of the page.
- Also,

$$1 = P[X \in \mathcal{X}, Y \in \mathcal{Y}] = P[\cup_{i,j}\{X = x_i, Y = y_j\}] =$$

$$= \sum_{i,j} P[X = x_i, Y = y_j] = \sum_{i,j} p_{XY}(x_i, y_j)$$

So, joint pmf must sum to 1.

Example

- An urn contains 3 red, 4 white, and 5 blue balls. 3 balls are picked at random.
- Let $X = \#$ white balls, $Y = \#$ red balls.
- To find $p_{XY}(i, j)$, we notice that there are $\binom{12}{3}$ ways of picking 3 of 12 balls.
- If $X = i$ and $Y = j$, then $\#$ blue balls is $3 - i - j$. There are:
  - $\binom{3}{i}$ ways of picking i red balls from 3 red balls,
  - $\binom{4}{j}$ ways of picking j white balls from 4 white balls,
  - $\binom{5}{3-i-j}$ ways of picking $3 - i - j$ blue balls from 5 blue balls.
- Therefore,

$$p_{XY}(i, j) = \frac{\binom{3}{i}\binom{4}{j}\binom{5}{3-i-j}}{\binom{12}{3}}$$

Continuous case

- We say that X and Y are continuous random variables if there exists a non-negative $f_{XY}(x,y)$ such that for every event $C \subset \mathbb{R}^2$

$$P[(X,Y) \in C] = \iint\limits_C f_{XY}(x,y)dxdy$$

$f_{XY}(x,y)$ is called the **joint probability density function**, or **joint pdf**.

- Since $P[X \in A, Y \in B] = P[(X,Y) \in \underbrace{A \times B}_{C}]$, then

$$P[X \in A, Y \in B] = \iint_{A \times B} f_{XY}(x,y)dxdy = \int_B \int_A f_{XY}(x,y)dxdy$$

- Also,

$$F_{XY}(a,b) = P[X \leq a,\ Y \leq b] = P[X \in (-\infty, a], Y \in (-\infty, b]] =$$

$$= \int_{-\infty}^{b} \int_{-\infty}^{a} f_{XY}(x,y)\ dxdy$$

- If we take partial derivatives with respect to a and b, we obtain

$$\boxed{f_{XY}(a,b) = \frac{\partial^2}{\partial a \partial b} F_{XY}(a,b)}$$

- Also,

$$\int_A f_X(x)dx = P[X \in A] = P[X \in A, Y \in (-\infty, \infty)] =$$

$$= \int_A \int_{-\infty}^{\infty} f_{XY}(x,y)\, dydx$$

and so we can *marginalize* to obtain

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)dy$$

- Similarly,

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)dx$$

Continuous case (cont.)

- We also have

$$1 = P[X \in (-\infty, \infty), Y \in (-\infty, \infty)] =$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x, y) dx dy$$

- Thus, for a joint pdf, volume under the surface is equal to 1.

Example

- The joint pdf of X and Y is given by

$$f_{XY}(x,y) = \begin{cases} 2e^{-x}e^{-2y}, & x > 0 \text{ and } y > 0 \\ 0, & \text{otherwise} \end{cases}$$

  Compute

  1. $P[X > 1, Y < 1]$
  2. $P[X < Y]$
  3. $P[X < a]$

- *Solution (1):*

$$P[X > 1, Y < 1] = P[X \in (1,\infty), Y \in (-\infty, 1)] =$$

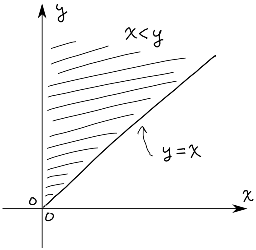
$$= \int_{-\infty}^{1}\int_{1}^{\infty} f_{XY}(x,y)\,dxdy = \int_{0}^{1}\int_{1}^{\infty} 2e^{-x}e^{-2y}\,dxdy =$$

$$= \int_{0}^{1} \left[-2e^{-x}e^{-2y}\right]_{x=1}^{x=\infty} dy = \int_{0}^{1} 2e^{-1}e^{-2y}\,dy =$$

$$= \left[-e^{-1}e^{-2y}\right]_{y=0}^{y=1} = e^{-1} - e^{-3}$$

- *Solution (2):*

$$P[X<Y]=\iint_{(x,y):x<y} f_{XY}(x,y)dxdy=\iint_{\substack{(x,y):x<y\\x>0\\y>0}} 2e^{-x}e^{-2y}dxdy$$



and, hence,

$$P[X<Y]=\int_0^\infty\int_0^y 2e^{-x}e^{-2y}dxdy=\int_0^\infty\left[-2e^{-x}e^{-2y}\right]_{x=0}^{x=y}dy=$$

$$=\int_0^\infty 2e^{-2y}-2e^{-3y}dy=\left[-e^{-2y}+\frac{2}{3}e^{-3y}\right]_0^\infty=1-\frac{2}{3}=\frac{1}{3}$$

- *Solution (3):*

$$P[X < a] = P[X \in (-\infty, a), Y \in (-\infty, \infty)] =$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{a} f_{XY}(x, y) dx dy = \int_{0}^{\infty} \int_{0}^{a} 2e^{-x} e^{-2y} dx dy =$$

$$= \int_{0}^{\infty} \left[-2e^{-x} e^{-2y}\right]_{x=0}^{x=a} dy = \int_{0}^{\infty} 2(1 - e^{-a}) e^{-2y} dy =$$

$$= -(1 - e^{-a}) e^{-2y} \Big|_{y=0}^{y=\infty} = (1 - e^{-a})$$

- Given $R > 0$, consider the joint pdf

$$f_{XY}(x,y) = \begin{cases} c, & \text{if } x^2 + y^2 \leq R^2 \\ 0, & \text{otherwise} \end{cases}$$

for some $c > 0$.

1. Find c.
2. Find the marginal pdfs of X and Y.
3. Let $D = \sqrt{X^2 + Y^2}$ be the distance of the pair (X, Y) from the origin. Find $P[D \leq a]$.
4. Find $E[D]$.

- *Note:* The pdf is the constant c on a disk of radius R, and 0 otherwise. This is called a *uniform distribution on the disk of radius R*.

Another example (cont.)

- *Solution (1):*

$$1 = \iint_{\mathbf{R}^2} f_{XY}(x,y)dxdy = \iint_{x^2+y^2\leq R^2} c\;dxdy = c \iint_{x^2+y^2\leq R^2} 1\;dxdy = c\cdot\pi R^2$$

So, $c = 1/\pi R^2$.

- *Solution (2):*

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)dy = \int_{y:x^2+y^2\leq R^2} c\;dy = \int_{y:y^2\leq R^2-x^2} c\;dy =$$

$$= \int_{-b}^{b} c\;dy = 2bc = \frac{2}{\pi R^2}\sqrt{R^2-x^2}$$

with $b = \sqrt{R^2-x^2}$ and assuming $x^2 \leq R^2$.

- If $x^2 > R^2$, then the set $\{y : y^2 \le R^2 - x^2\}$ is empty and the above integral is 0. Thus,

$$f_X(x) = \begin{cases} \frac{2}{\pi R^2}\sqrt{R^2 - x^2}, & x^2 \le R^2 \\ 0, & \text{otherwise} \end{cases}$$

- Similarly,

$$f_Y(y) = \begin{cases} \frac{2}{\pi R^2}\sqrt{R^2 - y^2}, & y^2 \le R^2 \\ 0, & \text{otherwise} \end{cases}$$

- *Solution (3):* Assuming $0 \le a \le R$,

$$P[D \le a] = P[X^2 + Y^2 \le a^2] = \iint\limits_{x^2+y^2 \le a^2} f_{XY}(x,y)\, dxdy =$$

$$= \iint\limits_{x^2+y^2 \le a^2} c\, dxdy = c \cdot \pi a^2 = \frac{a^2}{R^2}$$

- If $a > R$, then since $X^2 + Y^2$ cannot be larger than R^2, we have $P[D \le a] = 1$. Formally,

$$P[D \le a] = P[X^2 + Y^2 \le a^2] = \iint\limits_{x^2+y^2 \le a^2} f_{XY}(x,y)\, dxdy =$$

$$= \iint\limits_{x^2+y^2 \le R^2} c\, dxdy = c \cdot \pi R^2 = 1$$

- If $a < 0$, then since a distance cannot be negative, $P[D \le a] = 0$.

Another example (cont.)

- *Solution (4):* The pdf of D for $0 \leq a \leq R$ is

$$f_D(a) = \frac{d}{da}\frac{a^2}{R^2} = \frac{2a}{R^2}$$

and 0 otherwise. Therefore

$$E[D] = \int_{-\infty}^{\infty} a f_D(a) da = \int_0^R \frac{2a^2}{R^2} da = \frac{2R}{3}$$

Final example

- The joint pdf of X and Y is

$$f_{XY}(x,y) = \begin{cases} e^{-(x+y)}, & x > 0 \text{ and } y > 0 \\ 0, & \text{else} \end{cases}$$

  Find the pdf of $Z = X/Y$.

- *Solution:* X and Y only take positive values. So the ratio X/Y only takes positive values. Assuming $a > 0$, we have

$$F_Z(a) = P\left[X/Y \le a\right] = P\left[X \le aY\right] = \iint\limits_{(x,y):x \le ay} f_{XY}(x,y)dxdy =$$

$$= \int_0^\infty \int_0^{ay} e^{-x} e^{-y}\, dxdy = \int_0^\infty (1 - e^{-ay}) e^{-y}\, dy =$$

$$= \int_0^\infty e^{-y} - e^{-(1+a)y}\, dy = 1 - \frac{1}{1+a}$$

  and $F_Z(a) = 0$ for $a \le 0$.

- Therefore, $f_Z(a) = dF_Z(a)/da = (1+a)^{-2}$ for $a > 0$, and 0 otherwise.

Jointly distributed RVs

- We can define *joint distributions* for n random variables as

$$F_{X_1,X_2,...,X_n}(a_1, a_2, ..., a_n) = P[X_1 \leq a_1, X_2 \leq a_2, \ldots, X_n \leq a_n]$$

- If $X_1, X_2, \ldots, X_n$ are discrete, then we have a *joint probability mass function*

$$p_{X_1,X_2,...,X_n}(a_1, a_2, ..., a_n) = P[X_1 = a_1, X_2 = a_2, \ldots, X_n = a_n]$$

- We also have

$$P[X_2 = a_2, X_3 = a_3, \ldots, X_n = a_n] =$$

$$= \sum_{a_1} P[X_1 = a_1, X_2 = a_2, X_3 = a_3, \ldots, X_n = a_n]$$

and thus one can *marginalize* according to

$$p_{X_2,X_3,...,X_n}(a_2, a_3, ..., a_n) = \sum_{a_1} p_{X_1,X_2,X_3,...,X_n}(a_1, a_2, a_3, ..., a_n)$$

with

$$\sum_{a_1,a_2,\ldots,a_n} p_{X_1,X_2,...,X_n}(a_1, a_2, \ldots, a_n) = 1$$

- Random variables $X_1, X_2, \ldots, X_n$ are continuous if there is a non-negative $f_{X_1,X_2,\ldots,X_n}(x_1, \ldots, x_n)$ such that for all events $C \subset \mathbf{R}^n$

  $$P[(X_1, X_2, \ldots, X_n) \in C] = \underset{C}{\int \cdots \int} f_{X_1,\ldots,X_n}(x_1, \ldots, x_n) dx_1 \cdots dx_n$$

  so that

  $$P[X_1 \in A_1, \ldots, X_n \in A_n] = \int_{A_n} \cdots \int_{A_1} f_{X_1,\ldots,X_n}(x_1, \ldots, x_n) dx_1 \cdots dx_n$$

- $f_{X_1,X_2,\ldots,X_n}(x_1, \ldots, x_n)$ is referred to as an *n-dimensional joint pdf*.

- We also have

$$P[X_2 \in A_2, \dots, X_n \in A_n] = P[X_1 \in (-\infty, \infty), X_2 \in A_2, \dots, X_n \in A_n] =$$

$$= \int_{A_n} \cdots \int_{A_2} \int_{-\infty}^{\infty} f_{X_1, X_2, \dots, X_n}(x_1, x_2, \dots, x_n) dx_1 dx_2 \cdots dx_n$$

and thus one can *marginalize* according to

$$f_{X_2, \dots, X_n}(x_2, \dots, x_n) = \int_{-\infty}^{\infty} f_{X_1, X_2, \dots, X_n}(x_1, x_2, \dots, x_n) dx_1$$

- Finally,

$$1 = P[X_1 \in (-\infty, \infty), \dots, X_n \in (-\infty, \infty)] =$$

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_{X_1, \dots, X_n}(x_1, \dots, x_n) dx_1 \cdots dx_n$$

- Let X, Y, and Z have the joint pdf given by

$$f_{XYZ}(x,y,z) = \begin{cases} c, & x^2+y^2+z^2 < R^2 \\ 0, & \text{else} \end{cases}$$

where $c > 0$ is some constant.

1. Find c.
2. What is the marginal distribution $f_{XY}(x,y)$?

- *Note:* This pdf is a *uniform distribution on a sphere of radius R*.

- *Solution (1):* We can find c from

$$1 = \iiint\limits_{\mathbf{R}^3} f_{XYZ}(x,y,z)dxdydz = \iiint\limits_{(x,y,z):x^2+y^2+z^2<R^2} c\ dxdydz = c\frac{4}{3}\pi R^3$$

Consequently, $c = 3/(4\pi R^3)$.

- *Solution (2):* We marginalize out the random variable Z as

$$f_{XY}(x,y) = \int_{-\infty}^{\infty} f_{XYZ}(x,y,z)\ dz = \underbrace{\int_{-a}^{a} c\ dz}_{\text{with } a=\sqrt{R^2-x^2-y^2}} = 2ac =$$

$$= \frac{3}{2\pi R^3}\sqrt{R^2-x^2-y^2}$$

when $x^2+y^2 \leq R^2$, while $f_{XY}(x,y) = 0$ otherwise.

Independent random variables

- Recall that two events E and F are *independent* when

$$P[EF] = P[E]P[F]$$

- **Definition:** We say that the random variables X and Y are *independent* if, for any two sets A and B, we have

$$P[X \in A, Y \in B] = P[X \in A]P[Y \in B]$$

- If $A = (-\infty, a]$ and $B = (-\infty, b]$, then this implies

$$F_{XY}(a,b) = P[X \in A, Y \in B] = P[X \in A]P[Y \in B] = F_X(a)F_Y(b)$$

- In fact, the above two definitions can be shown to be equivalent.

Discrete case

- If X and Y are discrete, then X and Y are independent is equivalent to

$$p_{XY}(x,y) = p_X(x)p_Y(y), \quad \forall x, y$$

- To see that, let $A = \{x\}$ and $B = \{y\}$. Then the independence of X and Y suggests

$$p_{XY}(x,y) = P[X \in A, Y \in B] = P[X \in A]P[Y \in B] = p_X(x)p_Y(y)$$

- Accordingly, we have

$$P[X \in A, Y \in B] = \sum_{x \in A, y \in B} p_{XY}(x,y) = \sum_{x \in A, y \in B} p_X(x)p_Y(y) =$$

$$= \sum_{x \in A} p_X(x) \sum_{y \in B} p_Y(y) = P[X \in A]P[Y \in B]$$

- If X and Y are continuous, then X and Y independent is equivalent to

$$f_{XY}(x,y) = f_X(x) f_Y(y)$$

- To see that, we note that the independence of X and Y implies

$$f_{XY}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{XY}(x,y) = \frac{\partial^2}{\partial x \partial y} F_X(x) F_Y(y) = f_X(x) f_Y(y)$$

- Accordingly, we have

$$F_{XY}(x,y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f_{XY}(u,v)\, dudv = \int_{-\infty}^{y} \int_{-\infty}^{x} f_X(u) f_Y(v)\, dudv =$$

$$= \int_{-\infty}^{x} f_X(u)\, du \int_{-\infty}^{y} f_Y(v)\, dv = F_X(x) F_Y(y)$$

- *In words:* If X and Y are independent, then knowing the outcome of one does not change the probability of the outcomes of the other.

Example

- Say $n + m$ independent Binomial(p) trials are performed.
- Let $X = \#$ of 1s in first n trials and $Y = \#$ of 1s in last m trials.
- Since knowing the outcomes of the first n trials does not affect the distribution of outcomes of the last m trials, we have

$$P[X = k, Y = l] = \binom{n}{k} p^k (1-p)^{n-k} \cdot \binom{m}{l} p^l (1-p)^{m-l}$$

Another example

- Let X and Y have joint density

$$f_{XY}(x,y) = \begin{cases} 6e^{-2x}e^{-3y}, & x > 0,\ y > 0 \\ 0, & \text{otherwise} \end{cases}$$

Are X and Y independent?

- *Solution:* We compute the marginals $f_X(x)$ and $f_Y(y)$, and then verify that $f_{XY}(x,y) = f_X(x)f_Y(y)$.

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)dy = \begin{cases} \int_0^{\infty} 6e^{-2x}e^{-3y}\, dy, & x > 0 \\ 0, & x \le 0 \end{cases} =$$

$$= \begin{cases} 2e^{-2x}, & x > 0 \\ 0, & x \le 0 \end{cases}$$

Similarly,

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)dx = \begin{cases} 3e^{-3y}, & y > 0 \\ 0, & y \le 0 \end{cases}$$

So, the variables are independent, since $f_{XY}(x,y) = f_X(x)f_Y(y)$.

- Alternatively, we could have noticed that $f_{XY}(x,y) = h(x)g(y)$ with

$$h(x) = \begin{cases} e^{-2x}, & x > 0 \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad g(y) = \begin{cases} 6e^{-3y}, & y > 0 \\ 0, & \text{otherwise} \end{cases}$$

Hence ,

$$1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x)g(y)dxdy =$$

$$= \underbrace{\int_{-\infty}^{\infty} h(x)dx}_{C_1} \underbrace{\int_{-\infty}^{\infty} g(y)dy}_{C_2} = C_1C_2$$

Now,

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)dy = \int_{-\infty}^{\infty} h(x)g(y)dy = C_2h(x)$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)dx = \int_{-\infty}^{\infty} h(x)g(y)dx = C_1g(y)$$

and, consequently,

$$f_X(x)f_Y(y) = C_1C_2h(x)g(y) = h(x)g(y) = f_{XY}(x,y)$$

Important conclusion

- The second method shows is that if you can factor a given pdf as $f_{XY}(x,y) = h(x)g(y)$, then X and Y are independent.

- Of course, if X and Y are independent, then $f_{XY}(x,y)$ can be always be factored with the choice $h(x) = f_X(x)$ and $g(y) = f_Y(y)$.

Proposition: X and Y are independent if and only if $f_{XY}(x,y) = h(x)g(y)$ for some $h(x)$ and $g(y)$.

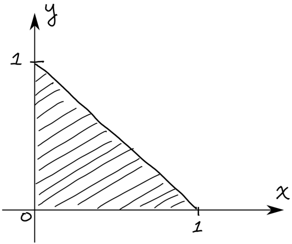
Example

- Let X and Y have joint density

$$f_{XY}(x,y) = \begin{cases} 24xy, & x>0,\ y>0,\ 0<x+y<1 \\ 0, & \text{otherise} \end{cases}$$

  Are X and Y independent?

- Solution: No. Below is the region where $f_{XY}(x,y) > 0$.



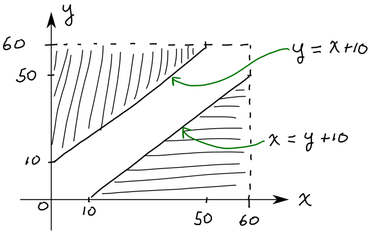
  This region cannot be the result of $h(x)g(y)$ for any choice of $h(x)$ and $g(y)$.

- Two people decide to meet. Each of them arrives independently and uniformly between noon and 1pm. What is the probability that the first to arrive has to wait longer than 10 min for the second the arrive?

- *Solution:* Let X and Y be the times at which both arrive in minutes past noon. The time of arrival of the first to arrive is $\min(X, Y)$. The time of arrival of the last to arrive is $\max(X, Y)$. We want to compute $P[\mathcal{E}]$, where

$$\mathcal{E} = \{\max(X, Y) > \min(X, Y) + 10\} =$$

$$= \{Y > X + 10\} \cup \{X > Y + 10\}$$



$$P[E] = 2P[Y > X+10] = 2 \iint_{\{y>x+10\}} f_{XY}(x,y)dxdy = 2 \iint_{\{y>x+10\}} f_X(x)f_Y(y)dxdy$$

$$= 2 \iint_{\substack{\{y>x+10, \\ 0<x<60, \\ 0<y<60\}}} \left(\frac{1}{60}\right)^2 dxdy = \frac{1}{1800}\int_{10}^{60}\int_{0}^{y-10} dxdy =$$

$$= \frac{1}{1800}\int_{10}^{60}(y-10)dy = \frac{25}{36}$$

Independence of multiple RVs

- The concept of independence can be extended to more than we variables as follows.
- **Definition:** We say that n random variables $X_1, X_2, \ldots, X_n$ are independent, if for any sets $A_1, A_2, ..., A_n$, we have

$$P[X_1 \in A_1, \ldots, X_n \in A_n] = \prod_{i=1}^{n} P[X_i \in A_i]$$

or, equivalently,

$$F_{X_1,\ldots,X_n}(a_1, \ldots, a_n) = \prod_{i=1}^{n} F_{X_i}(a_i)$$

for all $a_1, \ldots, a_n$.

- An infinite collection of random variables is independent if every finite subset are independent.

Independence of multiple RVs (cont.)

- Verifying that $X_1, \ldots, X_n$ are independent amounts to analyzing

$$\begin{aligned} & P[X_1 \le a_1, X_2 \le a_2, \ldots, X_n \le a_n] \\ & \quad = P[X_1 \le a_1] \times \\ & \quad \times P[X_2 \le a_2 \mid X_1 \le a_1] \\ & \quad \times P[X_3 \le a_3 \mid X_2 \le a_2, X_1 \le a_1] \\ & \qquad \vdots \\ & \quad \times P[X_n \le a_n \mid X_{n-1} \le a_{n-1}, \ldots, X_1 \le a_1] \end{aligned}$$

- So, we need to show that

$$\begin{aligned} P[X_2 \le a_2] &= P[X_2 \le a_2 \mid X_1 \le a_1] \\ P[X_3 \le a_3] &= P[X_3 \le a_3 \mid X_2 \le a_2, X_1 \le a_1] \\ & \vdots \\ P[X_n \le a_n] &= P[X_n \le a_n \mid X_{n-1} \le a_{n-1}, \ldots, X_1 \le a_1] \end{aligned}$$

- This is equivalent to showing X_2 is independent of X_1, X_3 is independent of X_1, X_2, X_4 is independent of X_1, X_2, X_3, etc.

Sums of independent RVs

- Say X and Y are independent continuous random variables. What is the pdf of $Z = X + Y$?

$$F_Z(z) = P[X + Y \leq z] = \iint\limits_{x+y\leq z} f_{XY}(x,y)dxdy =$$

$$= \iint\limits_{x+y\leq z} f_X(x)f_Y(y)dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{z-y} f_X(x)f_Y(y)dxdy =$$

$$= \int_{-\infty}^{\infty} f_Y(y)\int_{-\infty}^{z-y} f_X(x)dx\;dy = \int_{-\infty}^{\infty} f_Y(y)F_X(z-y)dy$$

- Differentiating with respect to z yields

$$f_Z(z) = \frac{d}{dz}\int_{-\infty}^{\infty} f_Y(y)F_X(z-y)dy = \int_{-\infty}^{\infty} f_Y(y)\frac{d}{dz}F_X(z-y)dy =$$

$$= \int_{-\infty}^{\infty} f_Y(y)f_X(z-y)dy$$

- Hence, the pdf of $Z = X + Y$ is obtained as the *convolution* of the pdfs $f_X(x)$ and $f_Y(y)$!

Example

- Suppose X and Y are independent random variables, both uniform on $(0, 1)$. What is the pdf of $Z = X + Y$?

- *Solution:* Here, Z can only take values between 0 and 2, and

$$f_X(a) = f_Y(a) = \begin{cases} 1, & 0 < a < 1 \\ 0, & \text{otherwise} \end{cases}$$

resulting in

$$f_Z(z) = \int_{-\infty}^{\infty} f_Y(y) f_X(z - y) dy = \int_0^1 f_X(z - y) dy$$

Thus, for $0 \leq z \leq 1$:

$$\int_0^1 f_X(z-y)dy = \int_0^z 1\ dy = z$$

For $1 \leq z \leq 2$:

$$\int_0^1 f_X(z-y)dy = \int_{z-1}^1 1\ dy = 2-z$$

Thus, we finally have

$$f_Z(z) = \begin{cases} z, & 0 \leq z \leq 1 \\ 2-z, & 1 \leq z \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

Normal (Gaussian) random variables

- **Proposition:** Let $X_1, X_2, \ldots, X_n$ be independent random variables. Let $X_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$ and let $Z = X_1 + X_2 + \cdots + X_n$. Then $Z \sim \mathcal{N}(\mu_Z, \sigma_Z^2)$, where

$$\mu_Z = \mu_1 + \mu_2 + \cdots + \mu_N$$
$$\sigma_Z^2 = \sigma_1^2 + \sigma_2^2 + \cdots + \sigma_N^2$$

- *Proof:* We prove the result for the case of 2 random variables X and Y[1]. Particularly, let $X \sim \mathcal{N}(0, \sigma^2)$ and $Y \sim \mathcal{N}(0, 1)$. Let's determine the density of $U = X + Y$

$$f_X(u-y) f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(u-y)^2}{2\sigma^2}\right\} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{y^2}{2}\right\} =$$

$$= \frac{1}{2\pi\sigma} \exp\left\{-\frac{(u-y)^2}{2\sigma^2} - \frac{y^2}{2}\right\} = \frac{1}{2\pi\sigma} \exp\left\{-\frac{u^2}{2(1+\sigma^2)} - c\left(y - \frac{u}{1+\sigma^2}\right)^2\right\}$$

where $c = (1+\sigma^2)/(2\sigma^2)$.

[1] The result for the case of n variables then follows by applying the result for 2 variables repeatedly.

- Therefore,

$$f_X(u-y)f_Y(y) = \frac{1}{2\pi\sigma}\exp\left\{\frac{-u^2}{2(1+\sigma^2)}\right\}\exp\left\{-c\left(y-\frac{u}{1+\sigma^2}\right)^2\right\}$$

and

$$f_U(u) = \int_{-\infty}^{\infty} f_X(u-y)f_Y(y)dy =$$

$$= \frac{1}{2\pi\sigma}\exp\left\{\frac{-u^2}{2(1+\sigma^2)}\right\}\int_{-\infty}^{\infty}\exp\left\{-c\left(y-\frac{u}{1+\sigma^2}\right)^2\right\}dy =$$

$$= C\exp\left\{\frac{-u^2}{2(1+\sigma^2)}\right\}$$

and C is some constant. But then U is normal with parameters $\mu_U = 0$ and $\sigma_U^2 = 1+\sigma^2$.

- Now, suppose $X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2)$ and $X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$. Then,

$$X_1 + X_2 = \sigma_2 \left(\frac{X_1 - \mu_1}{\sigma_2} + \frac{X_2 - \mu_2}{\sigma_2} \right) + \mu_1 + \mu_2$$

Since $(X_1 - \mu_1)/\sigma_2 \sim \mathcal{N}(0, \sigma_1^2/\sigma_2^2)$ and $(X_2 - \mu_2)/\sigma_2 \sim \mathcal{N}(0, 1)$, we have

$$Z = \frac{X_1 - \mu_1}{\sigma_2} + \frac{X_2 - \mu_2}{\sigma_2} \sim \mathcal{N}(0, 1 + \frac{\sigma_1^2}{\sigma_2^2})$$

and, consequently,

$$X_1 + X_2 = \sigma_2 Z + (\mu_1 + \mu_2) \sim \mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$$

Some definitions

- **Definition:** A random variable Y is called *lognormal* with parameters μ and σ if $\log Y$ is normal with parameter μ and σ^2, i.e., if

  $$Y = e^X,$$

  where $X \sim \mathcal{N}(\mu, \sigma^2)$.

- **Definition:** If the random variables $X_1, X_2, \ldots, X_n$ are *independent and identically distributed*, we say that they are *i.i.d.*, or *iid*.

Example

- Let $S(n)$ be the value of an investment at the end of week n.
- A model for the evolution of $S(n)$ is that the ratios $S(n)/S(n-1)$ are iid lognormal random variables with parameters μ and σ. What is the probability that
  1. the value increases in each of the next two weeks?
  2. the value at the end of two weeks is higher than it is today?
- *Solution (1):* Let U_1 and U_2 be independent normal variables with mean μ and σ^2. Let $Z \sim \mathcal{N}(0,1)$.

$$P\left[S(1) > S(0)\right] = P\left[\frac{S(1)}{S(0)} > 1\right] = P\left[\log\frac{S(1)}{S(0)} > 0\right] =$$

$$= P\left[U_1 > 0\right] = P\left[\frac{U_1 - \mu}{\sigma} > \frac{-\mu}{\sigma}\right] = P\left[Z > \frac{-\mu}{\sigma}\right] = 1 - \Phi(-\mu/\sigma)$$

Similarly,

$$P\left[S(2) \geq S(1)\right] = 1 - \Phi(-\mu/\sigma)$$

Example (cont.)

- Consequently, we have

$$P\left[S(1) > S(0), S(2) > S(1)\right] = P\left[\log\frac{S(1)}{S(0)} > 0, \log\frac{S(2)}{S(1)} > 0\right] =$$

$$= P\left[\log\frac{S(1)}{S(0)} > 0\right] P\left[\log\frac{S(2)}{S(1)} > 0\right] = (1 - \Phi(-\mu/\sigma))^2$$

- *Solution (2):*

$$P\left[S(2) > S(0)\right] = P\left[\frac{S(2)}{S(0)} > 1\right] = P\left[\frac{S(2)}{S(1)}\frac{S(1)}{S(0)} > 1\right] =$$

$$= P\left[\log\frac{S(2)}{S(1)} + \log\frac{S(1)}{S(0)} > 0\right] = P\left[U_2 + U_1 > 0\right] =$$

$$= P\left[\frac{U_2 + U_1 - 2\mu}{\sqrt{2\sigma^2}} > \frac{0 - 2\mu}{\sqrt{2\sigma^2}}\right] = P\left[Z > -\frac{2\mu}{\sqrt{2\sigma^2}}\right] =$$

$$= 1 - \Phi\left(-\frac{2\mu}{\sqrt{2\sigma^2}}\right)$$

Another example

- Let X and Y be independent Poisson random variables with parameters λ_1 and λ_2. What is the pmf of $Z = X + Y$?

- *Solution:*

$$P[Z = n] = P[X + Y = n] = P[\cup_{k=-\infty}^{\infty}\{X = k, Y = n - k\}] =$$

$$= \sum_{k=-\infty}^{\infty} P[X = k, Y = n - k] = \sum_{k=-\infty}^{\infty} P[X = k]P[Y = n - k] =$$

$$= \sum_{k=-\infty}^{\infty} p_X(k)p_Y(n-k) = \sum_{k=0}^{n} p_X(k)p_Y(n-k) = \sum_{k=0}^{n} \frac{\lambda_1^k}{k!}e^{-\lambda_1}\frac{\lambda_2^{n-k}}{(n-k)!}e^{-\lambda_2} =$$

$$= e^{-(\lambda_1+\lambda_2)}\sum_{k=0}^{n} \frac{\lambda_1^k}{k!}\frac{\lambda_2^{n-k}}{(n-k)!} = \frac{e^{-(\lambda_1+\lambda_2)}}{n!}\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\lambda_1^k\lambda_2^{n-k} =$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{n!}\sum_{k=0}^{n}\binom{n}{k}\lambda_1^k\lambda_2^{n-k} = \frac{e^{-(\lambda_1+\lambda_2)}}{n!}(\lambda_1+\lambda_2)^n$$

Therefore, Z is Poisson with parameter $\lambda_1 + \lambda_2$.

Yet another example

- Let X and Y be independent binomial random variables with parameters (n, p) and (m, p). What is the pmf of $Z = X + Y$?

- *Solution:* Let $X = \#$ ones in n repeated independent Bernoulli(p) trials and let $Y = \#$ ones in m repeated independent Bernoulli(p) trials. As X and Y are independent, $Z = \#$ ones in $n + m$ repeated independent Bernoulli(p) trials. Therefore,

$$P[Z = k] = \binom{n+m}{k} p^k (1-p)^{n+m-k}$$

for $k = 0, 1, \ldots, n + m$.

Conditional distributions

- In the *discrete case*, recall that for $P[F] > 0$,

$$P[E \mid F] = \frac{P[EF]}{P[F]}$$

- We want the distribution of X, conditioned on $Y = y$. The conditional pmf for X given Y is given by

$$p_{X|Y}(x|y) = P[X = x|Y = y] = \frac{P[X = x, Y = y]}{P[Y = y]} = \frac{p_{XY}(x, y)}{p_Y(y)}$$

- The conditional cdf for X given Y is given by

$$F_{X|Y}(x|y) = P[X \leq x|Y = y] = \frac{P[X \leq x, Y = y]}{P[Y = y]} =$$

$$= \sum_{a \leq x} \frac{P[X = a, Y = y]}{P[Y = y]} = \sum_{a \leq x} p_{X|Y}(a|y)$$

- If X and Y are independent, then

$$p_{X|Y}(x|y) = \frac{p_{XY}(x, y)}{p_Y(y)} = \frac{p_X(x)p_Y(y)}{p_Y(y)} = p_X(x)$$

Example

- Say the joint pmf of X and Y is

$$p_{XY}(0,0) = 0.4 \qquad p_{XY}(1,0) = 0.1$$
$$p_{XY}(0,1) = 0.2 \qquad p_{XY}(1,1) = 0.3$$

  Compute the conditional pmf of X given $Y = 1$.

- *Solution:*

$$p_Y(1) = \sum_x p_{XY}(x,1) = p_{XY}(0,1) + p_{XY}(1,1) = 0.5$$

  Therefore,

$$p_{X|Y}(0|1) = \frac{p_{XY}(0,1)}{p_Y(1)} = \frac{2}{5}$$

  and

$$p_{X|Y}(1|1) = \frac{p_{XY}(1,1)}{p_Y(1)} = \frac{3}{5}$$

Another example

- Say X and Y are independent Poisson with parameters λ_1 and λ_2. What is the conditional distribution for X given $X+Y=n$?

- *Solution:*

$$P[X=k|X+Y=n] = \frac{P[X=k, X+Y=n]}{P[X+Y=n]} =$$

$$= \frac{P[X=k, Y=n-k]}{P[X+Y=n]} = \frac{P[X=k]P[Y=n-k]}{P[X+Y=n]}$$

Since $X+Y$ is Poisson with parameter $\lambda_1+\lambda_2$, we have

$$P[X=k|X+Y=n] = \frac{\lambda_1^k e^{-\lambda_1}}{k!}\frac{\lambda_2^{n-k}e^{-\lambda_2}}{(n-k)!}\left[\frac{(\lambda_1+\lambda_2)^n e^{-(\lambda_1+\lambda_2)}}{n!}\right]^{-1} =$$

$$= \frac{n!}{k!(n-k)!}\frac{\lambda_1^k\lambda_2^{n-k}}{(\lambda_1+\lambda_2)^n} = \binom{n}{k}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-k}$$

and this is binomial with parameters n and $\lambda_1/(\lambda_1+\lambda_2)$.

Yet another example

- Consider n iid Bernoulli trials $X_1, X_2, \ldots, X_n$ with parameter p. Given that these trials result in k ones, show that each of the $\binom{n}{k}$ possible orderings are equally likely.

- *Solution:* Let $Z = X_1 + \ldots + X_n$. We are conditioning on $Z = k$. Also, let $x_1, x_2, \ldots, x_n$ be binary, and such that $x_1 + x_2 + \cdots + x_n = k$. Then,

$$P[X_1 = x_1, \ldots, X_n = x_n | Z = k] = \frac{P[X_1 = x_1, \ldots, X_n = x_n, Z = k]}{P[Z = k]} =$$

$$= \frac{P[X_1 = x_1, \ldots, X_n = x_n]}{P[Z = k]} = \frac{P[X_1 = x_1, \ldots, X_n = x_n]}{P[Z = k]} =$$

$$= \frac{p^k (1-p)^{n-k}}{\binom{n}{k} p^k (1-p)^{n-k}} = \frac{1}{\binom{n}{k}}$$

- If X and Y are continuous, for $f_Y(y) > 0$ the *conditional probability density function (pdf)* of X given $Y = y$ is defined to be
$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)}$$

- Then
$$P[X \in A|Y = y] = \int_A f_{X|Y}(x|y)dx$$
and choosing $A = (-\infty, a]$, we get
$$F_{X|Y}(a|y) = P[X \leq a|Y = y] = \int_{-\infty}^{a} f_{X|Y}(x|y)dx$$

- If X and Y are *independent*, then
$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{f_X(x)f_Y(y)}{f_Y(y)} = f_X(x)$$

Example

- The joint pdf of X and Y is given by

$$f_{XY}(x,y) = \begin{cases} \frac{12}{5}x(2-x-y), & 0<x<1,\ 0<y<1 \\ 0, & \text{otherwise} \end{cases}$$

For $0<y<1$, what is $f_{X|Y}(x|y)$?

- *Solution:* For $0<x<1,\ 0<y<1$, we have

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{f_{XY}(x,y)}{\int_{-\infty}^{\infty} f_{XY}(x,y)dx} =$$

$$= \frac{x(2-x-y)}{\int_0^1 x(2-x-y)dx} = \frac{x(2-x-y)}{\frac{2}{3}-\frac{y}{2}}$$

When $x \notin (0,1)$ but $0<y<1$, then $f_{XY}(x,y)=0$, so $f_{X|Y}(x|y)=0$.

Another example

- The joint pdf of X and Y is given by

$$f_{XY}(x,y) = \begin{cases} \frac{e^{-x/y}e^{-y}}{y}, & 0 < x < \infty,\ 0 < y < \infty \\ 0, & \text{otherwise} \end{cases}$$

Find $P[X > 1|Y = 1]$.

- *Solution:* First, for $y > 0$, we have

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{f_{XY}(x,y)}{\int_{-\infty}^{\infty} f_{XY}(x,y)dx} =$$

$$= \frac{\frac{1}{y}e^{-x/y}e^{-y}}{e^{-y}\int_0^{\infty} \frac{1}{y}e^{-x/y}dx} = \frac{\frac{1}{y}e^{-x/y}e^{-y}}{e^{-y}\cdot 1} = \frac{1}{y}e^{-x/y}$$

Therefore,

$$P[X > 1|Y = y] = \int_1^{\infty} f_{X|Y}(x|y)dx = \int_1^{\infty} \frac{1}{y}e^{-x/y}dx =$$

$$= -\left. e^{-x/y}\right|_1^{\infty} = \left. e^{-1/y}\right|_{y=1} = 1/e.$$

- **Definition:** Two random variables X and Y are *jointly Gaussian (normal)* or *bivariate Gaussian (normal)* with parameters μ_X, μ_Y, $\sigma_X > 0$, $\sigma_Y > 0$, and $-1 < \rho < 1$ when

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \times$$

$$\times \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2 - 2\rho\frac{(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y}\right]\right\}$$

- It is customary to denote

$$\boldsymbol{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \boldsymbol{\mu} = \begin{bmatrix} \mu_X \\ \mu_Y \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_Y^2 \end{bmatrix}$$

and then

$$\boxed{f_{XY}(x,y) = \frac{1}{2\pi|\Sigma|^{1/2}} \exp\left\{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^T\Sigma^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right\}}$$

with $\boldsymbol{\mu}$ called the **mean vector** of (X,Y) and Σ is called the **covariance matrix** of (X,Y). We say the pair $(X,Y) \sim \mathcal{N}(\boldsymbol{\mu},\Sigma)$.

Marginal distributions

- To find the marginal density

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y)dy$$

we first note that

$$\left(\frac{x-\mu_X}{\sigma_X}\right)^2 + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2 - 2\rho\frac{(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y} =$$

$$= \left(w - \frac{\rho(x-\mu_X)}{\sigma_X}\right)^2 + (1-\rho^2)\left(\frac{x-\mu_X}{\sigma_X}\right)^2$$

where $w = (y-\mu_Y)/\sigma_Y$.

$$f_X(x) =$$

$$C_1 \int_{-\infty}^{\infty} \exp\left\{\frac{-1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2 - 2\rho\frac{(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y}\right]\right\} dy =$$

$$= C_2 \int_{-\infty}^{\infty} \exp\left\{\frac{-1}{2(1-\rho^2)}\left[\left(w - \frac{\rho(x-\mu_X)}{\sigma_X}\right)^2 + (1-\rho^2)\left(\frac{x-\mu_X}{\sigma_X}\right)^2\right]\right\} dw =$$

$$= C_2 e^{-\frac{(x-\mu_X)^2}{2\sigma_X^2}} \int_{-\infty}^{\infty} \exp\left\{\frac{-1}{2(1-\rho^2)}\left(w - \frac{\rho(x-\mu_X)}{\sigma_X}\right)^2\right\} dw =$$

$$= C_2 e^{-\frac{(x-\mu_X)^2}{2\sigma_X^2}} \int_{-\infty}^{\infty} \exp\left\{\frac{-v^2}{2(1-\rho^2)}\right\} dv = \qquad \left(\text{with } \; v = w - \frac{\rho(x-\mu_X)}{\sigma_X}\right)$$

$$= C_3 e^{-\frac{(x-\mu_X)^2}{2\sigma_X^2}}$$

where C_1, C_2, and C_3 are constants that do not involve x.

- So, X is normal with mean μ_X and variance σ_X^2!
- Likewise, the marginal $f_Y(y)$ shows that $Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$.

Conditional distribution

- The knowledge of $f_{XY}(x,y)$ and of the marginal density $f_Y(y)$

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma_Y} e^{-\frac{(y-\mu_Y)^2}{2\sigma_Y^2}}$$

can be used to compute

$$\boxed{\begin{aligned} f_{X|Y}(x|y) &= \frac{f_{XY}(x,y)}{f_Y(y)} = \\ &= C_4 \exp\left\{ \frac{-1}{2\sigma_X^2(1-\rho^2)} \left[x - \left(\mu_X + \rho \frac{\sigma_X}{\sigma_Y}(y-\mu_Y) \right) \right]^2 \right\} \end{aligned}}$$

where C_4 is a constant that depends neither on x nor on y.

- One can see that $f_{X|Y}(x|y)$ is the pdf of X, with its mean and variance equal to $\mu_X + \rho\frac{\sigma_X}{\sigma_Y}(y-\mu_Y)$ and $\sigma_X^2(1-\rho^2)$, respectively.

Conditional distribution (cont.)

- Note that we have

$$f_{XY}(x,y) = f_X(x) f_Y(y) \Leftrightarrow f_{X|Y}(x|y) = f_X(x)$$

  and the latter happens when $\rho = 0$.

- Thus, for bivariate normal X and Y, X and Y are independent when $\rho = 0$.

- **Remark:** ρ is called the *correlation coefficient* between X and Y. When $\rho = 0$, the random variables X and Y are called *uncorrelated*.

- Given two random variables X_1 and X_2, let's consider

$$Y_1 = g_1(X_1, X_2), \quad Y_2 = g_2(X_1, X_2)$$

where we want to find the joint pdf of Y_1 and Y_2.

- **Assumption 1:** The system of equations

$$y_1 = g_1(x_1, x_2), \quad y_2 = g_2(x_1, x_2)$$

can be uniquely solved for x_1 and x_2 in terms of y_1 and y_2, i.e.,

$$x_1 = h_1(y_1, y_2), \quad x_2 = h_2(y_1, y_2)$$

- **Assumption 2:** The functions g_1 and g_2 have continuous partial derivates such that the determinant of the 2×2 matrix

$$J(x_1, x_2) = \begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{vmatrix} = \frac{\partial g_1}{\partial x_1}\frac{\partial g_2}{\partial x_2} - \frac{\partial g_1}{\partial x_2}\frac{\partial g_2}{\partial x_1} \neq 0$$

- Under these conditions, the pdf for Y_1 and Y_2 can be shown to be:

$$f_{Y_1 Y_2}(y_1, y_2) = f_{X_1 X_2}(x_1, x_2)\,|J(x_1, x_2)|^{-1}$$

where

$$x_1 = h_1(y_1, y_2)$$
$$x_2 = h_2(y_1, y_2).$$

Example

- Let X_1 and X_2 have joint pdf $f_{X_1X_2}(x_1,x_2)$. Let

$$Y_1 = X_1 + X_2$$
$$Y_2 = X_1 - X_2$$

Find the joint pdf $f_{Y_1Y_2}(y_1,y_2)$ of Y_1 and Y_2 in terms of $f_{X_1X_2}(x_1,x_2)$.

- *Solution:* Solving the pair of linear equations

$$y_1 = x_1 + x_2, \quad y_2 = x_1 - x_2$$

we get

$$x_1 = \frac{1}{2}y_1 + \frac{1}{2}y_2, \quad x_2 = \frac{1}{2}y_1 - \frac{1}{2}y_2$$

We also have

$$J(x_1,x_2) = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2 \neq 0$$

and therefore

$$f_{Y_1Y_2}(y_1,y_2) = \frac{1}{2} f_{X_1X_2}\left(\frac{1}{2}y_1 + \frac{1}{2}y_2, \frac{1}{2}y_1 - \frac{1}{2}y_2\right)$$

- What is $f_{Y_1Y_2}(y_1, y_2)$ in the last example if

$$f_{X_1X_2}(x_1, x_2) = \begin{cases} 1, & 0 < x_1 < 1,\ 0 < x_2 < 1 \\ 0, & \text{otherwise} \end{cases}$$

<u>Note:</u> X_1 and X_2 are independent and uniform on $(0, 1)$.

- *Solution:* From the last equation on Slide 62, we have

$$f_{Y_1Y_2}(y_1, y_2) = \begin{cases} \frac{1}{2}, & 0 < \frac{1}{2}y_1 + \frac{1}{2}y_2 < 1,\ 0 < \frac{1}{2}y_1 - \frac{1}{2}y_2 < 1 \\ 0, & \text{otherwise} \end{cases}$$

Final example

- What is $f_{Y_1 Y_2}(y_1, y_2)$ if now

$$f_{X_1 X_2}(x_1, x_2) = \begin{cases} \lambda_1 \lambda_2 e^{-\lambda_1 x_1} e^{-\lambda_2 x_2}, & 0 \leq x_1, \ 0 \leq x_2 \\ 0, & \text{otherwise} \end{cases}$$

Note: X_1 and X_2 are independent exponential random variables with parameters λ_1 and λ_2.

- *Solution:* Using the same equation, we obtain

$$f_{Y_1 Y_2}(y_1, y_2) =$$

$$= \begin{cases} \frac{1}{2} \lambda_1 \lambda_2 e^{-\lambda_1 (\frac{1}{2} y_1 + \frac{1}{2} y_2)} e^{-\lambda_2 (\frac{1}{2} y_1 - \frac{1}{2} y_2)}, & 0 \leq \frac{1}{2} y_1 + \frac{1}{2} y_2, \ 0 \leq \frac{1}{2} y_1 - \frac{1}{2} y_2 \\ 0, & \text{othewise} \end{cases}$$