ECE 203 – Section 7
Limit theorems

- Weak Law of Large Numbers (WLLN)
- Central Limit Theorem (CLT)
- Strong Law of Large Numbers (SLLN)

The slides have been prepared based on the lecture notes of Prof. Patrick Mitran.

- **Proposition:** Let $X_1, X_2, \ldots$ be a sequence of iid random variables with $E[X_i] = \mu$. Then, for any $\epsilon \geq 0$:

$$P\left[\left|\frac{X_1 + X_2 + \cdots + X_n}{n} - \mu\right| \geq \epsilon\right] \to 0, \quad \text{as } n \to \infty$$

This is the **Weak Law of Large Numbers (WLLN)**.

- **Example:** A fair coin has a 0 on one side and a 1 on the other. You conduct a sequence of independent trials that consists of repeatedly flipping the coin. Let Z_n be the fraction of flips that result in the number 1 after n flips. What can you say about the probability that Z_n is between 0.499 and 0.501 as $n \to \infty$?

- *Solution:* Let X_i be the outcome of the i-th flip. Note that

$$Z_n = \frac{X_1 + \cdots + X_n}{n}$$

So, by the WLLN: $P[|Z_n - 0.5| < 0.001] \to 1$, as $n \to \infty$.

Central Limit Theorem

- **The Central Limit Theorem (CLT):** Let $X_1, X_2, \ldots$ be a sequence of iid random variables having mean μ and variance σ^2. Then, the distribution of

$$Z_n = \frac{X_1 + X_2 + \ldots + X_n - n\mu}{\sigma\sqrt{n}}$$

tends to the standard normal as $n \to \infty$. Specifically,

$$P[Z_n \le a] \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-u^2/2} du, \quad \text{as } n \to \infty$$

- Note, that the CLT is often stated in the form

$$Z_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{X_i - \mu}{\sigma}$$

- The CLT can be used to approximate probabilities.

Example

- An astronomer makes iid measurements $X_1, X_2, \ldots$ of the distance to a star. Assume that each X_i has mean d (the true distance) and variance 4 light-years^2. How many measurements are needed to be 95% certain that the average of the measurements is within ± 0.5 lightyears of the true value d?
- *Solution:* Let $Z_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (X_i - d)/2$.By the CLT, when n is large, this is approximately $\mathcal{N}(0,1)$.

$$P\left[-0.5 \leq \left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) - d \leq 0.5\right] =$$

$$= P\left[-0.5 \cdot \frac{\sqrt{n}}{2} \leq \frac{1}{\sqrt{n}}\sum_{i=1}^{n} \frac{X_i - d}{2} \leq 0.5 \cdot \frac{\sqrt{n}}{2}\right] =$$

$$P\left[-\frac{\sqrt{n}}{4} \leq Z_n \leq \frac{\sqrt{n}}{4}\right] \approx \Phi\left(\frac{\sqrt{n}}{4}\right) - \Phi\left(-\frac{\sqrt{n}}{4}\right) = 2\Phi\left(\frac{\sqrt{n}}{4}\right) - 1$$

For this to be at least 0.95, we need $\Phi\left(\frac{\sqrt{n}}{4}\right) \geq 0.975$ or, from the Table of the $\Phi(\cdot)$ function, $\sqrt{n}/4 \geq 1.96$. The smallest integer than makes this true is $n = 62$. Thus, with 62 observations, Z_n would be well approximated by a Gaussian.

- **The Strong Law of Large Numbers:** Let $X_1, X_2, \cdots$ be iid with common mean $E[X_i] = \mu$. Then

$$P\left[\lim_{n\to\infty} \frac{X_1 + X_2 + \cdots + X_n}{n} = \mu\right] = 1$$