ECE 203: Probability Theory and Statistics I
Fall 2020 Final Exam

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- This exam consists of 8 problems and 10 pages, including the declaration of integrity and 1 table. Each page is numbered.
- Q1 is the declaration of integrity. Failure to complete this declaration will result in a grade of 0 on the exam.
- You have a 24 hour window (9am on 14 Dec to 9am on 15 Dec) during which you must start, complete and submit this exam. From the moment you start the exam, you will have the lesser of 3 hours or until 9am on 15 Dec to submit your exam.
- Non-graphing, non-programmable calculators are allowed, but will not be helpful. If the answer to a question is 5, writing $\sqrt{100}/2$ will get you full marks. Access to Matlab or similar computational software is prohibited.
- You may use i) any personal notes you take, as long as you composed them yourself prior to the exam (this includes notes based on the lectures/videos, tutorials, office hours, textbook, or other book), ii) any content available on the ECE203 Learn website (including all videos) and ECE203 Piazza website, iii) the course textbook.
- You may not use the internet other than i) to access the ECE203 course webpage on Learn and ECE203 Piazza discussions, ii) to access the textbook (if ecopy) iii) to access crowdmark, or iv) to send email to me or receive email from me.
- You may use a computer, tablet or phone for only the following purposes: i) to create/scan/upload your solutions, ii) to access crowdmark, iii) to access Learn and Piazza, iv) to access your personal notes (if you took these electronically), v) to access the textbook, vi) to send/receive email to/from me, or vii) to be used as a basic calculator following the calculator rule above. Use of any file sharing services such as chegg.com is prohibited.
- You may not communicate directly or indirectly with your classmates or anyone else except for me.
- Questions are allowed but will be answered only if you cannot understand the statement of a problem. You can reach me by email (pmitran@uwaterloo.ca) from 9am to 6pm on 14 Dec.
- All answers must be written legibly. We reserve our right to reduce your grade if your answer is not written in a legible manner.
- A final correct answer does not mean much to us, if the corresponding approach is not clear and sensible. Please explain your solutions and convince us that your solutions make sense.

Q1: [1 point] DECLARATION OF INTEGRITY IN EXAMINATIONS AND TESTS
Course: ECE 203 Probability Theory and Statistics I
Term: Fall 2020

I declare that I have read and followed the instructions listed on the cover page of the ECE203 Final Exam.

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Signature	**ID Number**	**Date**

Note: If you are unable to print this page, it is sufficient to write out yourself "I declare that I have read and followed the instructions listed on the cover page of the ECE203 Final Exam.", then sign, write your ID number, date the statement, and upload this.

Q2: [10] Consider the following game.

- **Person A flips two coins, and records the number of heads.**
- **Person B rolls a 4-sided die (with sides numbered 1 to 4), and records the number.**
- **The person who recorded the larger number wins. If the same number is recorded by both persons, the game ends in a tie.**

Each coin flip is equally likely to come up heads or tails. Each of the 4 sides of the die are equally likely. The two coin flips and the die roll are independent.

Let T be the event that there is a tie, and W the event that person A wins.

[2] a) What is the sample space S, and the size $|S|$ of the sample space?

[2] b) What is $P[W]$ and $P[T]$?

[3] c) Given that person A wins, what is the conditional probability that person B recorded a 2?

[3] d) Given person B recorded a 2, what is the conditional probability of a tie?

Q3: [15] Let X have the probability density function (pdf)

$$f_X(x) = \begin{cases} \frac{1+x}{2} & -1 < x < 1 \\ 0 & \textbf{else} \end{cases}$$

[2] a) What is the cumulative distribution function (cdf) $F_X(x)$ of X?

[3] b) What is $E[X]$?

[3] c) What is $Var[X]$?

[4] d) What is the cdf $F_Y(y)$ of $Y = X^2$?

[3] e) If $Z = 1 - 2X$, what is the covariance between X and Z?

Q4: [11] Let X and Y have joint probability density function (pdf):

$$f_{XY}(x,y) = \begin{cases} \frac{e^{-(x-1)}}{y^2} & x > 1, y > 1 \\ 0 & \textbf{else} \end{cases}$$

[3] a) What is $P[X^2 + Y^2 = 4]$?

[4] b) What is the cumulative distribution function (cdf) of $Z = XY$?

[4] c) Let $U = X + 2Y$ and $V = X - 2Y$. What is the joint pdf of U and V?

Note: In part a), it is indeed an equal sign.

Q5: [8] Let X be a Bernoulli random variable with parameter 1/2. Let N be a discrete random variable that takes integer outcomes and

$$p_{N|X}(n|x) = \begin{cases} \frac{\lambda_0^n}{n!}e^{-\lambda_0} & x = 0,\ n \in \{0,1,2,\ldots\} \\ \frac{\lambda_1^n}{n!}e^{-\lambda_1} & x = 1,\ n \in \{0,1,2,\ldots\} \\ 0 & n \notin \{0,1,2,\ldots\} \end{cases}$$

[4] a) What is $E[X|N=n]$?

[4] b) What is $Var[N]$?

Q6: [10] Consider the following game.

You roll a 4-sided die with sides numbered from 1 to 4. If the outcome

- **is a 1, you get 1 dollar, and get to roll again.**
- **is a 2, you get 2 dollars, and get to roll again.**
- **is a 3, you get 3 dollars, and get to roll again.**
- **is a 4, you stop rolling.**

For example, if you roll the sequence 2, 1, 2, 3, 4, then you win $2 + 1 + 2 + 3 = 8$ **dollars and then stop rolling.**

Let X **be the number of times you get to roll the die.**
Let Y **be the amount of money you win from the game.**

[2] a) What is $E[X]$**?**

[4] **b) What is** $P[Y = 2]$**?**

[4] **c) What is** $E[Y]$**?**

Q7: [8] Let $Y_1, Y_2, Y_3, \ldots$ be a sequence of independent and identically distributed continuous random variables. You do not know the mean μ of the Y_i's, but you do know that the variance is some real number between 9 and 11. You plan to estimate the mean μ by calculating

$$\frac{Y_1 + Y_2 + \cdots + Y_n}{n}. \tag{1}$$

You want your estimate to be within 0.1 of the true value of μ with probability of at least 0.99.

What value of n do you use?

Note: full credit will be given if you can justify why your finite n is large enough, even if it is not the smallest n that will meet the constraints.

Q8: [9 points] In each case, must the conclusion always be true? [Answer yes or no for each case].

If you claim yes, it is always true, then prove why it is always true. If you claim no, give an example where the conditions are satisfied, but the conclusion is false.

[3] a) Let X and Y be independent random variables, and $g(x)$ any function. Then $Z = g(X)$ and Y are independent.

[3] b) If X is a random variable, then $E[X^4] \geq (E[X^2])^2$.

[3] c) Let A, B and C be events with $P[A|B] > 0$, $P[A|C] > 0$ and $P[BC] > 0$. Then $P[A|BC] > 0$.

Table of $\Phi(x)$. [This table is also in Section 5.4 of the textbook]

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.50000	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.52790	0.53188	0.53586
0.1	0.53983	0.54380	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.62930	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.65910	0.66276	0.66640	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.70540	0.70884	0.71226	0.71566	0.71904	0.72240
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.75490
0.7	0.75804	0.76115	0.76424	0.76730	0.77035	0.77337	0.77637	0.77935	0.78230	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.86650	0.86864	0.87076	0.87286	0.87493	0.87698	0.87900	0.88100	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.90320	0.90490	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.92220	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.94520	0.94630	0.94738	0.94845	0.94950	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.96080	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.97320	0.97381	0.97441	0.97500	0.97558	0.97615	0.97670
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.98030	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.98300	0.98341	0.98382	0.98422	0.98461	0.98500	0.98537	0.98574
2.2	0.98610	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.98840	0.98870	0.98899
2.3	0.98928	0.98956	0.98983	0.99010	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.99180	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.99430	0.99446	0.99461	0.99477	0.99492	0.99506	0.99520
2.6	0.99534	0.99547	0.99560	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.99720	0.99728	0.99736
2.8	0.99744	0.99752	0.99760	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.99900
3.1	0.99903	0.99906	0.99910	0.99913	0.99916	0.99918	0.99921	0.99924	0.99926	0.99929
3.2	0.99931	0.99934	0.99936	0.99938	0.99940	0.99942	0.99944	0.99946	0.99948	0.99950
3.3	0.99952	0.99953	0.99955	0.99957	0.99958	0.99960	0.99961	0.99962	0.99964	0.99965
3.4	0.99966	0.99968	0.99969	0.99970	0.99971	0.99972	0.99973	0.99974	0.99975	0.99976