ECE 203: Probability Theory and Statistics I
Fall 2020 Final Exam

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- This exam consists of ABC problems and XYZ pages, including the declaration of integrity and 1 table. Each page is numbered.
- Q1 is the declaration of integrity. Failure to complete this declaration will result in a grade of 0 on the exam.
- You have a 24 hour window (9am on 14 Dec to 9am on 15 Dec) during which you must start, complete and submit this exam. From the moment you start the exam, you will have the lesser of 3 hours or until 9am on 15 Nov to submit your exam.
- Non-graphing, non-programmable calculators are allowed, but will not be helpful. If the answer to a question is 5, writing $\sqrt{100}/2$ will get you full marks. Access to Matlab or similar computational software is prohibited.
- You may use i) any personal notes you take, as long as you composed them yourself prior to the exam (this includes notes based on the lectures/videos, tutorials, office hours, textbook, or other book), ii) any content available on the ECE203 Learn website (including all videos) and ECE203 Piazza website, iii) the course textbook.
- You may not use the internet other than i) to access the ECE203 course webpage on Learn and ECE203 Piazza discussions, ii) to access the textbook (if ecopy) iii) to access crowdmark, or iv) to send email to me or receive email from me.
- You may use a computer, tablet or phone for only the following purposes: i) to create/scan/upload your solutions, ii) to access crowdmark, iii) to access Learn and Piazza, iv) to access your personal notes (if you took these electronically), v) to access the textbook, vi) to send/receive email to/from me, or vii) to be used as a basic calculator following the calculator rule above. Use of any file sharing services such as chegg.com is prohibited.
- You may not communicate directly or indirectly with your classmates or anyone else except for me.
- Questions are allowed but will be answered only if you cannot understand the statement of a problem. You can reach me by email (pmitran@uwaterloo.ca) from 9am to 6pm on 14 Dec.
- All answers must be written legibly. We reserve our right to reduce your grade if your answer is not written in a legible manner.
- A final correct answer does not mean much to us, if the corresponding approach is not clear and sensible. Please explain your solutions and convince us that your solutions make sense.

Q1: [1 point] DECLARATION OF INTEGRITY IN EXAMINATIONS AND TESTS
Course: ECE 203 Probability Theory and Statistics I
Term: Fall 2020

I declare that I have read and followed the instructions listed on the cover page of the ECE203 Final Exam.

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Signature	**ID Number**	**Date**

Note: If you are unable to print this page, it is sufficient to write out yourself "I declare that I have read and followed the instructions listed on the cover page of the ECE203 Final Exam.", then sign, write your ID number, date the statement, and upload this.

Q2: [10] Consider the following game.

- **Person A flips two fair coins, and records the number of heads.**
- **Person B rolls a 4-sided die (with sides numbered 1 to 4), and records the number.**
- **The person who recorded the larger number wins. If the same number is recorded by both persons, the game ends in a tie.**

Each coin flip is equally likely to come up heads or tails. Each of the 4 sides of the die are equally likely. The two coin flips and die roll are independent.

Let T be the event that there is a tie, and W the event that person A wins.

[2] a) What is the sample space S, and the size $|S|$ of the sample space?

[2] b) What is $P[W]$ and $P[T]$?

[3] c) Given that person A wins, what is the conditional probability that person B recorded a 2?

[3] d) Given person B recorded a 2, what is the conditional probability that of a tie?

Solution:

a) The sample space $S = \{h,t\} \times \{h,t\} \times \{1,2,3,4\}$.

So, $S = \{(h,h,1),(h,t,1),(t,h,1),(t,t,1),(h,h,2),\ldots,(t,t,4)\}$. One could also (lazily) write $S = \{hh1, ht1, \ldots, th4, tt4\}$.

There are $2 \times 2 \times 4 = 16$ outcomes in S.

One could also claim that the sample space $S = \{0,1,2\} \times \{1,2,3,4\}$ by only counting the number of heads in the coin tosses instead of the outcomes themselves. Then there are 12 outcomes in the sample space. But these outcomes are now not equally likely, which makes the computations in parts b), c) and d) more difficult.

b) With a sample space of size 16, all 16 outcomes are equally likely. We have

$$W = \{(h,h,1)\}$$
$$T = \{(t,h,1),(h,t,1),(h,h,2)\}$$

so

$$P[W] = |W|/|S| = 1/16$$
$$P[T] = |T|/|S| = 3/16$$

c) Let the event that B recorded a 2 be

$$B_2 = \{(h,h,2),(h,t,2),(t,h,2),(t,t,2)\}$$

$$P[B_2|W] = \frac{P[B_2W]}{P[W]} = \frac{P[\emptyset]}{P[\{(h,h,1)\}]} = 0$$

d)

$$P[T|B_2] = \frac{P[TB_2]}{P[B_2]} = \frac{P[(h,h,2)]}{P[\{(h,h,2),(h,t,2),(t,h,2),(t,t,2)\}]} = 1/4$$

Q3: [15] Let X have the probability density function (pdf)

$$f_X(x) = \begin{cases} \frac{1+x}{2} & -1 < x < 1 \\ 0 & \textbf{else} \end{cases}$$

[2] a) What is the cumulative distribution function (cdf) of X?

[3] b) What is $E[X]$?

[3] c) What is $Var[X]$?

[4] d) What is the cdf $f_Y(y)$ of $Y = X^2$?

[3] e) If $Z = 1 - 2X$, what is the covariance between X and Z?

Solution:

a) For $x \leq -1$, $F_X(x) = 0$ and for $x \geq 1$, $F_X(x) = 1$. Hence, that leaves $-1 < x < 1$:

$$\begin{aligned} F_X(x) &= \int_{-\infty}^{x} f_X(u)du \\ &= \int_{-1}^{x} \frac{1}{2} + \frac{u}{2} du \\ &= \left[\frac{1}{2}u + \frac{u^2}{4}\right]_{-1}^{x} \\ &= \frac{1}{2}x + \frac{x^2}{4} - (\frac{-1}{2} + \frac{1}{4}) \\ &= \frac{1}{4} + \frac{x}{2} + \frac{x^2}{4} \end{aligned}$$

b)

$$\begin{aligned} E[X] &= \int_{-\infty}^{\infty} x f_X(x)dx \\ &= \int_{-1}^{1} \frac{x}{2} + \frac{x^2}{2} dx \\ &= \left[\frac{x^2}{4} + \frac{x^3}{6}\right]_{-1}^{1} \\ &= \frac{1}{4} + \frac{1}{6} - \frac{1}{4} - \frac{-1}{6} \\ &= \frac{1}{3} \end{aligned}$$

c)

$$
\begin{aligned}
E[X^2] &= \int_{-\infty}^{\infty} x^2 f_X(x)dx \\
&= \int_{-1}^{1} \frac{x^2}{2} + \frac{x^3}{2} dx \\
&= \left[\frac{x^3}{6} + \frac{x^4}{8}\right]_{-1}^{1} \\
&= \frac{1}{6} + \frac{1}{8} - \frac{-1}{6} - \frac{1}{8} \\
&= \frac{1}{3}
\end{aligned}
$$

So,

$$
Var[X] = E[X^2] - (E[X])^2 = 1/3 - (1/3)^2 = 2/9
$$

d) Let $Y = X^2$. There are 3 cases. Case $y < 0$:

$$
F_Y(y) = P[Y \leq y] = P[X^2 \leq y] = 0
$$

since y is strictly negative and $X^2 \geq 0$.

Case $y > 1$:

$$
F_Y(y) = P[Y \leq y] = P[X^2 \leq y] = P[-\sqrt{y} \leq X \leq \sqrt{y}] = 1
$$

since X is between -1 and 1.

Case $0 \leq y \leq 1$:

$$
\begin{aligned}
F_Y(y) &= P[Y \leq y] \\
&= P[X^2 \leq y] \\
&= P[-\sqrt{y} \leq X \leq \sqrt{y}] \\
&= P[-\sqrt{y} < X \leq \sqrt{y}] \\
&= F_X(\sqrt{y}) - F_X(-\sqrt{y}) \\
&= \frac{1}{4} + \frac{\sqrt{y}}{2} + \frac{y}{4} - (\frac{1}{4} + \frac{-\sqrt{y}}{2} + \frac{y}{4}) \\
&= \sqrt{y}
\end{aligned}
$$

Alternatively,

$$
\begin{aligned}
F_Y(y) &= P[Y \leq y] \\
&= P[X^2 \leq y] \\
&= P[-\sqrt{y} \leq X \leq \sqrt{y}] \\
&= \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{2} + \frac{x}{2} dx \\
&= \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{2} dx \qquad\qquad \text{since } \int_{-a}^{a} x dx = 0 \text{ for any value of } a \\
&= \sqrt{y}
\end{aligned}
$$

e)

$$
\begin{aligned}
Cov[X, Z] &= Cov[X, 1 - 2X] \\
&= Cov[X, 1] - Cov[X, 2X] \\
&= Cov[X, 1] - 2Cov[X, X] \\
&= 0 - 2Var[X] \\
&= -\frac{4}{9}
\end{aligned}
$$

Alternatively, let $\mu = E[X]$. Then

$$
\begin{aligned}
E[Z] &= E[1 - 2X] = 1 - E[X] = 1 - 2\mu \\
Cov[X, Z] &= E[XZ] - E[X]E[Z] \\
&= E[X(1 - 2X)] - \mu(1 - 2\mu) \\
&= E[X - 2X^2)] - \mu(1 - 2\mu) \\
&= E[X] - 2E[X^2] - \mu(1 - 2\mu) \\
&= \mu - 2E[X^2] - \mu(1 - 2\mu) \\
&= -2(E[X^2] - \mu^2) \\
&= -2(\frac{1}{3} - \frac{1}{9}) \\
&= -\frac{4}{9}
\end{aligned}
$$

Q4: [11] Let X and Y have joint probability density function (pdf):

$$f_{XY}(x,y) = \begin{cases} \frac{e^{-(x-1)}}{y^2} & x > 1, y > 1 \\ 0 & \textbf{else} \end{cases}$$

[3] a) What is $P[X^2 + Y^2 = 4]$?

[4] b) What is the cumulative distribution function (cdf) of $Z = XY$?

[4] c) Let $U = X + 2Y$ and $V = X - 2Y$. What is the joint pdf of U and V?

Solution:

a) We want to compute:

$$\begin{aligned} P[X^2 + Y^2 = 4] &= \iint\limits_{x^2+y^2=4} f_{XY}(x,y)\, dxdy \\ &= \iint\limits_{\substack{x^2+y^2=4 \\ x>1,y>1}} f_{XY}(x,y)\, dxdy \\ &=^{(a)} \int_1^{\sqrt{3}} \int_{\sqrt{4-y^2}}^{\sqrt{4-y^2}} f_{XY}(x,y)\, dxdy \\ &= \int_1^{\sqrt{3}} 0\, dy \\ &= 0. \end{aligned}$$

where step (a) is because $x^2 + y^2 = 4$ subject to $x > 1$ and $y > 1$ has only one solution for x in terms of y: $x = \sqrt{4 - y^2}$.

The result can be explained as follows: since the region of integration is the curve $x^2 + y^2 = 4$ which has no area, therefore, the double integral above is 0. Simpling writing this last sentence is sufficient for full-credit. [This is similar to how, for a continuous rv X, a point has probability 0, i.e. $P[X = a] = 0$.]

b) Since $X > 1$ and $Y > 1$, then $P[Z \le 1] = 0$. So it remains to consider the case of $z > 1$:

$$\begin{aligned}
P[Z \le z] &= P[XY \le z] \\
&= \iint\limits_{xy \le z} f_{XY}(x, y) dx dy \\
&= e \int_1^z \int_1^{z/x} \frac{e^{-x}}{y^2} dy dx \\
&= e \int_1^z e^{-x} \int_1^{z/x} \frac{1}{y^2} dy dx \\
&= e \int_1^z e^{-x} \left[\frac{-1}{y} \right]_{y=1}^{y=z/x} dx \\
&= e \int_1^z e^{-x} \left[1 - \frac{x}{z} \right] dx \\
&= e \int_1^z e^{-x} dx - \frac{e}{z} \int_1^z x e^{-x} dx \\
&= e[-e^{-x}]_1^z - \frac{e}{z}[-xe^{-x} - e^{-x}]_1^z \\
&= e[e^{-1} - e^{-z}] + \frac{e}{z}[xe^{-x} + e^{-x}]_1^z \\
&= 1 - ee^{-z} + \frac{e}{z}[ze^{-z} + e^{-z} - 2e^{-1}] \\
&= 1 - ee^{-z} + [ee^{-z} + ee^{-z}/z - 2ee^{-1}/z] \\
&= 1 + e^{-(z-1)}/z - 2/z
\end{aligned}$$

c) Here we have

$$u = x + 2y$$
$$v = x - 2y$$

So,

$$J = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 1 & -2 \end{vmatrix} = -4$$

So $|J| = 4$ and $x = (u + v)/2$ while $y = (u - v)/4$. Therefore

$$\begin{aligned}
f_{UV}(u, v) &= f_{X,Y}\left(\frac{u+v}{2}, \frac{u-v}{4} \right) |J|^{-1} \\
&= \begin{cases} \frac{4e^{-(\frac{u+v}{2} - 1)}}{(u-v)^2} & u + v > 2, u - v > 4 \\ 0 & \text{else} \end{cases}
\end{aligned}$$

Q5: [8] Let X be Bernoulli random variable with parameter 1/2. Let N be a discrete random variable that takes integer outcomes and

$$p_{N|X}(n|x) = \begin{cases} \frac{\lambda_0^n}{n!}e^{-\lambda_0} & x = 0,\ n \in \{0, 1, 2, \ldots\} \\ \frac{\lambda_1^n}{n!}e^{-\lambda_1} & x = 1,\ n \in \{0, 1, 2, \ldots\} \\ 0 & \textbf{else} \end{cases}$$

[4] a) What is $E[X|N = n]$?

[4] b) What is $Var[N]$?

Solution:

a)

$$\begin{aligned} E[X|N = n] &= 1 \times P[X = 1|N = n] + 0 \times P[X = 0|N = n] \\ &= P[X = 1|N = n] \\ &= \frac{P[X = 1, N = n]}{P[N = n]} \\ &= \frac{P[N = n|X = 1]P[X = 1]}{P[N = n|X = 1]P[X = 1] + P[N = n|X = 0]P[X = 0]} \\ &= \frac{P[N = n|X = 1]\frac{1}{2}}{P[N = n|X = 1]\frac{1}{2} + P[N = n|X = 0]\frac{1}{2}} \\ &= \frac{P[N = n|X = 1]}{P[N = n|X = 1] + P[N = n|X = 0]} \\ &= \frac{\frac{\lambda_1^n}{n!}e^{-\lambda_1}}{\frac{\lambda_1^n}{n!}e^{-\lambda_1} + \frac{\lambda_0^n}{n!}e^{-\lambda_0}} \\ &= \frac{\lambda_1^n e^{-\lambda_1}}{\lambda_1^n e^{-\lambda_1} + \lambda_0^n e^{-\lambda_0}} \end{aligned}$$

b) First we get the first and second moments of N:

$$\begin{aligned} E[N] &= E[N|X = 0]P[X = 0] + E[N|X = 1]P[X = 1] \\ &=^{(a)} \frac{\lambda_0}{2} + \frac{\lambda_1}{2} \\ E[N^2] &= E[N^2|X = 0]P[X = 0] + E[N^2|X = 1]P[X = 1] \\ &=^{(b)} \lambda_0(\lambda_0 + 1)\frac{1}{2} + \lambda_1(\lambda_1 + 1)\frac{1}{2} \\ &= \frac{\lambda_0^2 + \lambda_1^2}{2} + \frac{\lambda_0 + \lambda_1}{2} \end{aligned}$$

where (a) follows since given $X = x$, N is Poisson with parameter λ_x, and (b) follows since the second moment of a Poisson random variable with parameter λ is $\lambda(\lambda + 1)$ (see Example 7c in Section 4.7 of the textbook).

Hence

$$\begin{aligned} Var[N] &= E[N^2] - (E[N])^2 \\ &= \lambda_0^2/2 + \lambda_1^2/2 + (\lambda_0 + \lambda_1)/2 - (\lambda_0 + \lambda_1)^2/4 \\ &= \lambda_0^2/4 + \lambda_1^2/4 + \lambda_0/2 + \lambda_1/2 - \lambda_0\lambda_1/2 \\ &= (\frac{\lambda_0 - \lambda_1}{2})^2 + \frac{\lambda_0 + \lambda_1}{2} \end{aligned}$$

Q6: [10] Consider the following game.

You roll a 4-sided die with sides numbered from 1 to 4. If the outcome

- **is a 1, you get 1 dollar, and get to roll again.**
- **is a 2, you get 2 dollars, and get to roll again.**
- **is a 3, you get 3 dollars, and get to roll again.**
- **is a 4, you stop rolling.**

For example, if you roll the sequence 2, 1, 2, 3, 4, then you win $2 + 1 + 2 + 3 = 8$ **dollars and then stop rolling.**

Let X **be the number of times you get to roll the die.**
Let Y **be the amount of money you win from the game.**

[**2**] **a) What is** $E[X]$**?**

[4] **b) What is** $P[Y = 2]$**?**

[4] **c) What is** $E[Y]$**?**

Solution:

a) X is a geometric random variable with parameter $p = 1/4$: If we roll a 4 (with probability 1/4), then we stop, otherwise we keep rolling (with probability 3/4). So $E[X] = 1/p = 4$.

b) There are only two ways to win 2 dollars: We win 2 dollars on the 1st roll and then roll a 4. Or we win 1 dollar on the 1st roll, followed by another dollar on the second roll and then roll a 4. So the event that we win 2 dollar is $E = \{24, 114\}$.

$$P[E] = P[\{24, 114\}] = P[24] + P[114] = \frac{1}{4}\frac{1}{4} + \frac{1}{4}\frac{1}{4}\frac{1}{4} = \frac{5}{64}$$

c) This is similar to example 5c in Section 7.5. Let $E[Y]$ be the expected winning from playing the game and let Z be the outcome of the 1st roll of the die. Then

$$\begin{aligned} E[Y] &= E[Y|Z=1]P[Z=1] + E[Y|Z=2]P[Z=2] + E[Y|Z=3]P[Z=3] + E[Y|Z=4]P[Z=4] \\ &= \frac{1}{4}\left(E[Y|Z=1] + E[Y|Z=2] + E[Y|Z=3] + E[Y|Z=4]\right) \end{aligned}$$

and, note that

$$\begin{aligned} E[Y|Z=1] &= 1 + E[Y] \\ E[Y|Z=2] &= 2 + E[Y] \\ E[Y|Z=3] &= 3 + E[Y] \\ E[Y|Z=4] &= 0 \end{aligned}$$

so

$$
\begin{aligned}
E[Y] &= \frac{1}{4}\left(1 + E[Y] + 2 + E[Y] + 3 + E[Y]\right) \\
&= \frac{6}{4} + \frac{3}{4}E[Y]
\end{aligned}
$$

so $E[Y] = 6$.

Q7: [8] Let $Y_1, Y_2, Y_3, \ldots$ be a sequence of independent and identically distributed continuous random variables. You do not know the mean μ of the Y_i's, but you do know that the variance is some number between 9 and 11. You plan to estimate the mean μ by calculating

$$\frac{Y_1 + Y_2 + \cdots + Y_n}{n}. \tag{1}$$

You want your estimate to be within 0.1 of the true value of μ with probability of at least 0.99.

What value of n do you use?

Note: full credit will be given if you can justify why your finite n is large enough, even if it is not the smallest n that will meet the constraints.

Solution:

Method 1: Use Chebyshev inequality

Let $Z_n = (Y_1 + \cdots + Y_n)/n$. Then $E[Z_n] = \mu$ and we want

$P[|Z_n - \mu| < 0.1] > 0.99$

or equivalently

$P[|Z_n - \mu| \geq 0.1] \leq 0.01$

Now, by Chebyshev,

$$\begin{aligned} P[|Z_n - \mu| \geq 0.1] &\leq \frac{E[|Z_n - \mu|^2]}{(0.1)^2} \\ &= \frac{\sigma^2}{0.01n} = \frac{100\sigma^2}{n} \end{aligned}$$

and we want $\frac{100\sigma^2}{n} \leq 0.01$ which implies $n \geq \sigma^2 \times 10^4$. Since all we know is that σ^2 is between 9 and 11, then we must pick $n \geq 11 \times 10^4 = 1.1 \times 10^5$.

Method 2: Use the central limit theorem as in example 3a of Section 8.3. This approach uses an approximation, but full credit is given for this approach as well.

Note that $Var[Z_n] = \sigma^2/n$ and we want

$$\begin{aligned} P[-0.1 < Z_n - \mu < 0.1] &= P\left[\frac{-0.1}{\sqrt{\sigma^2/n}} < \frac{Z_n - \mu}{\sqrt{\sigma^2/n}} < \frac{0.1}{\sqrt{\sigma^2/n}}\right] \\ &= P\left[\frac{-0.1\sqrt{n}}{\sigma} < \frac{Z_n - \mu}{\sqrt{\sigma^2/n}} < \frac{0.1\sqrt{n}}{\sigma}\right] \end{aligned}$$

Let $U \sim \mathcal{N}(0,1)$. If n is large, by the CLT:

$$\begin{aligned} P[-0.1 < Z_n - \mu < 0.1] &\approx P\left[\frac{-0.1\sqrt{n}}{\sigma} < U < \frac{0.1\sqrt{n}}{\sigma}\right] \\ &= 2\Phi\left(\frac{0.1\sqrt{n}}{\sigma}\right) - 1 \end{aligned}$$

So we want

$$2\Phi\left(\frac{0.1\sqrt{n}}{\sigma}\right) - 1 > 0.99$$

or equivalently

$$\Phi\left(\frac{0.1\sqrt{n}}{\sigma}\right) > 0.995$$

From the Φ table, we get

$$\frac{0.1\sqrt{n}}{\sigma} \geq 2.58$$

or equivalently

$$\sqrt{n} \geq 25.8\sigma$$

so we need $n \geq (25.8)^2\sigma^2$. Since we know that $9 \leq \sigma^2 \leq 11$, then we pick

$$n \geq (25.8)^2 \times 11 = 7322.04$$

so we need n to be at least $n = 7323$. Full credit for getting $n \geq (25.8)^2 \times 11$ or any value greater than this provided the method was valid.

Q8: [9 points] In each case, must the conclusion always be true? [Answer yes or no for each case].

If you claim yes, it is always true, then prove why it is always true. If you claim no, give an example where the conditions are satisfied, but the conclusion is false.

[3] a) Let X and Y be independent random variables, and $g(x)$ any function. Then $Z = g(X)$ and Y are independent.

[3] b) If X is a random variable, then $E[X^4] \geq (E[X^2])^2$.

[3] c) Let A, B and C be events with $P[A|B] > 0$, $P[A|C] > 0$ and $P[BC] > 0$. Then $P[A|BC] > 0$.

Solution:

a) Yes, this is always true. Let $A = \{x \in \mathbb{R} \mid g(x) \leq z\}$. One way to see this is that

$$
\begin{aligned}
P[Z \leq z, Y \leq y] &= P[g(X) \leq z, Y \leq y] \\
&= P[X \in A, Y \in (-\infty, y]\ ] \\
&= P[X \in A]P[Y \in (-\infty, y]\ ] && \text{since } X \text{ and } Y \text{ are independent} \\
&= P[g(X) \leq z]P[Y \in (-\infty, y]\ ] \\
&= P[Z \leq z]P[Y \leq y]
\end{aligned}
$$

So, from the result in Section 6.2 between (2.1) and (2.2), Z and Y are independent.

b) Yes, this is always true. Let $U = X^2$. Note that $Var[U] = E[U^2] - (E[U])^2 \geq 0$. So $E[U^2] \geq (E[U])^2$. Using $U = X^2$, we get $E[X^4] \geq (E[X^2])^2$.

c) No, this is not always true. Roll a fair six-sided die, and let $A = \{1, 2\}$, $B = \{2, 3\}$, $C = \{1, 3\}$. Then

$P[BC] = P[\{3\}] = 1/6 > 0$

$P[A|B] = P[AB]/P[B] = 1/2$

$P[A|C] = P[AC]/P[C] = 1/2$

But,

$P[A|BC] = P[ABC]/P[BC] = P[\emptyset]/P[3] = 0/(1/6) = 0.$

Table of $\Phi(x)$:

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.50000	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.52790	0.53188	0.53586
0.1	0.53983	0.54380	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.62930	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.65910	0.66276	0.66640	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.70540	0.70884	0.71226	0.71566	0.71904	0.72240
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.75490
0.7	0.75804	0.76115	0.76424	0.76730	0.77035	0.77337	0.77637	0.77935	0.78230	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.86650	0.86864	0.87076	0.87286	0.87493	0.87698	0.87900	0.88100	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.90320	0.90490	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.92220	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.94520	0.94630	0.94738	0.94845	0.94950	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.96080	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.97320	0.97381	0.97441	0.97500	0.97558	0.97615	0.97670
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.98030	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.98300	0.98341	0.98382	0.98422	0.98461	0.98500	0.98537	0.98574
2.2	0.98610	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.98840	0.98870	0.98899
2.3	0.98928	0.98956	0.98983	0.99010	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.99180	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.99430	0.99446	0.99461	0.99477	0.99492	0.99506	0.99520
2.6	0.99534	0.99547	0.99560	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.99720	0.99728	0.99736
2.8	0.99744	0.99752	0.99760	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.99900
3.1	0.99903	0.99906	0.99910	0.99913	0.99916	0.99918	0.99921	0.99924	0.99926	0.99929
3.2	0.99931	0.99934	0.99936	0.99938	0.99940	0.99942	0.99944	0.99946	0.99948	0.99950
3.3	0.99952	0.99953	0.99955	0.99957	0.99958	0.99960	0.99961	0.99962	0.99964	0.99965
3.4	0.99966	0.99968	0.99969	0.99970	0.99971	0.99972	0.99973	0.99974	0.99975	0.99976