ECE 203: Probability Theory and Statistics I
Fall 2020 Test#1

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- This test consists of ABC problems and XYZ pages, including the declaration of integrity. Each page is numbered.
- Q1 is the declaration of integrity. Failure to complete this declaration will result in a grade of 0 on the test.
- You have a 24 hour window (10am on 8 Oct to 10am on 9 Oct) during which you must start, complete and submit this test. From the moment you start the test, you will have the lesser of 2 hours or until 10am on 9 Oct to submit your test.
- Non-graphing, non-programmable calculators are allowed, but will not be helpful. If the answer to a question is 5, writing $\sqrt{100}/2$ will get you full marks. Access to Matlab or similar computational software is prohibited.
- You may use i) any personal notes you take, as long as you composed them yourself prior to the test (this includes notes based on the lectures/videos, tutorials, office hours, textbook, or other book), ii) any content available on the ECE203 Learn website (including all videos) and ECE203 Piazza website, iii) the course textbook.
- You may not use the internet other than i) to access the course ECE203 webpage on Learn and ECE203 Piazza discussions, ii) to access the textbook (if ecopy) iii) to access crowdmark, or iv) to send email to me or receive email from me.
- You may use a computer, tablet or phone for only the following purposes: i) to create/scan/upload your solutions, ii) to access crowdmark, iii) to access Learn and Piazza, iv) to access your personal notes (if you took these electronically), v) to access the textbook, vi) to send/receive email to/from me, or vii) to be used as a basic calculator following the calculator rule above. Use of any file sharing services such as chegg.com is prohibited.
- You may not communicate directly or indirectly with your classmates or anyone else except for me.
- Questions are allowed but will be answered only if you cannot understand the statement of a problem. You can reach me by email (pmitran@uwaterloo.ca) from 10am to 6pm on 8 Oct.
- All answers must be written legibly. We reserve our right to reduce your grade if your answer is not written in a legible manner.
- A final correct answer does not mean much to us, if the corresponding approach is not clear and sensible. Please explain your solutions and convince us that your solutions make sense.

Q1: [1 point] DECLARATION OF INTEGRITY IN EXAMINATIONS AND TESTS
Course: ECE 203 Probability Theory and Statistics I
Term: Fall 2020

I declare that I have read and followed the instructions listed on the cover page of ECE203 Test#1.

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Signature	**Date**

Note: If you are unable to print this page, it is sufficient to write out yourself "I declare that I have read and followed the instructions listed on the cover page of ECE203 Test#1.", then sign and date the statement, and upload this.

Q2: [12 points] Consider the following (unfair) game.

- **Person A rolls a 4-sided die.**
- **Person B rolls a 6-sided die.**
- **The person who rolled the larger number wins. If the same number is rolled by both persons, the game ends in a tie.**

Let T be the event that there is a tie, and W the event that person A wins.

[2] a) Enumerate the elements of T and W.

[2] b) What are $P[W]$ and $P[T]$?

[2] c) What is $P[W^cT^c]$?

[3] d) Given person B rolled a 2, what is the conditional probability that person A wins?

[3] e) Given person A wins, what is the conditional probability that person B rolled a 2?

Note: Assume that all 24 possible outcomes of the two dice are equally likely.

Solution:

a) Listing the outcomes of the two dice rolls with the 4 sided outcome first,

$T = \{(1,1),(2,2),(3,3),(4,4)\}$

$W = \{(2,1),(3,1),(3,2),(4,1),(4,2),(4,3)\}$

[One could also list the outcoms with the result of the 6-sided die first]

b) Since all $|S| = 24$ outcomes of S are equally likely, and $|T| = 4$ and $|W| = 6$, then

$P[T] = |T|/|S| = 4/24$

$P[W] = |W|/|S| = 6/24$

c) We could list the elements of W^cT^c. This would be a little tedious, but we would find there are 14 cases. Alternatively,

$$P[W^cT^c] = 1 - P[(W^cT^c)^c] = 1 - P[W \cup T] = 1 - P[W] - P[T] = 1 - \frac{6}{24} - \frac{4}{24} = \frac{14}{24}.$$

d) Let $F = \{\text{person } B \text{ rolls a } 2\} = \{(1,2),(2,2),(3,2),(4,2)\}$. Then

$$P[W|F] = \frac{P[FW]}{P[F]} = \frac{P[\{(3,2),(4,2)\}]}{4/24} = \frac{2/24}{4/24} = 1/2$$

e)

$$P[F|W] = \frac{P[FW]}{P[W]} = \frac{P[\{(3,2),(4,2)\}]}{6/24} = \frac{2/24}{6/24} = 1/3$$

Q3: [8 points] Two teams, called A and B, are scheduled to play each other. Ties cannot occur, and at the end of the game, one of the two teams will be the winner.

The weather on the day that they play may be either warm or cold, and the probability that it will be warm is 0.7. If it is cold, team A has a probability of winning of 0.4. If it is warm, team A has probability of winning of 0.8.

[4] **a) What is the probability that team A wins the game?**

[4] **b) Given team B wins, what is the conditional probability that the weather was cold?**

Solution: Define the events:

$A = \{\text{team A wins}\}$

$B = \{\text{team B wins}\} = A^c$

$W = \{\text{weather is warm}\}$

$C = \{\text{weather is cold}\} = W^c$

We know that:

$$P[W] = 0.7$$
$$P[C] = P[W^c] = 1 - P[W] = 0.3$$
$$P[A|C] = 0.4$$
$$P[B|C] = P[A^c|C] = 1 - P[A|C] = 0.6$$
$$P[A|W] = 0.8$$
$$P[B|W] = P[A^c|W] = 1 - P[A|W] = 0.2$$

a) We use the law of total probability:

$$P[A] = P[A|C]P[C] + P[A|W]P[W] = 0.4 \times 0.3 + 0.8 \times 0.7 = 0.12 + 0.56 = 0.68$$

b) By Baye's thm

$$P[C|B] = \frac{P[BC]}{P[B]} = \frac{P[B|C]P[C]}{P[B|C]P[C] + P[B|W]P[W]} = \frac{0.6 \times 0.3}{0.6 \times 0.3 + 0.2 \times 0.7} = \frac{0.18}{0.32}$$

Q4: [10 points] A type of computer memory requires 3 transistors to create 1 bit of memory, and all 3 transistors that comprise the bit of memory need to function correctly for the bit to function correctly.

A module with 10^6 bits is created, and thus this module has 3×10^6 transistors.

Each of the 3×10^6 transistors is, independently, defective with probability $p = 10^{-6}$.

[3] a) If X is the number of defective transistors on the module, what is the probability mass function (pmf) of X?

[3] b) If Y is the number of defective bits on the module, what is the probability mass function (pmf) of Y?

[4] c) Can Y be well approximated by a Poisson distribution? If so, explain why and give the parameter λ. If not, explain why not.

Note: For a) and b), do not make any approximations.

Solution:

a) There are $n = 3 \times 10^6$ transistors, and each functions or is defective with probability $p = 10^{-6}$, independently of any other transistor. So X follows a binomial distribution with parameters n and p. The pmf is

$$P[X = k] = \begin{cases} \binom{3\times 10^6}{k}(1-10^{-6})^{n-k}(10^{-6})^k & k \in \{0, 1, \ldots, 3 \times 10^6\} \\ 0 & \text{else} \end{cases}$$

b) A bit functions if and only if the 3 transistors that comprise it function. And those 3 transistors function independently of any other transistors on the module. So each bit on the module functions or is defective independently of whether the other bits on the module function or are defective. [This argument can be formalized, but it is not necessary to do so for full credit.] Let the probability that a bit is defective be q.

So the pmf of Y is binomial with parameters $m = 10^6$ and q:

$$P[Y = k] = \begin{cases} \binom{10^6}{k}(1-q)^{n-k}q^k & k \in \{0, 1, \ldots, 10^6\} \\ 0 & \text{else} \end{cases}$$

It remains to determine q. This is the probability that one bit is defective. Since we need all 3 transistors to function for the bit to function, then the probability that a bit functions is

$$P[\{\text{a bit functions}\}] = 1 - q = (1 - 10^{-6})^3$$

so

$q = 1 - (1 - 10^{-6})^3 = 3 \times 10^{-6} - 3 \times 10^{-12} + 10^{-18}$

Note: parts a) and b) did not allow for any approximations; $q \neq 3p$ and the exact expression for q is $q = 1 - (1-p)^3 = 3p - 3p^2 + p^3$. The 2 extra terms account for "double counting" the ways that 1 or more transistors are defective, and could also be obtained from the inclusion/exclusion principle.

c) Yes, Y can be well approximated. This is because $n = 10^6$ is large, $q = 3 \times 10^{-6} - 3 \times 10^{-12} + 10^{-18}$ is small, and $nq = 10^6(1 - (1 - 10^{-6})^3)$ is moderate (nq is approximately 3).

So Y can be approximated by a Poisson distribution with parameter

$$\lambda = nq = 10^6 \times (1 - (1 - 10^{-6})^3) = 3 - 3 \times 10^{-6} + 10^{-12}$$

If one wants to further approximate, Y can also be approximated by a Poisson distribution with parameter $\lambda = 3$ since the difference in the PMF of a Poisson distribution with parameter $3 - 3 \times 10^{-6} + 10^{-12}$ and another with parameter 3 is small.

Q5: [10 points] A sensor node sends a request to a server every t_0 seconds until the server responds, with the first request at time $t = 0$. Each request is either received instantly by the server, or lost forever (and thus never received by the server). Each request is, independently, lost forever with probability p.

As soon as the server receives a request, it then responds instantly with no delay, and the sensor node stops sending requests.

There are two design options:

A) Send a request every $t_0 = 1$ seconds until the server responds. In this case, $p = 0.5$.

B) Send a request every $t_0 = 2$ seconds until the server responds. In this case, $p = 0.2$.

Let T be the time when the server responds.

[5] a) For option A, find $E[T]$. For option B, find $E[T]$.

[5] b) For option B only, what is $Var[T]$?

Solution: Let X be the number of attempts/requests by the node until the server responds, and T the time until the server responds. So $X = 1$ means the server responds on the 1st attempt/request at time 0, and in this case $T = 0$. Likewise, $X = 2$ means the server responds on the 2nd attempt at time t_0, and in this case, $T = t_0$. In general,

$$T = t_0 \times (X - 1).$$

Also, we note that X is the number of trials needed to obtain a success where each trial is independently a success with probability $q = 1-p$, so X is a geometric random variable with parameter $q = 1-p = 0.5$ for option A, and $q = 1 - p = 0.8$ for option B.

a) Here, we have that

$E[T] = E[t_0(X - 1)] = t_0 E[X - 1] = t_0(E[X] - 1),$

and, since X is a geometric rv with parameter q, $E[X] = 1/q$ (from the textbook or course notes). Therefore

$$E[T] = t_0(1/q - 1) = t_0(1 - q)/q$$

Option A) with $t_0 = 1$s and $q = 0.5$, this gives $E[T] = 1(1 - 0.5)/0.5 = 1s$.
Option B) with $t_0 = 2$s and $q = 0.8$, this gives $E[T] = 2(1 - 0.8)/0.8 = 0.4/0.8 = 0.5s$.

So, option B has the shorter expected time to respond.

b) Here, we have that

$Var[T] = Var[t_0(X-1)] = t_0^2 Var[X-1] = t_0^2 Var[X]$.

So, for $t_0 = 2$, and $Var[X] = (1-q)/q^2$, we get

$Var[T] = t_0^2 Var[X] = 2^2 \dfrac{(1-0.8)}{(0.8)^2} = \dfrac{0.8}{(0.8)^2} = 1.25s^2$

The units of s and s^2 are not needed for full credit on either part.

Q6: [9 points] In each case, must the conclusion always be true? [Answer yes or no for each case].

If you claim yes, it is always true, then prove why it is always true. If you claim no, give an example where the conditions are satisfied, but the conclusion is false.

[3] a) If $A \subset B \subset C$, then $P[B|C] > P[A|B]$.

[3] b) If $A \subset B$ and $P[A \cap B \cap C] = P[A]P[B]P[C]$, then A, B and C are independent.

[3] c) If E and F are events with $P[E] = 1$, then E and F are independent.

Solution:

a) No, it is possible that this is false. Consider rolling a fair 6 sided die, and let $A = \{1\}, B = \{1, 2\}, C = \{1, 2, 3, 4, 5, 6\}$. Then $A \subset B \subset C$ and

$P[B|C] = 2/6$

$P[A|B] = 1/2$

b) No, it is possible that this is false. Consider rolling a fair 6 sided die, and let $A = \{1\}, B = \{1, 2, 3\}, C = \emptyset$.

Then $A \subset B$, and $P[A \cap B \cap C] = 0 = P[A]P[B]P[C]$. But $P[B|A] = 1 \neq 1/2 = P[B]$ so A and B are not independent, and hence A, B and C are not independent.

c) Yes, always true. Note, we don't know if $P[F] = 0$ or not. So we can't do any reasoning that requires dividing by $P[F]$. One way to show this is to first show that because $P[E] = 1$, that $P[F] = P[EF]$. Then

$$P[EF] = P[F] = 1 \times P[F] = P[E]P[F],$$

so E and F are independent. To show that $P[F] = P[EF]$:

$$P[F] = P[(E \cup E^c)F] = P[EF] + P[E^cF] \tag{1}$$

Now, $P[E^c] = 0$. So $0 \leq P[E^cF] \leq P[E^c] = 0$ since $E^cF \subset E^c$. Therefore, $P[E^cF] = 0$ and from (1): $P[F] = P[EF]$.

There are other ways to show that E and F are independent. Another would be to show that $G = E^c$ and F are independent where $P[G] = 0$ now. Then, by Proposition 4.1 of Chapter 3 of the textbook, $G^c = E$ and F are independent.