ECE 203: Probability Theory and Statistics I Fall 2020 Test#2

Instructor: Patrick Mitran

- This test consists of 6 problems and 7 pages, including the declaration of integrity. Each page is numbered.
- Q1 is the declaration of integrity. Failure to complete this declaration will result in a grade of 0 on the test.
- You have a 24 hour window (10am on 12 Nov to 10am on 13 Nov) during which you must start, complete and submit this test. From the moment you start the test, you will have the lesser of 2 hours or until 10am on 13 Nov to submit your test.
- Non-graphing, non-programmable calculators are allowed, but will not be helpful. If the answer to a question is 5, writing $\sqrt{100}/2$ will get you full marks. Access to Matlab or similar computational software is prohibited.
- You may use i) any personal notes you take, as long as you composed them yourself prior to the test (this includes notes based on the lectures/videos, tutorials, office hours, textbook, or other book), ii) any content available on the ECE203 Learn website (including all videos) and ECE203 Piazza website, iii) the course textbook.
- You may not use the internet other than i) to access the course ECE203 webpage on Learn and ECE203 Piazza discussions, ii) to access the textbook (if ecopy) iii) to access crowdmark, or iv) to send email to me or receive email from me.
- You may use a computer, tablet or phone for only the following purposes: i) to create/scan/upload your solutions, ii) to access crowdmark, iii) to access Learn and Piazza, iv) to access your personal notes (if you took these electronically), v) to access the textbook, vi) to send/receive email to/from me, or vii) to be used as a basic calculator following the calculator rule above. Use of any file sharing services such as chegg.com is prohibited.
- You may not communicate directly or indirectly with your classmates or anyone else except for me.
- Questions are allowed but will be answered only if you cannot understand the statement of a problem. You can reach me by email (pmitran@uwaterloo.ca) from 10am to 6pm on 12 Nov.
- All answers must be written legibly. We reserve our right to reduce your grade if your answer is not written in a legible manner.
- A final correct answer does not mean much to us, if the corresponding approach is not clear and sensible. Please explain your solutions and convince us that your solutions make sense.

Q1: [1 point] DECLARATION OF INTEGRITY IN EXAMINATIONS AND TESTS Course: ECE 203 Probability Theory and Statistics I Term: Fall 2020

I declare that I have read and followed the instructions listed on the cover page of ECE203 Test#2.

Signature

ID Number

Date

Note: If you are unable to print this page, it is sufficient to write out yourself "I declare that I have read and followed the instructions listed on the cover page of ECE203 Test#2.", then sign, write your ID number, and date the statement, and upload this.

Q2: [7 points] Let the probability density function (pdf) of X be

$$f_X(x) = \begin{cases} c(4 - x^2) & a < x < b \\ 0 & else \end{cases}$$

where a, b are some constants and c > 0 is another constant.

[3] a) What is the smallest possible value of a? What is the largest possible value of b?

[4] b) If a = -1 and b = 1, what is c?

Q3: [12 points] Let the probability density function (pdf) of Y be

$$f_Y(y) = \begin{cases} \frac{1+y^3}{2} & -1 < y < 1\\ 0 & \text{else} \end{cases}$$

- [3] a) What is $F_Y(y)$?
- [3] b) What is P[|Y| < 1/2]?
- [3] c) What is $E[Y^n]$ where $n \ge 0$ is an integer?
- [3] d) What is Var[Y]?

Q4: [10 points] Let the probability density function (pdf) of Z be

$$f_Z(z) = \begin{cases} \frac{1}{z^2} & z \ge 1\\ 0 & \text{else} \end{cases}$$

- [4] a) What is the pdf of $Y = \sqrt{Z}$?
- [4] b) Let $a \ge 1$ and $b \ge 1$. What is $P[Z > ab \mid Z > a]$?
- [2] c) Is the random variable Z memoryless? Justify your answer.

Q5: [10 points] Let X and Y be independent random variables such that X is exponential with parameter 1 and Y is exponential with parameter 2 (i.e., $X \sim \text{Exp}(1)$ and $Y \sim \text{Exp}(2)$).

- [5] a) Let Z = X/2 + Y. What is the pdf $f_Z(z)$?
- [5] b) Let $U = \max(X, Y)$. What is the cdf $F_U(u)$?

Q6: [10 points] The lifetime X of a hard drive is a random variable whose distribution depends on the usage factor U of the drive. A drive that is used more tends to have a shorter lifetime. Specifically, the conditional pdf of the lifetime is given by

$$f_{X|U}(x|u) = \begin{cases} u \exp(-ux) & x > 0\\ 0 & \text{else} \end{cases}$$

Assume now that U is a random variable with pdf:

$$f_U(u) = \begin{cases} \frac{1}{u} & 1 \le u \le e \\ 0 & \text{else} \end{cases}$$

[5] a) What is the pdf $f_X(x)$ of X?

[5] b) Suppose that the drive has lasted for x > 0 units of time. What is the conditional pdf $f_{U|X}(u|x)$?