ECE 203: Probability Theory and Statistics I
Fall 2020 Test#2

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- This test consists of ABC problems and XYZ pages, including the declaration of integrity. Each page is numbered.
- Q1 is the declaration of integrity. Failure to complete this declaration will result in a grade of 0 on the test.
- You have a 24 hour window (10am on 12 Nov to 10am on 13 Nov) during which you must start, complete and submit this test. From the moment you start the test, you will have the lesser of 2 hours or until 10am on 13 Nov to submit your test.
- Non-graphing, non-programmable calculators are allowed, but will not be helpful. If the answer to a question is 5, writing $\sqrt{100}/2$ will get you full marks. Access to Matlab or similar computational software is prohibited.
- You may use i) any personal notes you take, as long as you composed them yourself prior to the test (this includes notes based on the lectures/videos, tutorials, office hours, textbook, or other book), ii) any content available on the ECE203 Learn website (including all videos) and ECE203 Piazza website, iii) the course textbook.
- You may not use the internet other than i) to access the course ECE203 webpage on Learn and ECE203 Piazza discussions, ii) to access the textbook (if ecopy) iii) to access crowdmark, or iv) to send email to me or receive email from me.
- You may use a computer, tablet or phone for only the following purposes: i) to create/scan/upload your solutions, ii) to access crowdmark, iii) to access Learn and Piazza, iv) to access your personal notes (if you took these electronically), v) to access the textbook, vi) to send/receive email to/from me, or vii) to be used as a basic calculator following the calculator rule above. Use of any file sharing services such as chegg.com is prohibited.
- You may not communicate directly or indirectly with your classmates or anyone else except for me.
- Questions are allowed but will be answered only if you cannot understand the statement of a problem. You can reach me by email (pmitran@uwaterloo.ca) from 10am to 6pm on 12 Nov.
- All answers must be written legibly. We reserve our right to reduce your grade if your answer is not written in a legible manner.
- A final correct answer does not mean much to us, if the corresponding approach is not clear and sensible. Please explain your solutions and convince us that your solutions make sense.

Q1: [1 point] DECLARATION OF INTEGRITY IN EXAMINATIONS AND TESTS
Course: ECE 203 Probability Theory and Statistics I
Term: Fall 2020

I declare that I have read and followed the instructions listed on the cover page of ECE203 Test#2.

_______________________	_______________________	_______________________
Signature	**ID Number**	**Date**

Note: If you are unable to print this page, it is sufficient to write out yourself "I declare that I have read and followed the instructions listed on the cover page of ECE203 Test#2.", then sign and date the statement, and upload this.

Q2: [7 points] Let the pdf of X be

$$f_X(x) = \begin{cases} c(4-x^2) & a < x < b \\ 0 & else \end{cases}$$

where a, b are some constants and $c > 0$ is another constant.

[3] a) What is the smallest possible value of a? What is the largest possible value of b?

[4] b) If $a = -1$ and $b = 1$, what is c?

Solution: a) A pdf $f_X(x)$ must satisfy $f_X(x) \geq 0$. So

$c(4 - x^2) \geq 0$

and since $c > 0$, this implies $-2 \leq x \leq 2$. So the smallest a is -2, and the largest b is +2.
b) A pdf must integrate to 1, so

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} f_X(x)dx \\ &= \int_{-1}^{1} c(4-x^2)dx \\ &= c\int_{-1}^{1} (4-x^2)dx \\ &= c[4x - x^3/3]_{-1}^{1} \\ &= c[4(1-(-1)) - (1/3 - (-1/3))] \\ &= c[8 - 2/3] \end{aligned}$$

So,

$$c = \frac{1}{8 - 2/3} = \frac{3}{22}$$

Q3: [12 points] Let the pdf of Y be

$$f_Y(y) = \begin{cases} \frac{1+y^3}{2} & -1 < y < 1 \\ 0 & \textbf{else} \end{cases}$$

[3] a) What is $F_Y(y)$?

[3] b) What is $P[|Y| < 1/2]$?

[3] c) What is $E[Y^n]$ where $n \geq 0$ is some integer?

[3] d) What is $Var[Y]$?

Solution: a) Note that the pdf is 0 for $y \leq -1$ and $y \geq 1$. So $F_Y(y) = 0$ for $y \leq -1$ and $F_Y(y) = 1$ for $y \geq 1$. It remains to compute the cdf for $-1 < y < 1$. So, assume $-1 < y < 1$:

$$\begin{aligned} F_Y(y) &= \int_{-\infty}^{y} f_Y(u)du \\ &= \int_{-1}^{y} f_Y(u)du \\ &= \frac{1}{2}\int_{-1}^{y} 1 + u^3 du \\ &= \frac{1}{2}[u + u^4/4]_{-1}^{y} du \\ &= \frac{1}{2}(y - (-1) + (y^4 - 1)/4) \\ &= 3/8 + y/2 + y^4/8 \end{aligned}$$

So,

$$F_Y(y) = \begin{cases} 0 & y \leq -1 \\ 3/8 + y/2 + y^4/8 & -1 < y < 1 \\ 1 & 1 \leq y \end{cases}$$

b) First approach:

$$
\begin{aligned}
P[|Y| < 1/2] &= P[-1/2 < Y < 1/2] \\
&= \int_{-1/2}^{1/2} f_Y(y)dy \\
&= \frac{1}{2}\int_{-1/2}^{1/2} 1 + y^3 dy \\
&= \frac{1}{2}\int_{-1/2}^{1/2} 1dy + \frac{1}{2}\int_{-1/2}^{1/2} y^3 dy \\
&= \frac{1}{2}\int_{-1/2}^{1/2} 1dy \\
&= 1/2
\end{aligned}
$$

Second approach

$$
\begin{aligned}
P[|Y| < 1/2] &= P[-1/2 < Y < 1/2] \\
&= P[Y < 1/2] - P[Y \leq -1/2] \\
&= P[Y \leq 1/2] - P[Y \leq -1/2] \qquad \text{Since } Y \text{ is a continuous rv.} \\
&= F_Y(1/2) - F_Y(-1/2) \\
&= 3/8 + (1/2)/2 + (1/2)^4/8 - (3/8 + (-1/2)/2 + (-1/2)^4/8) \\
&= (1/2)/2 - (-1/2)/2 \\
&= 1/2
\end{aligned}
$$

c)

$$
\begin{aligned}
E[Y^n] &= \int_{-\infty}^{\infty} u^n f_Y(u)du \\
&= \int_{-1}^{1} u^n f_Y(u)du \\
&= \frac{1}{2}\int_{-1}^{1} u^n + u^{n+3} du \\
&= \frac{1}{2}\left[\frac{u^{n+1}}{n+1} + \frac{u^{n+4}}{n+4}\right]_{-1}^{1} \\
&=^{(a)} \frac{1}{2}\left[\frac{1}{n+1} + \frac{1}{n+4}\right] - \frac{1}{2}\left[\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n+4}}{n+4}\right] \\
&= \begin{cases} \frac{1}{n+1} & n \text{ is even} \\ \frac{1}{n+4} & n \text{ is odd} \end{cases}
\end{aligned}
$$

Full credit is given for getting to (a).

d)

$$
\begin{aligned}
Var[Y] &= E[Y^2] - (E[Y])^2 \\
&= \frac{1}{3} - \left(\frac{1}{5}\right)^2 \\
&= \frac{22}{75}
\end{aligned}
$$

Q4: [10 points] Let the pdf of Z be

$$f_Z(z) = \begin{cases} \frac{1}{z^2} & z \geq 1 \\ 0 & \textbf{else} \end{cases}$$

[4] a) What is the pdf of $Y = \sqrt{Z}$?

[4] b) Let $a \geq 1$ and $b \geq 1$. What is $P[Z > ab \mid Z > a]$?

[2] c) Is the random variable Z memoryless? Justify your answer.

Solution: a) First, we get the CDF of Z. Assuming $a \geq 1$:

$$F_Z(a) = P[Z \leq a] = \int_{-\infty}^{a} f_Z(z)dz = \int_{1}^{a} \frac{1}{z^2}dz = \left.\frac{-1}{z}\right|_1^a = 1 - \frac{1}{a}$$

Also, $F_Z(a) = 0$ for $a < 1$. Now, for $a \geq 1$:

$$P[Y \leq a] = P[\sqrt{Z} \leq a] = P[Z \leq a^2] = 1 - \frac{1}{a^2}$$

Also, for $a < 1$, $P[Y \leq a] = P[\sqrt{Z} \leq a] = 0$ since $PZ \geq 1] = 1$. So,

$$\begin{aligned} f_Y(a) &= \frac{d}{da}P[Y \leq a] \\ &= \begin{cases} \frac{2}{a^3} & 1 \leq a \\ 0 & \text{else} \end{cases} \end{aligned}$$

b) First, note that $P[Z > a] = 1 - P[Z \leq a] = 1 - (1 - 1/a) = 1/a$.

$$\begin{aligned} P[Z > ab \mid Z > a] &=^{(a)} \frac{P[Z > ab, Z > a]}{P[Z > a]} \\ &=^{(b)} \frac{P[Z > ab]}{P[Z > a]} \\ &= \frac{1/(ab)}{1/a} \\ &= 1/b \\ &= P[Z > b] \end{aligned}$$

where (a) follows since $P[A|B] = P[AB]/P[B]$, and (b) follows since $\{Z > ab\} \subset \{Z > a\}$ since $ab \geq a$ (because $a \geq 1$ and $b \geq 1$). Full credit for any step after step (b).

c) To be memoryless, we would have to show that for $a \geq 0$ and $b \geq 0$, that $P[Z > a + b|Z > a] = P[Z > b]$. We have not shown this is part b).

No, it is not memoryless.

Justification 1: As explained in example 5c of Chapter 5 in the textbook, the exponential distribution is the only (continuous) distribution that is memoryless. Since Z is continuous and is not exponential, it is not memoryless.

Justification 2: Alternatively, you can do a direct computation (for $a \geq 1$ and $b \geq 1$):

$$\begin{aligned} P[Z > a + b \mid Z > a] &= \frac{P[Z > a + b, Z > a]}{P[Z > a]} \\ &= \frac{P[Z > a + b]}{P[Z > a]} \\ &= \frac{1/(a+b)}{1/a} \\ &= \frac{1}{1 + b/a} \\ &\neq P[Z > b] \end{aligned}$$

where we don't have equality in the last line because the last line is a function of b only, while the preceeding line depends on both a and b.

Q5: [10 points] Let X and Y be independent random variables such that X is exponential with parameter 1 and Y is exponential with parameter 2 (i.e., $X \sim \mathsf{Exp}(1)$ and $Y \sim \mathsf{Exp}(2)$).

[5] a) Let $Z = X/2 + Y$. What is the pdf of Z?

[5] b) Let $U = \max(X, Y)$. What is the cdf of U?

Solution: a) We are told that:

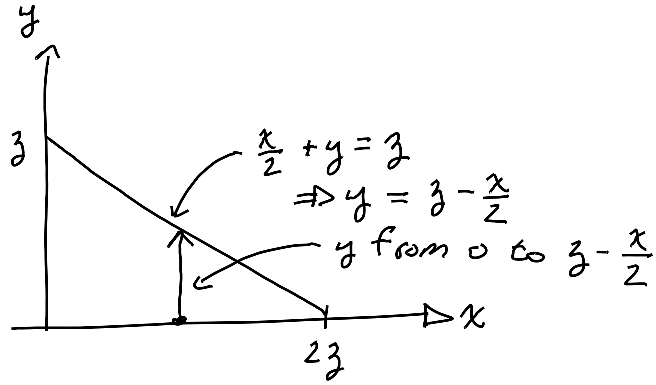
$$f_X(x) = \begin{cases} e^{-x} & x > 0 \\ 0 & \text{else} \end{cases} \qquad f_Y(y) = \begin{cases} 2e^{-2y} & y > 0 \\ 0 & \text{else} \end{cases}$$

and since X and Y are independent:

$$\begin{aligned} f_{XY}(x,y) &= f_X(x) f_Y(y) \\ &= \begin{cases} 2e^{-x}e^{-2y} & x > 0, y > 0 \\ 0 & \text{else} \end{cases} \end{aligned}$$

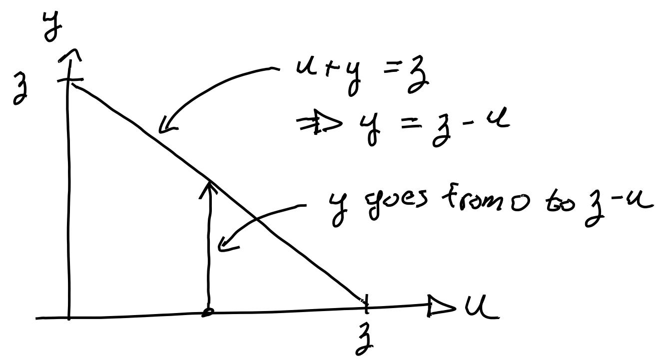
a) Method 1: For $z > 0$, let $A = \{(x,y) \in \mathbb{R}^2 | x > 0, y > 0, x/2 + y < z\}$. Then

$$\begin{aligned}
P[Z \leq z] &= P[X/2 + Y \leq z] \\
&= P[(X,Y) \in A] \\
&= \iint_A f_{XY}(x,y)dxdy \\
&= \iint\limits_{\substack{x/2+y\leq z \\ x>0, y>0}} 2e^{-x}e^{-2y}dxdy \\
&= \int_0^{2z} \int_0^{z-x/2} 2e^{-2y}e^{-x}dydx \\
&= \int_0^{2z} e^{-x} \int_0^{z-x/2} 2e^{-2y}dydx \\
&= \int_0^{2z} e^{-x}[-e^{-2y}]_{y=0}^{y=z-x/2}dx \\
&= \int_0^{2z} e^{-x}[1-e^{-2(z-x/2)}]dx \\
&= \int_0^{2z} e^{-x} - e^{-2z}dx \\
&= [-e^{-x} - xe^{-2z}]_{x=0}^{x=2z} \\
&= 1 - e^{-2z} - 2ze^{-2z}
\end{aligned}$$



Method 2: Let $U = X/2$. Then, following the method of example 1d of chapter 5, U is $\mathsf{Exp}(2)$. So $Z = U + Y$, and $U = X/2$ is independent of Y since X is independent of Y. Therefore, for $z > 0$:

$$
\begin{aligned}
P[Z \le z] &= P[U + Y \le z] \\
&= \int_0^z \int_0^{z-u} 4e^{-2y} e^{-2u} dy du \\
&= \int_0^z 2e^{-2u} \int_0^{z-u} 2e^{-2y} dy du \\
&= \int_0^z 2e^{-2u} [-e^{-2y}]_{y=0}^{z-u} du \\
&= \int_0^z 2e^{-2u} [1 - e^{-2(z-u)}] du \\
&= \int_0^z 2e^{-2u} - 2e^{-2z} du \\
&= [-e^{-2u}]_0^z - 2ze^{-2z} \\
&= 1 - e^{-2z} - 2ze^{-2z}
\end{aligned}
$$



From either method: the pdf is, for $z \ge 0$:

$$
\begin{aligned}
f_Z(z) &= \frac{d}{dz} P[Z \le z] \\
&= \frac{d}{dz} 1 - e^{-2z} - e^{-2z} 2z \\
&= 4ze^{-2z}
\end{aligned}
$$

For $z < 0$, $P[Z \le z] = 0$ since X and Y are non-negative. So the pdf $f_Z(z) = 0$ for $z < 0$. So

$$
f_Z(z) = \begin{cases} 4ze^{-2z} & z \ge 0 \\ 0 & z < 0 \end{cases}
$$

b) For $u \ge 0$:

$$
\begin{aligned}
P[U \le u] &= P[\max(X, Y) \le u] \\
&= P[X \le u, Y \le u] \\
&= P[X \le u] P[Y \le u] \qquad \text{since } X \text{ and } Y \text{ are independent} \\
&= (1 - e^{-u})(1 - e^{-2u}) \\
&= 1 - e^{-u} - e^{-2u} + e^{-3u}
\end{aligned}
$$

while for $u < 0$, $P[U \le u] = 0$ since $X \ge 0$ and $Y \ge 0$. So

$$
F_U(u) = \begin{cases} 1 - e^{-u} - e^{-2u} + e^{-3u} & u \ge 0 \\ 0 & u < 0 \end{cases}
$$

Q6: [10 points] The lifetime X of a hard drive is a random variable whose distribution depends on the usage factor U of the drive. A drive that is used more tends to have a shorter lifetime. Specifically, the conditional pdf of the lifetime is given by

$$f_{X|U}(x|u) = \begin{cases} u\exp(-ux) & x > 0 \\ 0 & \textbf{else} \end{cases}$$

Assume now that U is a random variable with pdf:

$$f_U(u) = \begin{cases} \frac{1}{u} & 1 \le u \le e \\ 0 & \textbf{else} \end{cases}$$

[5] a) What is the pdf $f_X(x)$ of X?

[5] b) Suppose that the drive has lasted for $x > 0$ units of time. What is the conditional pdf $f_{U|X}(u|x)$?

Solution: a) We have that

$$\begin{aligned} f_{XU}(x,u) &= f_{X|U}(x|u)f_U(u) \\ &= \begin{cases} u\exp(-ux) \times \frac{1}{u} & x > 0, 1 \le u \le e \\ 0 & \text{else} \end{cases} \\ &= \begin{cases} \exp(-ux) & x > 0, 1 \le u \le e \\ 0 & \text{else} \end{cases} \end{aligned}$$

So

$$f_X(x) = \int_{-\infty}^{\infty} f_{XU}(x,u)du$$

and we have two cases. Case 1: $x > 0$:

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{XU}(x,u)du \\ &= \int_1^e f_{XU}(x,u)du \\ &= \int_1^e \exp(-ux)du \\ &= [-\exp(-ux)/x]_{u=1}^{u=e} \\ &= \exp(-x)/x - \exp(-ex)/x \\ &= \frac{\exp(-x) - \exp(-ex)}{x} \end{aligned}$$

Case 2: $x \leq 0$: Here $f_{XU}(x,u) = 0$, so

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{XU}(x,u)du \\ &= \int_{-\infty}^{\infty} 0du \\ &= 0 \end{aligned}$$

Combined, we have

$$f_X(x) = \begin{cases} \frac{\exp(-x)-\exp(-ex)}{x} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

b) For $x > 0$:

$$f_{U|X}(u|x) = \frac{f_{XU}(x,u)}{f_X(x)}$$

Again, there are 2 cases. Case 1: $1 \leq u \leq e$:

$$\begin{aligned} f_{U|X}(u|x) &= \frac{\exp(-ux)}{\frac{\exp(-x)-\exp(-ex)}{x}} \\ &= \frac{x\exp(-ux)}{\exp(-x)-\exp(-ex)} \end{aligned}$$

Case 2: $u \notin [1,e]$:

$$\begin{aligned} f_{U|X}(u|x) &= \frac{f_{XU}(x,u)}{f_X(x)} \\ &= \frac{0}{f_X(x)} \\ &= 0 \end{aligned}$$

So,

$$f_{U|X}(u|x) = \begin{cases} \frac{x\exp(-ux)}{\exp(-x)-\exp(-ex)} & 1 \leq u \leq e \\ 0 & \text{else} \end{cases}$$