ECE 203: Probability Theory and Statistics I
Fall 2021 Final Exam

Instructor: Patrick Mitran

- This exam consists of 8 problems and 10 pages, including the declaration of integrity and 1 table. Each page is numbered.
- Q1 is the declaration of integrity. Failure to complete this declaration will result in a grade of 0 on the test.
- You have a 24 hour window (9am on 11 Dec to 9am on 12 Dec) during which you must start, complete and submit this test. From the moment you start the test, you will have the lesser of 3 hours or until 9am on 12 Dec to submit your exam.
- Non-graphing, non-programmable calculators are allowed, but are not necessary. If the answer to a question is 5, writing $\sqrt{100}/2$ will get you full marks. Access to Matlab or similar computational software is prohibited.
- You may use i) any personal notes you take, as long as you composed them yourself prior to the test (this includes notes based on the lectures/videos, tutorials, office hours, textbook, or other book), ii) any content available on the ECE203 Learn website (including all videos) and ECE203 Piazza website, iii) the course textbook.
- You may not use the internet other than i) to access the ECE203 course webpage on Learn and ECE203 Piazza discussions, ii) to access the textbook (if ecopy) iii) to access crowdmark, or iv) to send email to me or receive email from me.
- You may use a computer, tablet or phone for only the following purposes: i) to create/scan/upload your solutions, ii) to access crowdmark, iii) to access Learn and Piazza, iv) to access your personal notes (if you took these electronically), v) to access the textbook, vi) to send/receive email to/from me, or vii) to be used as a basic calculator following the calculator rule above. Use of any file sharing services such as chegg.com is prohibited.
- You may not communicate directly or indirectly with your classmates or anyone else except for me. Do not post on piazza during the test period.
- Questions are allowed but will be answered only if you cannot understand the statement of a problem. You can reach me by email (pmitran@uwaterloo.ca) from 9am to 5pm on 11 Dec.
- All answers must be written legibly. We reserve our right to reduce your grade if your answer is not written in a legible manner.
- A final correct answer does not mean much to us if the corresponding approach is not clear and sensible. Please explain your solutions and convince us that your solutions make sense.

Q1: [1 point] DECLARATION OF INTEGRITY IN EXAMINATIONS AND TESTS
Course: ECE 203 Probability Theory and Statistics I
Term: Fall 2021

I declare that I have read and followed the instructions listed on the cover page of the ECE203 Final Exam.

_____	_____	_____
Signature	**ID Number**	**Date**

Note: If you are unable to print this page, it is sufficient to write out yourself "I declare that I have read and followed the instructions listed on the cover page of the ECE203 Final Exam.", then sign, write your ID number, date the statement, and upload this.

Q2: [9 points] You have a 4-sided die with sides numbered from 1 to 4. You perform the following experiment:

- **You roll the 4-sided die and record the number.**
- **If you recorded a 2, 3 or a 4, the experiment ends.**
- **If you recorded a 1, you roll the die a second time, record the second number as well, and the experiment ends.**

Assume that the die is fair, and that rolls are independent.
Let N be the number of rolls, and X the sum of all recorded numbers.

[2] a) Define the sample space S of the experiment.

[3] b) Find the probability mass function (pmf) $p_X(x)$ of X.

[4] c) Find $P[N = 1|X = 3]$ and $P[N = 2|X = 3]$.

Q3: [8 points] Alice has a coin that flips heads with probability p. She flips the coin $n > 0$ times (assume the flips are independent).
If exactly k of the n flips are heads, she wins $\$1$. Otherwise she wins nothing.

[6] a) If $0 < k < n$, what should p be to maximize Alice's expected winnings?

[2] b) If $k = n$, what should p be to maximize Alice's expected winnings? Explain why this answer makes sense.

Q4: [9 points] A building has 2 rooms, called R1 and R2.
Room R1 has two doors, called A and B. If a person in room R1:

- **chooses door A, they return to room R1 after 1 hour.**
- **chooses door B, they exit the building after 2 hours.**

Room R2 has 3 doors, called C, D and E. If a person in room R2:

- **chooses door C, they return to room R2 after 3 hours.**
- **chooses door D, they return to room R2 after 4 hours.**
- **chooses door E, they exit the building after 5 hours.**

Bob always chooses randomly from among the doors in front of him with each door chosen equally likely.

[3] a) If Bob starts in room R1, how long does it take, on average, for Bob to exit the building?

[3] b) If Bob starts in room R2, how long does it take, on average, for Bob to exit the building?

[3] c) If Bob starts in room R1 with probability 1/3 and starts in room R2 with probability 2/3, how long does it take, on average, for Bob to exit the building?

Q5: [9 points] Let X and Y be bivariate Gaussian random variables with:

- $E[X] = 1$, $E[Y] = 2$,
- $Var[X] = 4$, $Var[Y] = 9$
- $Cov[X, Y] = -2$.

[2] a) What is the probability density function (pdf) $f_X(x)$ of X?

[2] b) What is $P[\, Y < E[Y] \,]$?

[3] c) What are the mean and variance of $Z = 2X - Y + 1$?

[2] d) What is the probability density function (pdf) $f_Z(z)$ of Z?

Q6: [9 points] The waiting time X for a processor to start processing a job has the cumulative distribution function (CDF)

$$F_X(x) = \begin{cases} 1 - e^{-x^3/\tau^3} & x \geq 0 \\ 0 & \textbf{else}, \end{cases}$$

where $\tau > 0$ is a constant. The cost of waiting is $Y = aX^3$ where $a > 0$ is another constant.

[3] a) Let $t > 0$ and $s > 0$. What is $P[X > s + t \mid X > s]$? Does X have the memoryless property?

[3] b) Does Y have the memoryless property?

[3] c) What is the probability density function (pdf) $f_X(x)$ of X?

Q7: [10 points] Let the pair of discrete random variables X and Y have joint probability mass function (pmf)

$$p_{XY}(x,y) = \begin{cases} \frac{1}{12} & x \in \{1,2\}, y \in \{1,2\} \\ \frac{1}{6} & x \in \{3,4\}, y \in \{3,4\} \\ 0 & \textbf{else.} \end{cases}$$

[3] a) Are X and Y independent? Explain your answer.

[3] b) Find the marginal pmf $p_X(x)$ of X.

[4] c) Find the pmf of $Z = \max(X, Y)$.

Q8: [10 points]

[5] a) Let $X_1, X_2, \ldots$ be an infinite sequence of independent and identically distributed Bernoulli random variables with parameter $p = 1/2$. Find

$$\lim_{n\to\infty} P\left[\sum_{i=1}^{n}\left(X_i - \frac{1}{2}\right) \leq \sqrt{n}\right].$$

b) Let $X \sim U(0,1)$ and $Y \sim U(0,1)$, and X and Y are independent.
[2] i) Find $E[X^Y]$.
[3] ii) Find $Var[X^Y]$.

Table of $\Phi(x)$:

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.50000	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.52790	0.53188	0.53586
0.1	0.53983	0.54380	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.62930	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.65910	0.66276	0.66640	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.70540	0.70884	0.71226	0.71566	0.71904	0.72240
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.75490
0.7	0.75804	0.76115	0.76424	0.76730	0.77035	0.77337	0.77637	0.77935	0.78230	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.86650	0.86864	0.87076	0.87286	0.87493	0.87698	0.87900	0.88100	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.90320	0.90490	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.92220	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.94520	0.94630	0.94738	0.94845	0.94950	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.96080	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.97320	0.97381	0.97441	0.97500	0.97558	0.97615	0.97670
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.98030	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.98300	0.98341	0.98382	0.98422	0.98461	0.98500	0.98537	0.98574
2.2	0.98610	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.98840	0.98870	0.98899
2.3	0.98928	0.98956	0.98983	0.99010	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.99180	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.99430	0.99446	0.99461	0.99477	0.99492	0.99506	0.99520
2.6	0.99534	0.99547	0.99560	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.99720	0.99728	0.99736
2.8	0.99744	0.99752	0.99760	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.99900
3.1	0.99903	0.99906	0.99910	0.99913	0.99916	0.99918	0.99921	0.99924	0.99926	0.99929
3.2	0.99931	0.99934	0.99936	0.99938	0.99940	0.99942	0.99944	0.99946	0.99948	0.99950
3.3	0.99952	0.99953	0.99955	0.99957	0.99958	0.99960	0.99961	0.99962	0.99964	0.99965
3.4	0.99966	0.99968	0.99969	0.99970	0.99971	0.99972	0.99973	0.99974	0.99975	0.99976