ECE 203: Probability Theory and Statistics I
Fall 2021 Final Exam

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- This exam consists of ABC problems and XYZ pages, including the declaration of integrity and 1 table. Each page is numbered.
- Q1 is the declaration of integrity. Failure to complete this declaration will result in a grade of 0 on the test.
- You have a 24 hour window (9am on 11 Dec to 9am on 12 Dec) during which you must start, complete and submit this test. From the moment you start the test, you will have the lesser of 3 hours or until 9am on 12 Dec to submit your test.
- Non-graphing, non-programmable calculators are allowed, but are not necessary. If the answer to a question is 5, writing $\sqrt{100}/2$ will get you full marks. Access to Matlab or similar computational software is prohibited.
- You may use i) any personal notes you take, as long as you composed them yourself prior to the test (this includes notes based on the lectures/videos, tutorials, office hours, textbook, or other book), ii) any content available on the ECE203 Learn website (including all videos) and ECE203 Piazza website, iii) the course textbook.
- You may not use the internet other than i) to access the ECE203 course webpage on Learn and ECE203 Piazza discussions, ii) to access the textbook (if ecopy) iii) to access crowdmark, or iv) to send email to me or receive email from me.
- You may use a computer, tablet or phone for only the following purposes: i) to create/scan/upload your solutions, ii) to access crowdmark, iii) to access Learn and Piazza, iv) to access your personal notes (if you took these electronically), v) to access the textbook, vi) to send/receive email to/from me, or vii) to be used as a basic calculator following the calculator rule above. Use of any file sharing services such as chegg.com is prohibited.
- You may not communicate directly or indirectly with your classmates or anyone else except for me. Do not post on piazza during the test period.
- Questions are allowed but will be answered only if you cannot understand the statement of a problem. You can reach me by email (pmitran@uwaterloo.ca) from 9am to 5pm on 11 Dec.
- All answers must be written legibly. We reserve our right to reduce your grade if your answer is not written in a legible manner.
- A final correct answer does not mean much to us if the corresponding approach is not clear and sensible. Please explain your solutions and convince us that your solutions make sense.

Q1: [1 point] DECLARATION OF INTEGRITY IN EXAMINATIONS AND TESTS
Course: ECE 203 Probability Theory and Statistics I
Term: Fall 2021

I declare that I have read and followed the instructions listed on the cover page of the ECE203 Final Exam.

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Signature	**ID Number**	**Date**

Note: If you are unable to print this page, it is sufficient to write out yourself "I declare that I have read and followed the instructions listed on the cover page of the ECE203 Final Exam.", then sign, write your ID number, date the statement, and upload this.

Q2: [9 points] You have a 4-sided die with sides numbered from 1 to 4. You perform the following experiment:

- **You roll the 4-sided die and record the number.**
- **If you recorded a 2, 3 or a 4, the experiment ends.**
- **If you recorded a 1, you roll the die a second time, record the second number as well, and the experiment ends.**

Assume that the die is fair, and that rolls are independent.
Let N be the number of rolls, and X the sum of all recorded numbers.

[2] a) Define the sample space S of the experiment.

[3] b) Find the probability mass function (pmf) $p_X(x)$ for X.

[4] c) Find $P[N = 1|X = 3]$ and $P[N = 2|X = 3]$.

Solutions:

a)

$$S = \{(1, 1), (1, 2), (1, 3), (1, 4), (2), (3), (4)\}$$

where a pair (a, b) denotes the first roll was a and the second was b, and a singleton (a) means a single roll that was a. There are 7 outcomes.

b) We have

$$P[(1, 1)] = P[(1, 2)] = P[(1, 3)] = P[(1, 4)] = 1/16$$
$$P[(2)] = P[(3)] = P[(4)] = 1/4$$

X can take integer values between 2 and 5:

$$p_X(2) = P[\{(1, 1), (2)\}] = 1/16 + 1/4 = 5/16$$
$$p_X(3) = P[\{(1, 2), (3)\}] = 1/16 + 1/4 = 5/16$$
$$p_X(4) = P[\{(1, 3), (4)\}] = 1/16 + 1/4 = 5/16$$
$$p_X(5) = P[(1, 4)] = 1/16$$

and $p_X(x) = 0$ otherwise.

c)

$$P[N = 1|X = 3] = \frac{P[N = 1, X = 3]}{P[X = 3]} = \frac{P[(3)]}{5/16} = \frac{4/16}{5/16} = 4/5$$
$$P[N = 2|X = 3] = 1 - P[N = 1|X = 3] = 1/5$$

Q3: [8 points] Alice has a coin that flips heads with probability p. She flips the coin $n > 0$ times (assume the flips are independent).
If exactly k of the n flips are heads, she wins \$1. Otherwise she wins nothing.

[6] a) If $0 < k < n$, what should p be to maximize Alice's expected winnings?

[2] b) If $k = n$, what should p be to maximize Alice's expected winnings? Explain why this answer makes sense.

Solutions: a) Let X be the number of heads, and Y Alice's winnings. Then

$$\begin{aligned} E[Y] &= 1 \times P[X = k] + 0 \times P[X \neq 1] \\ &= P[X = 1] \\ &= \frac{n!}{(n-k)!k!} p^k (1-p)^{n-k} \end{aligned}$$

So, we want to pick p to maximize this, which is equivalent to maximizing $p^k(1-k)^{n-k}$:

$$\begin{aligned} 0 &= \frac{d}{dp} p^k (1-p)^{n-k} \\ &= kp^{k-1}(1-p)^{n-k} - p^k(n-k)(1-p)^{n-k-1} \\ &= p^{k-1}(1-p)^{n-k-1}[k(1-p) - (n-k)p] \end{aligned}$$

We rule out $p = 0$ and $p = 1$ as solutions since then Alice has 0 probability of winning \$1. This leaves

$$k(1-p) - (n-k)p = 0$$

which implies $p = k/n$.

b) In the special case that $k = n$, $p = 1$ since then all coins flip heads and Alice will always get n out of n heads.

Q4: [9 points] A building has 2 rooms, called R1 and R2.
Room R1 has two doors, called A and B. If a person in room R1:

- **chooses door A, they return to room R1 after 1 hour.**
- **chooses door B, they exit the building after 2 hours.**

Room R2 has 3 doors, called C, D and E. If a person in room R2:

- **chooses door C, they return to room R2 after 3 hours.**
- **chooses door D, they return to room R2 after 4 hours.**
- **chooses door E, they exit the building after 5 hours.**

Bob always chooses randomly from among the doors in front of him with each door chosen equally likely.

[3] a) If Bob starts in room R1, how long does it take, on average, for Bob to exit the building?

[3] b) If Bob starts in room R2, how long does it take, on average, for Bob to exit the building?

[3] c) If Bob starts in room R1 with probability 1/3 and starts in room R2 with probability 2/3, how long does it take, on average, for Bob to exit the building?

Solutions: This is a variation of Example 5c of Section 7.5 of the textbook.

a) Let X be the time it takes to go from room R1 to exit and let Z denote the door Bob chooses when in room R1. Then

$$\begin{aligned}E[X] &= E[X|Z = A]P[Z = A] + E[X|Z = B]P[Z = B]\\ &= (E[X] + 1)P[Z = A] + 2P[Z = B]\\ &= (E[X] + 1)\frac{1}{2} + 2\frac{1}{2}\end{aligned}$$

From which we solve that $E[X] = 3$.

b) Let Y be the time that it takes to go from room R2 to exit, and W the door that Bob chooses when in room R2. Then

$$\begin{aligned}E[Y] &= E[Y|W = C]P[W = C] + E[Y|W = D]P[W = D] + E[Y|W = E]P[W = E]\\ &= (E[Y] + 3)\frac{1}{3} + (E[Y] + 4)\frac{1}{3} + 5\frac{1}{3}\\ &= \frac{2}{3}E[Y] + \frac{12}{3}\end{aligned}$$

which results in $E[Y] = 12$.

c) Let U be the time the exit. Then

$$\begin{aligned} E[U] &= E[U|R1]P[R1] + E[U|R2]P[R2] \\ &= 3 \times \frac{1}{3} + 12 \times \frac{2}{3} \\ &= 9 \end{aligned}$$

Full credit if the solution used incorrect answers from a) and b), but is otherwise correct in how it used these answers.

Q5: [9 points] Let X and Y be bivariate Gaussian random variables with:

- $E[X] = 1$, $E[Y] = 2$,
- $Var[X] = 4$, $Var[Y] = 9$
- $Cov[X, Y] = -2$.

[2] a) What is the probability density function (pdf) of X?

[2] b) What is $P[\, Y < E[Y]\,]$?

[3] c) What are the mean and variance of $Z = 2X - Y + 1$?

[2] d) What is the probability density function (pdf) of Z?

Solutions:

a) Since X and Y are jointly Gaussian, X will be Gaussian. It has mean 1 and variance 4. So

$$\begin{aligned} f_X(x) &= \frac{1}{\sqrt{2\pi}\sigma_X} e^{-\frac{(x-\mu_X)^2}{2\sigma_X^2}} \\ &= \frac{1}{\sqrt{2\pi}2} e^{-\frac{(x-1)^2}{8}} \end{aligned}$$

b) Since Y is Gaussian, $P[Y < E[Y]] = \Phi(0) = 1/2$.

c)

$$\begin{aligned} E[Z] &= E[2X - Y + 1] = 2E[X] - E[Y] + 1 = 2 - 2 + 1 = 1 \\ Var[Z] &= Cov[Z, Z] \\ &= Cov[2X - Y + 1, 2X - Y + 1] \\ &= Cov[2X - Y, 2X - Y + 1] \\ &= Cov[2X - Y, 2X - Y] \\ &= 4Cov[X, X] - 2Cov[X, Y] - 2Cov[Y, X] + Cov[Y, Y] \\ &= 4 \times 4 - 2(-2) - 2(-2) + 9 \\ &= 16 + 4 + 4 + 9 \\ &= 33 \end{aligned}$$

d) Since Z is a linear function of X and Y (which are jointly Gaussian) plus a constant, Z is Gaussian. Since from part c) its mean is 1 and variance is 33, we have

$$f_Z(z) = \frac{1}{\sqrt{2\pi}\sqrt{33}} e^{-\frac{(z-1)^2}{2\times 33}}$$

[If a student reuses incorrect mean/variance from part c), but correctly states the Gaussian pdf with these incorrect mean/variance, full credit for part d).]

Q6: [9 points] The waiting time X for a processor to start processing a job has the cumulative distribution function (CDF)

$$F_X(x) = \begin{cases} 1 - e^{-x^3/\tau^3} & x \geq 0 \\ 0 & \textbf{else,} \end{cases}$$

where $\tau > 0$ is a constant. The cost of waiting is $Y = aX^3$ where $a > 0$ is another constant.

[3] a) Let $t > 0$ and $s > 0$. What is $P[X > s + t | X > s]$? Does X have the memoryless property?

[3] b) Does Y have the memoryless property?

[3] c) What is the probability density function (pdf) of X?

Solutions:

a) $P[X > x] = 1 - F_X(x) = \exp(-x^3/\tau^3)$. So

$$\begin{aligned} P[X > s + t | X > s] &= \frac{P[X > s + t, X > s]}{P[X > s]} \\ &= \frac{P[X > s + t]}{P[X > s]} \\ &= \frac{\exp(-(s+t)^3/\tau^3)}{\exp(-s^3/\tau^3)} \\ &= \exp(-(3s^2t + 3st^2 + t^3)/\tau^3) \end{aligned}$$

Since this is not $\exp(-t^3/\tau^3)$, X is not memoryless.

One can also conclude that X is not memoryless since it is not exponential. However, that only answers part of the question.

b) First we find the CDF of Y:

$$\begin{aligned} P[Y \leq y] &= P[aX^3 \leq y] \\ &= P[X \leq \sqrt[3]{y/a}] \\ &= 1 - e^{-(\sqrt[3]{y/a})^3/\tau^3} \\ &= 1 - e^{-y/a\tau^3} \end{aligned}$$

So Y is exponential with parameter $\lambda = 1/a\tau^3$. Since Y is exponential, it is memoryless (as shown on page 213 of the texbook).

c) We get the pdf of X by differentiating the CDF:

$$\begin{aligned} f_X(x) &= \frac{d}{dx} F_X(x) \\ &= \begin{cases} 0 & x < 0 \\ \frac{3x^2}{\tau^3} e^{-x^3/\tau^3} & x \geq 0 \end{cases} \end{aligned}$$

Q7: [10 points] Let the pair of discrete random variables X and Y have joint probability mass function (pmf)

$$p_{XY}(x,y) = \begin{cases} \frac{1}{12} & x \in \{1,2\}, y \in \{1,2\} \\ \frac{1}{6} & x \in \{3,4\}, y \in \{3,4\} \\ 0 & \textbf{else.} \end{cases}$$

[3] a) Are X and Y independent? Explain your answer.

[3] b) Find the marginal pmf $p_X(x)$ of X.

[4] c) Find the pmf of $Z = \max(X,Y)$.

Solutions:

a) No. First,

$$p_X(x) = \sum_y p_{XY}(x,y) > 0 \qquad \text{for } x \in \{1,2,3,4\}$$
$$p_Y(y) = \sum_x p_{XY}(x,y) > 0 \qquad \text{for } y \in \{1,2,3,4\}$$

So $p_{XY}(x,y)$ cannot be equal to $p_X(x)p_Y(y)$ for all x and y. One way to see this is that $p_{XY}(1,3) = 0$. But $p_X(1)p_Y(3) > 0$.

b)

$$p_X(x) = \sum_y p_{XY}(x,y) = \begin{cases} \frac{1}{6} & x \in \{1,2\} \\ \frac{1}{3} & x \in \{3,4\} \\ 0 & \text{else} \end{cases}$$

since in the first case we are summing $\frac{1}{12}$ two times and in the second case, we are summing $\frac{1}{6}$ two times.

c) For $k \notin \{1,2,3,4\}$, $p_Z(k) = 0$.

Note that

$\{Z = 1\} = \{(1,1)\}$

$\{Z = 2\} = \{(1,2),(2,2),(2,1)\}$

$\{Z = 3\} = \{(3,3)\}$

$\{Z = 4\} = \{(3,4),(4,4),(4,3))\}$

So

$$
\begin{aligned}
P[Z = 1] &= \frac{1}{12} \\
P[Z = 2] &= \frac{1}{12} \times 3 = \frac{1}{4} \\
P[Z = 3] &= \frac{1}{6} \\
P[Z = 4] &= \frac{1}{6} \times 3 = \frac{1}{2}
\end{aligned}
$$

Q8: [10 points]

[5] a) Let $X_1, X_2, \ldots$ be an infinite sequence of independent Bernoulli random variables with parameter $p = 1/2$. Find

$$\lim_{n\to\infty} P\left[\sum_{i=1}^{n}\left(X_i - \frac{1}{2}\right) \leq \sqrt{n}\right].$$

b) Let $X \sim U(0,1)$ and $Y \sim U(0,1)$, and X and Y are independent.
[2] i) Find $E[X^Y]$.
[3] ii) Find $Var[X^Y]$.

Solutions: a) We start by re-arranging terms inside:

$$\lim_{n\to\infty} P\left[\sum_{i=1}^{n}\left(X_i - \frac{1}{2}\right) \leq \sqrt{n}\right] = \lim_{n\to\infty} P\left[\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(X_i - \frac{1}{2}\right) \leq 1\right]$$

Note that $E[X_i] = 1/2$ and the variance $Var[X_i - \frac{1}{2}] = Var[X_i] = 1/2 \times (1 - 1/2) = 1/4$. So, if we divide on both sides by $\sigma = \sqrt{1/4} = 1/2$, we can apply the CLT:

$$\begin{aligned}\lim_{n\to\infty} P\left[\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(X_i - \frac{1}{2}\right) \leq 1\right] &= \lim_{n\to\infty} P\left[\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\left(X_i - \frac{1}{2}\right)}{1/2} \leq 2\right] \\ &= P[Z \leq 2] \\ &= \Phi(2).\end{aligned}$$

where Z is a standard Normal rv.

b)

$$
\begin{aligned}
E[X^Y] &= \int_0^1 \int_0^1 x^y dx dy \\
&= \int_0^1 \frac{x^{y+1}}{y+1}|_0^1 dy \\
&= \int_0^1 \frac{1}{y+1} dy \\
&= \ln(y+1)|_0^1 \\
&= \ln 2
\end{aligned}
$$

$$
\begin{aligned}
E[(X^Y)^2] &= E[X^{2Y}] \\
&= \int_0^1 \int_0^1 x^{2y} dx dy \\
&= \int_0^1 \frac{x^{2y+1}}{2y+1}|_0^1 dy \\
&= \int_0^1 \frac{1}{2y+1} dy \\
&= \frac{1}{2} \ln(2y+1)|_0^1 \\
&= \frac{1}{2} \ln 3
\end{aligned}
$$

$$
\begin{aligned}
Var[X^Y] &= E[(X^Y)^2] - (E[X^Y])^2 \\
&= \frac{1}{2} \ln 3 - (\ln 2)^2
\end{aligned}
$$

Table of $\Phi(x)$:

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.50000	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.52790	0.53188	0.53586
0.1	0.53983	0.54380	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.62930	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.65910	0.66276	0.66640	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.70540	0.70884	0.71226	0.71566	0.71904	0.72240
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.75490
0.7	0.75804	0.76115	0.76424	0.76730	0.77035	0.77337	0.77637	0.77935	0.78230	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.86650	0.86864	0.87076	0.87286	0.87493	0.87698	0.87900	0.88100	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.90320	0.90490	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.92220	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.94520	0.94630	0.94738	0.94845	0.94950	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.96080	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.97320	0.97381	0.97441	0.97500	0.97558	0.97615	0.97670
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.98030	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.98300	0.98341	0.98382	0.98422	0.98461	0.98500	0.98537	0.98574
2.2	0.98610	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.98840	0.98870	0.98899
2.3	0.98928	0.98956	0.98983	0.99010	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.99180	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.99430	0.99446	0.99461	0.99477	0.99492	0.99506	0.99520
2.6	0.99534	0.99547	0.99560	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.99720	0.99728	0.99736
2.8	0.99744	0.99752	0.99760	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.99900
3.1	0.99903	0.99906	0.99910	0.99913	0.99916	0.99918	0.99921	0.99924	0.99926	0.99929
3.2	0.99931	0.99934	0.99936	0.99938	0.99940	0.99942	0.99944	0.99946	0.99948	0.99950
3.3	0.99952	0.99953	0.99955	0.99957	0.99958	0.99960	0.99961	0.99962	0.99964	0.99965
3.4	0.99966	0.99968	0.99969	0.99970	0.99971	0.99972	0.99973	0.99974	0.99975	0.99976