ECE 203: Probability Theory and Statistics I
Fall 2021 Test#1

Instructor: Patrick Mitran

- This test consists of ABC problems and XYZ pages, including the declaration of integrity. Each page is numbered.
- Q1 is the declaration of integrity. Failure to complete this declaration will result in a grade of 0 on the test.
- You have a 24 hour window (10am on 7 Oct to 10am on 8 Oct) during which you must start, complete and submit this test. From the moment you start the test, you will have the lesser of 2 hours or until 10am on 8 Oct to submit your test.
- Non-graphing, non-programmable calculators are allowed, but will not be helpful. If the answer to a question is 5, writing $\sqrt{100}/2$ will get you full marks. Access to Matlab or similar computational software is prohibited.
- You may use i) any personal notes you take, as long as you composed them yourself prior to the test (this includes notes based on the lectures/videos, tutorials, office hours, textbook, or other book), ii) any content available on the ECE203 Learn website (including all videos) and ECE203 Piazza website, iii) the course textbook.
- You may not use the internet other than i) to access the ECE203 course webpage on Learn and ECE203 Piazza discussions, ii) to access the textbook (if ecopy) iii) to access crowdmark, or iv) to send email to me or receive email from me.
- You may use a computer, tablet or phone for only the following purposes: i) to create/scan/upload your solutions, ii) to access crowdmark, iii) to access Learn and Piazza, iv) to access your personal notes (if you took these electronically), v) to access the textbook, vi) to send/receive email to/from me, or vii) to be used as a basic calculator following the calculator rule above. Use of any file sharing services such as chegg.com is prohibited.
- You may not communicate directly or indirectly with your classmates or anyone else except for me.
- Questions are allowed but will be answered only if you cannot understand the statement of a problem. You can reach me by email (pmitran@uwaterloo.ca) from 10am to 6pm on 8 Oct.
- All answers must be written legibly. We reserve our right to reduce your grade if your answer is not written in a legible manner.
- A final correct answer does not mean much to us, if the corresponding approach is not clear and sensible. Please explain your solutions and convince us that your solutions make sense.

Q1: [1 point] DECLARATION OF INTEGRITY IN EXAMINATIONS AND TESTS
Course: ECE 203 Probability Theory and Statistics I
Term: Fall 2021

I declare that I have read and followed the instructions listed on the cover page of the ECE203 Test#1.

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Signature	**ID Number**	**Date**

Note: If you are unable to print this page, it is sufficient to write out yourself "I declare that I have read and followed the instructions listed on the cover page of the ECE203 Test#1.", then sign, write your ID number, date the statement, and upload this.

Q2: [10] Alice and Bob are scheduled to meet at 7pm. Neither will arrive early and neither will arrive later than 8pm.

a) Mathematically define the sample space S. Sketch the sample space.

b) Mathematically define and sketch the following events:

i) the event E that Bob is exactly twice as late as Alice.

ii) the event F that at least one of the two is at least 30 min late.

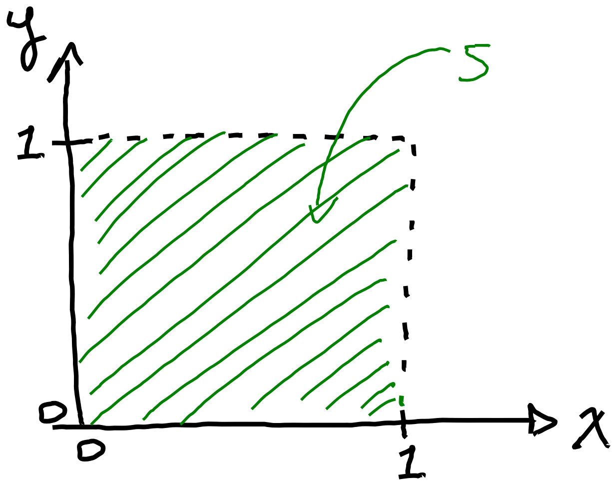
iii) the event G that at least one of the two is at least 30 min late and Bob is exactly twice as late as Alice.

Note: make sure to clearly label all key values in your sketches.

Solution:

a) The sample space $S = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1, 0 \leq y \leq 1\}$.

Writing $<$ instead of $\leq$ is also correct. Using units of minutes so that $0 < x < 60$ is fine too. One may also write $7 < x < 8$ to refer to the actual time of arrival instead of the time relative to 7pm, however, this makes part b.i) a bit more tricky. Omitting $\mathbb{R}^2$ is incorrect since $S = \{(x, y) \in \mathbb{N}^2 \mid 0 \leq x \leq 1, 0 \leq y \leq 1\}$ is incorrect. Another valid answer would be $S = \{(x, y) \in \mathbb{N}^2 \mid 0 \leq x \leq 60, 0 \leq y \leq 60\}$, but then one should be careful when plotting the sample space since the sample space is now 61^2 discrete points and not a continuous region.

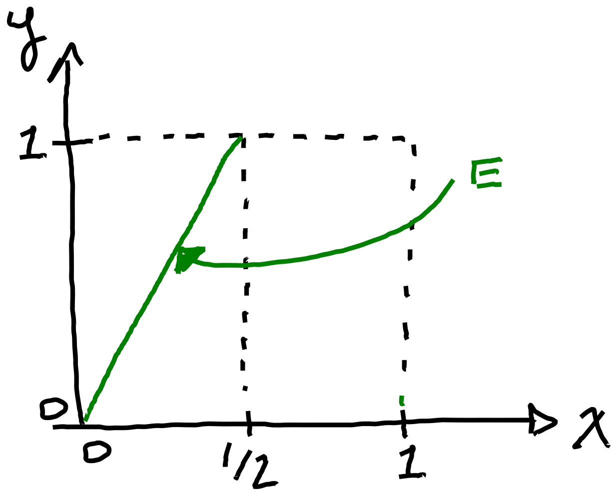


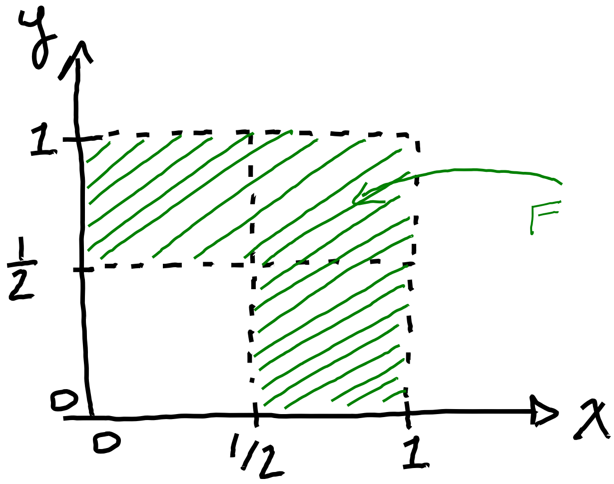
b) Assuming Alice is the first coordinate x in the sample space, then

$E = \{(x, y) \in S \mid y = 2x\}$.

$F = \{(x, y) \in S \mid \max(x, y) \geq 0.5\}$.

$G = E \cap F = \{(x, y) \in S \mid \max(x, y) \geq 0.5, y = 2x\}$.


y
1
0
0
1/2
1
x
E


y
1
1/2
0
0
1/2
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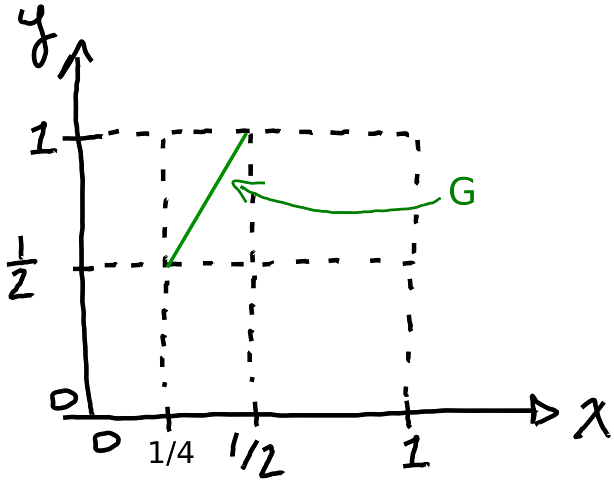
x
F


y
1
1/2
0
0
1/4
1/2
1
x
G

Q3: [10] An integrated circuit (IC) manufacturer has two plants, called P1 and P2. Once manufactured, each IC is tested and categorized into one of three quality levels, denoted A, B and C.

If an IC is manufactured at P1, the probability that it is:

- **of quality level A is 0.2**
- **of quality level B is 0.3**
- **of quality level C is 0.5**

If an IC is manufactured at P2, the probability that it is:

- **of quality level A is 0.1**
- **of quality level B is 0.2**
- **of quality level C is 0.7**

1/3 of all ICs are manufactured at P1, and the remaining 2/3 at P2.

a) What is the probability that a randomly chosen IC is of quality A?

b) If a randomly chosen IC is of quality B, what is the probability that it is from P1?

c) If a randomly chosen IC is not of quality C, what is the probability that it is not from P1?

Note: Each manufactured IC is equally likely to be the randomly chosen one.

Solution:

a) $P[A] = P[A|P_1]P[P_1] + P[A|P_2]P[P_2] = 0.2 \times 1/3 + 0.1 \times 2/3 = 0.4/3.$

b)

$$\begin{aligned} P[P_1|B] &= \frac{P[BP_1]}{P[B]} \\ &= \frac{P[B|P_1]P[P_1]}{P[B|P_1]P[P_1] + P[B|P_2]P[P_2]} \\ &= \frac{0.3 \times 1/3}{0.3 \times 1/3 + 0.2 \times 2/3} \\ &= \frac{0.3}{0.3 + 0.2 \times 2} \\ &= \frac{3}{7} \end{aligned}$$

c)

$$
\begin{aligned}
P[P_1^c|C^c] &= P[P_2|C^c] \\
&= \frac{P[C^c P_2]}{P[C^c]} \\
&= \frac{P[C^c|P_2]P[P_2]}{P[C^c|P_1]P[P_1] + P[C^c|P_2]P[P_2]} \\
&= \frac{(1 - P[C|P_2])P[P_2]}{(1 - P[C|P_1])P[P_1] + (1 - P[C|P_2])P[P_2]} \\
&= \frac{(1 - 0.7)\frac{2}{3}}{(1 - 0.5)\frac{1}{3} + (1 - 0.7)\frac{2}{3}} \\
&= \frac{6}{11}
\end{aligned}
$$

Q4: [10] Alice and Bob play the following game:

1. **Alice flips a fair coin. If it is heads, Alice wins and the game is over. Otherwise, go to step 2).**
2. **Bob rolls a fair 4-sided die. If he gets a 4, Bob wins and the game is over. Otherwise, go back to step 1).**

The game starts at step 1), and continues until someone wins.

a) What is the probability that Alice wins with exactly n coin flips?
b) What is the probability that Alice wins?
c) Given Alice wins, what is the probability that she won by flipping the coin exactly 3 times?
Note: The coin and die are fair. All coin flips and die rolls are independent.

Solution:

a) Let A_n be the event that Alice wins in exactly n coin flips. Then A_n is the event that the first $n-1$ flips are tails, the first $n-1$ die rolls are not 4, and the nth coin toss is heads. So

$$P[A_n] = (1/2)^{n-1}(3/4)^{n-1}(1/2) = (1/2) \times (3/8)^{n-1}$$

b) Let A be the event that Alice wins. Then $A = A_1 \cup A_2 \cup A_3 \cup \cdots$.

$$\begin{aligned}
P[A] &= P\left[A_1 \cup A_2 \cup A_3 \cup \cdots\right] \\
&= P[A_1] + P[A_2] + P[A_3] + \cdots \qquad \text{[since the } A_n \text{ are disjoint]} \\
&= \frac{1}{2}\left(1 + \left(\frac{3}{8}\right) + \left(\frac{3}{8}\right)^2 + \cdots\right) \\
&= \frac{1}{2}\frac{1}{1-3/8} \\
&= \frac{1}{2}\frac{8}{5} \\
&= \frac{4}{5}
\end{aligned}$$

c) We want

$$
\begin{aligned}
P[A_3|A] &= \frac{P[A_3 \cap A]}{P[A]} \\
&= \frac{P[A_3]}{P[A]} \\
&= \frac{(1/2) \times (3/8)^2}{4/5} \\
&= \frac{45}{512}
\end{aligned}
$$

Q5: The current I through a 1 kΩ resistor is a random variable:

$$P[I = 1\ \mathbf{mA}] = 0.2$$
$$P[I = 2\ \mathbf{mA}] = 0.8$$

a) What is the average voltage accross the resistor? What is the variance of the voltage accross the resistor?
b) What is the average power dissipated by the resistor? What is the variance of the power dissipated by the resistor?
Note: Recall that $V = RI$ and $P = RI^2$. Make sure to specify the units of your answers.

Solution:

a) Here, $R = 1\ k\Omega$. When $I = 1$ mA, $V = 1$ Volt. When $I = 2$ mA, then $V = 2$ Volt. So

$$\begin{aligned} E[V] &= 0.2 \times 1 + 0.8 \times 2 = 1.8 \text{ Volt} \\ E[V^2] &= 0.2 \times 1^2 + 0.8 \times 2^2 = 3.4 \text{ Volt}^2 \\ Var[V] &= E[V^2] - (E[V])^2 = 3.4 - (1.8)^2 \text{ Volt}^2 = 0.16 \text{ Volt}^2 \end{aligned}$$

Writing $Var[V] = 0.2 \times 1^2 + 0.8 \times 2^2 - (0.2 \times 1 + 0.8 \times 2)^2$ Volt^2 receives full credit. Here I wrote Volt instead of V to avoid confusing the variable V with the units V, but writing 1.8V receives full credit too.

Writing units of (k$\Omega \times$mA)^2 instead of V^2 is fine too.

b) When $I = 1$ mA, then $P = 1$ mW. When $I = 2$ mA, then $P = 4$ mW. So

$$\begin{aligned} E[P] &= 0.2 \times 1 + 0.8 \times 4 = 3.4 \text{ mW} \\ E[P^2] &= 0.2 \times 1^2 + 0.8 \times 4^2 = 13.0 \text{ mW}^2 \\ Var[P] &= E[P^2] - (E[P])^2 = 13.0 - (3.4)^2 \text{ mW}^2 = 1.44 \text{ mW}^2 \end{aligned}$$

Writing 3.4×10^{-3} W and 1.44×10^{-6} W^2 is also valid.

Q6:

a) Let X be a discrete random variable. If $E[X+X^2]=7$ and $E[X+3X^2]=17$, is it possible to compute $E[X]$? If so, provide the value of $E[X]$ and the details of how you obtained this answer. If not, explain why not.

b) For i) and ii) below, in each case, must the conclusion always be true? [Answer yes or no for each case]. If you claim yes, it is always true, then prove why it is always true. If you claim no, give an example where the conditions are satisfied, but the conclusion is false.

i) If events A and B are independent, then events A^c and AB^c are independent.

ii) Let A, B and C be 3 events with $P[C]>0$. Then $P[A\cup B|C]=P[A|C]+P[B|C]-P[A\cap B|C]$.

Solutions:

a) We are given

$$7 = E[X+X^2] = \sum_x (x+x^2)p_X(x) = \sum_x xp_X(x) + \sum_x x^2 p_X(x) = E[X] + E[X^2]$$
$$17 = E[X+3X^2] = \sum_x (x+3x^2)p_X(x) = \sum_x xp_X(x) + 3\sum_x x^2 p_X(x) = E[X] + 3E[X^2]$$

From this, we can solve that $E[X^2]=5$ and $E[X]=2$.

i) Events A^c and AB^c could be independent, but they do not have to be. So the answer is no. As an example, flip two independent fair coins and let

$$A = \{\text{first flip is heads}\}$$
$$B = \{\text{second flip is heads}\}$$

Then A and B are independent, but

$$A^c = \{th, tt\}$$
$$AB^c = \{ht\}$$

and these are not independent because

$$P[AB^c] = 1/4 \neq 0 = P[AB^c|A^c].$$

ii) This is always true:

$$
\begin{aligned}
P[A \cup B|C] &= \frac{P[(A \cup B)C]}{P[C]} \\
&= \frac{P[(AC) \cup (BC)]}{P[C]} \\
&= \frac{P[AC] + P[BC] - P[(AC)(BC)]}{P[C]} \\
&= \frac{P[AC] + P[BC] - P[ABC]}{P[C]} \\
&= P[A|C] + P[B|C] - P[AB|C]
\end{aligned}
$$

Alternatively, any probability function $P[\cdot]$ will satisfy the inclusion/exclusion principle $P[A \cup B] = P[A] + P[B] - P[A \cap B]$. Since conditional probability $P[\cdot|C]$ is a probability function, it will also satisfy the inclusion/exclusion principle: $P[A \cup B|C] = P[A|C] + P[B|C] - P[A \cap B|C]$.