ECE 203: Probability Theory and Statistics I
Fall 2021 Test#2

Instructor: Patrick Mitran

- This test consists of 6 problems and 8 pages, including the declaration of integrity and 1 table. Each page is numbered.
- Q1 is the declaration of integrity. Failure to complete this declaration will result in a grade of 0 on the test.
- You have a 24 hour window (10am on 11 Nov to 10am on 12 Nov) during which you must start, complete and submit this test. From the moment you start the test, you will have the lesser of 2 hours or until 10am on 11 Nov to submit your test.
- Non-graphing, non-programmable calculators are allowed, but are not necessary. If the answer to a question is 5, writing $\sqrt{100}/2$ will get you full marks. Access to Matlab or similar computational software is prohibited.
- You may use i) any personal notes you take, as long as you composed them yourself prior to the test (this includes notes based on the lectures/videos, tutorials, office hours, textbook, or other book), ii) any content available on the ECE203 Learn website (including all videos) and ECE203 Piazza website, iii) the course textbook.
- You may not use the internet other than i) to access the ECE203 course webpage on Learn and ECE203 Piazza discussions, ii) to access the textbook (if ecopy) iii) to access crowdmark, or iv) to send email to me or receive email from me.
- You may use a computer, tablet or phone for only the following purposes: i) to create/scan/upload your solutions, ii) to access crowdmark, iii) to access Learn and Piazza, iv) to access your personal notes (if you took these electronically), v) to access the textbook, vi) to send/receive email to/from me, or vii) to be used as a basic calculator following the calculator rule above. Use of any file sharing services such as chegg.com is prohibited.
- You may not communicate directly or indirectly with your classmates or anyone else except for me. Do not post on piazza during the test period.
- Questions are allowed but will be answered only if you cannot understand the statement of a problem. You can reach me by email (pmitran@uwaterloo.ca) from 10am to 6pm on 11 Nov.
- All answers must be written legibly. We reserve our right to reduce your grade if your answer is not written in a legible manner.
- A final correct answer does not mean much to us if the corresponding approach is not clear and sensible. Please explain your solutions and convince us that your solutions make sense.

Q1: [1 point] DECLARATION OF INTEGRITY IN EXAMINATIONS AND TESTS
Course: ECE 203 Probability Theory and Statistics I
Term: Fall 2021

I declare that I have read and followed the instructions listed on the cover page of the ECE203 Test#2.

_____	_____	_____
Signature	**ID Number**	**Date**

Note: If you are unable to print this page, it is sufficient to write out yourself "I declare that I have read and followed the instructions listed on the cover page of the ECE203 Test#2.", then sign, write your ID number, date the statement, and upload this.

Q2: [8 points]

[4] a) The profit X in dollars from an investment is a Normal random variable with the following properties:

- $E[X] = 100$**,**
- $P[X > 0] = 0.6$

What is the probability that the investment generates a profit of at least 200 dollars?

[4] b) Let U and V be two independent random variables with $U \sim \mathcal{N}(1, 4)$ and $V \sim \mathcal{N}(3, 12)$. Which of the following three cases is true:

A) $P[U > 2] < P[U + V > 6]$

B) $P[U > 2] = P[U + V > 6]$

C) $P[U > 2] > P[U + V > 6]$

Explain your answer.

Note: If you use the $\Phi()$ table, use nearest values.

Q3: [10 points]

[4] a) Let X have the following probability density function (PDF):

$$f_X(x) = \begin{cases} \frac{x}{\sigma^2} e^{-x^2/(2\sigma^2)} & x \geq 0 \\ 0 & \textbf{else}. \end{cases}$$

If $s > 0$ and $t > 0$, what is $P[X > s + t \mid X > s]$? Does X have the memoryless property?

b) Let Y be uniform on $(-1, 1)$, and $Z = Y^3$.

[2] i) What is $E[\cos Y]$?

[2] ii) What is the PDF $f_Z(z)$ of Z?

[2] iii) What is the variance of Z?

Q4: [10 points] Let the pair of random variables X and Y have joint probability density function (PDF)

$$f_{XY}(x,y) = \begin{cases} 1/3 & 0 \le x < 1, 0 \le y < 1 \\ 2/3 & 1 \le x < 2, 1 \le y < 2 \\ 0 & \textbf{else.} \end{cases}$$

Let the pair of random variables U and V have joint PDF

$$f_{UV}(u,v) = \begin{cases} 1/2 & 0 \le u < 1, 0 \le v < 1 \\ 1/2 & 1 \le u < 2, 1 \le v < 2 \\ 0 & \textbf{else.} \end{cases}$$

[3] a) Are X and Y independent? Are U and V independent? Explain your answers.

[3] b) Find the PDF $f_X(x)$ of X.

[4] c) Find the cumulative distribution function (CDF) of $Z = \max(X, Y)$.

Q5: [10 points] Let X and Y be independent random variables with $X \sim U(-1,1)$ and $Y \sim$ Exp(λ).

[3] a) What is $P[X = Y]$?

[3] b) If $b > 0$, what is $P[X > bY]$?

c) Let $Z = X^2/Y^2$.

[2] i) Show that if $a > 0$, then $P[Z \leq a] = 1 - 2P[X > \sqrt{a}Y]$

[2] ii) Find the cumulative distribution function (CDF) of Z.

Hint: You may use the results of previous parts of this problem to solve ii)

Q6: [8 points] In each case below, must the conclusion always be true? [Answer yes or no for each case].
If you claim yes, it is always true, then prove why it is always true. If you claim no, give an example where the conditions are satisfied, but the conclusion is false.

[4] a) Let X be a discrete random variable, and $Y = |X|$. Then X and Y are not independent.

[4] b) If X is a continuous random variable, then $E[X^3] = (E[X])^3$.

Table of $\Phi(x)$:

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.50000	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.52790	0.53188	0.53586
0.1	0.53983	0.54380	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.62930	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.65910	0.66276	0.66640	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.70540	0.70884	0.71226	0.71566	0.71904	0.72240
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.75490
0.7	0.75804	0.76115	0.76424	0.76730	0.77035	0.77337	0.77637	0.77935	0.78230	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.86650	0.86864	0.87076	0.87286	0.87493	0.87698	0.87900	0.88100	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.90320	0.90490	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.92220	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.94520	0.94630	0.94738	0.94845	0.94950	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.96080	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.97320	0.97381	0.97441	0.97500	0.97558	0.97615	0.97670
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.98030	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.98300	0.98341	0.98382	0.98422	0.98461	0.98500	0.98537	0.98574
2.2	0.98610	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.98840	0.98870	0.98899
2.3	0.98928	0.98956	0.98983	0.99010	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.99180	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.99430	0.99446	0.99461	0.99477	0.99492	0.99506	0.99520
2.6	0.99534	0.99547	0.99560	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.99720	0.99728	0.99736
2.8	0.99744	0.99752	0.99760	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.99900
3.1	0.99903	0.99906	0.99910	0.99913	0.99916	0.99918	0.99921	0.99924	0.99926	0.99929
3.2	0.99931	0.99934	0.99936	0.99938	0.99940	0.99942	0.99944	0.99946	0.99948	0.99950
3.3	0.99952	0.99953	0.99955	0.99957	0.99958	0.99960	0.99961	0.99962	0.99964	0.99965
3.4	0.99966	0.99968	0.99969	0.99970	0.99971	0.99972	0.99973	0.99974	0.99975	0.99976