ECE 203: Probability Theory and Statistics I
Fall 2021 Test#2

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- This test consists of ABC problems and XYZ pages, including the declaration of integrity and 1 table. Each page is numbered.
- Q1 is the declaration of integrity. Failure to complete this declaration will result in a grade of 0 on the test.
- You have a 24 hour window (10am on 11 Nov to 10am on 12 Nov) during which you must start, complete and submit this test. From the moment you start the test, you will have the lesser of 2 hours or until 10am on 11 Nov to submit your test.
- Non-graphing, non-programmable calculators are allowed, but are not necessary. If the answer to a question is 5, writing $\sqrt{100}/2$ will get you full marks. Access to Matlab or similar computational software is prohibited.
- You may use i) any personal notes you take, as long as you composed them yourself prior to the test (this includes notes based on the lectures/videos, tutorials, office hours, textbook, or other book), ii) any content available on the ECE203 Learn website (including all videos) and ECE203 Piazza website, iii) the course textbook.
- You may not use the internet other than i) to access the ECE203 course webpage on Learn and ECE203 Piazza discussions, ii) to access the textbook (if ecopy) iii) to access crowdmark, or iv) to send email to me or receive email from me.
- You may use a computer, tablet or phone for only the following purposes: i) to create/scan/upload your solutions, ii) to access crowdmark, iii) to access Learn and Piazza, iv) to access your personal notes (if you took these electronically), v) to access the textbook, vi) to send/receive email to/from me, or vii) to be used as a basic calculator following the calculator rule above. Use of any file sharing services such as chegg.com is prohibited.
- You may not communicate directly or indirectly with your classmates or anyone else except for me. Do not post on piazza during the test period.
- Questions are allowed but will be answered only if you cannot understand the statement of a problem. You can reach me by email (pmitran@uwaterloo.ca) from 10am to 6pm on 11 Nov.
- All answers must be written legibly. We reserve our right to reduce your grade if your answer is not written in a legible manner.
- A final correct answer does not mean much to us if the corresponding approach is not clear and sensible. Please explain your solutions and convince us that your solutions make sense.

Q1: [1 point] DECLARATION OF INTEGRITY IN EXAMINATIONS AND TESTS
Course: ECE 203 Probability Theory and Statistics I
Term: Fall 2021

I declare that I have read and followed the instructions listed on the cover page of the ECE203 Test#2.

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Signature	**ID Number**	**Date**

Note: If you are unable to print this page, it is sufficient to write out yourself "I declare that I have read and followed the instructions listed on the cover page of the ECE203 Test#2.", then sign, write your ID number, date the statement, and upload this.

Q2: [8 points]

[4] a) The profit X in dollars from an investment is a Normal random variable with the following properties:

- $E[X] = 100$,
- $P[X > 0] = 0.6$

What is the probability that the investment generates a profit of at least 200 dollars?

[4] b) Let U and V be two independent random variables with $U \sim \mathcal{N}(1, 4)$ and $V \sim \mathcal{N}(3, 12)$. Which of the following three cases is true:

A) $P[U > 2] < P[U + V > 6]$

B) $P[U > 2] = P[U + V > 6]$

C) $P[U > 2] > P[U + V > 6]$

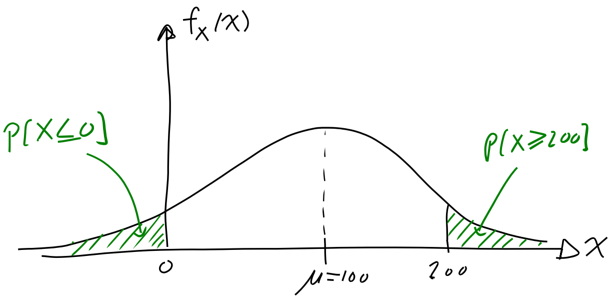
Explain your answer.

Note: If you use the $\Phi()$ table, use nearest values.

Solution: a) We are given $X \sim \mathcal{N}(\mu, \sigma^2)$. Also $\mu = E[X] = 100$. Furthermore

$$\begin{aligned}
P[X \geq 200] &= P[X - \mu \geq (200 - \mu)] \\
&=^{(a)} P[X - \mu \leq -(200 - \mu)] \\
&= P[X - 100 \leq -100] \\
&= P[X \leq 0] \\
&= 1 - P[X > 0] \\
&= 1 - 0.6 = 0.4
\end{aligned}$$

where (a) follows since the pdf of a Gaussian is symmetric about its mean $\mu = 100$, and therefore $P[X - \mu \geq a] = P[X - \mu \leq -a]$. The figure below illustrates the symmetry, but is not required for full credit.



A harder way to solve the problem is to use the Φ function. Let $Z \sim \mathcal{N}(0,1)$. Then

$$\begin{aligned} P[X>0] &= P[\frac{X-100}{\sigma} > \frac{-100}{\sigma}] \\ &= P[Z > \frac{-100}{\sigma}] \\ &= 1 - P[Z \leq \frac{-100}{\sigma}] \\ &= 1 - \Phi(-100/\sigma) \\ &= \Phi(100/\sigma) \end{aligned}$$

One could now look up $\Phi(100/\sigma) = 0.6$ in the table to get $100/\sigma = 0.25$ ($\Phi(0.25)$ is closer to 0.6 than $\Phi(0.26)$) from which $\sigma = 400$. Then

$$\begin{aligned} P[X \geq 200] &= P[\frac{X-100}{\sigma} \geq \frac{200-100}{\sigma}] \\ &= P[Z \geq \frac{100}{\sigma}] \\ &= 1 - P[Z < \frac{100}{\sigma}] \\ &= 1 - \Phi(100/\sigma) \\ &= 1 - 0.6 = 0.4 \end{aligned}$$

b) From Proposition 3.2 of Section 6.3.3 of the textbook, $U+V$ is a normal random variable with mean $1+3=4$ and variance $4+12=16$. So

$$\begin{aligned} P[U>2] &= P[\frac{U-1}{\sqrt{4}} > \frac{2-1}{\sqrt{4}}] = P[Z > 1/2] \\ P[U+V>6] &= P[\frac{U+V-4}{\sqrt{16}} > \frac{6-4}{\sqrt{16}}] = P[Z > 1/2] \end{aligned}$$

So $P[U>2] = P[U+V>6]$ and case B) is true.

Q3: [10 points]

[4] a) Let X have the following pdf:

$$f_X(x) = \begin{cases} \frac{x}{\sigma^2} e^{-x^2/(2\sigma^2)} & x \geq 0 \\ 0 & \textbf{else}. \end{cases}$$

What is $P[X > s + t \mid X > s]$? Does X have the memoryless property?

b) Let Y be uniform on $(-1, 1)$, and $Z = Y^3$.

[2] i) What is $E[\cos Y]$?

[2] ii) What is the pdf $f_Z(z)$ of Z?

[2] iii) What is the variance of Z?

Solutions:

a) We first find $P[X > a]$:

$$\begin{aligned}
P[X > a] &= \int_a^\infty f_X(x)dx \\
&= \int_a^\infty \frac{x}{\sigma^2} \exp\left(-\frac{x^2}{2\sigma^2}\right) dx
\end{aligned}$$

$$\begin{aligned}
u &= x^2/(2\sigma^2) \\
du &= \frac{x}{\sigma^2} dx \\
x = a \to u &= \frac{a^2}{2\sigma^2} \\
x = \infty \to u &= \infty
\end{aligned}$$

$$\begin{aligned}
&= \int_{\frac{a^2}{2\sigma^2}}^\infty e^{-u} du \\
&= \left[-e^{-u}\right]_{u=\frac{a^2}{2\sigma^2}}^{u=\infty} \\
&= e^{-a^2/(2\sigma^2)}
\end{aligned}$$

Hence,

$$\begin{aligned}
P[X > s + t \mid X > s] &= \frac{P[\{X > s + t\} \cap \{X > s\}]}{P[X > s]} \\
&= \frac{P[X > s + t]}{P[X > s]} \\
&= \frac{e^{-(s+t)^2/(2\sigma^2)}}{e^{-s^2/(2\sigma^2)}} \\
&= e^{-(t^2+2st)/(2\sigma^2)}
\end{aligned}$$

Simplifying to the last line isn't necessary for full credit. X does not have the memoryless property since this is not the same as $P[X > t] = e^{-t^2/(2\sigma^2)}$.

Another way to see that X does not have the memoryless property is that among all continuous random variables, only the exponential distribution has the memoryless property (see textbook p. 213). Since X is a continuous rv, but is not exponentialy distributed, X does not have the memoryless property.

b) The pdf of Y is

$$f_Y(y) = \begin{cases} 1/2 & -1 < y < 1 \\ 0 & \text{else} \end{cases}$$

i) So

$$E[\cos Y] = \int_{-\infty}^{\infty} f_Y(y) \cos y \, dy = \frac{1}{2} \int_{-1}^{1} \cos y \, dy = \frac{1}{2} [\sin y]_{-1}^{1} = \frac{1}{2} \sin 1 - \frac{1}{2} \sin(-1) = \sin 1$$

ii) The CDF of Y is

$$F_Y(y) = \begin{cases} 0 & y < -1 \\ \frac{(y+1)}{2} & -1 \le y \le 1 \\ 1 & 1 < y \end{cases}$$

So,

$$\begin{aligned} F_Z(z) = P[Z \le z] &= P[Y^3 \le z] \\ &= P[Y \le z^{1/3}] \\ &= F_Y(z^{1/3}) \\ &= \begin{cases} 0 & z < -1 \\ \frac{(z^{1/3}+1)}{2} & -1 \le z \le 1 \\ 1 & 1 < z \end{cases} \end{aligned}$$

where we used the fact that $z = y^3 = \pm 1$ implies $y = \pm 1$. Differentiating, we get:

$$\begin{aligned} f_Z(z) &= \frac{d}{dz} F_Z(z) \\ &= \begin{cases} 0 & z < -1 \\ \frac{z^{-2/3}}{6} & -1 \le z \le 1 \\ 0 & 1 < z \end{cases} \end{aligned}$$

iii) We want $Var[Z] = E[Z^2] - (E[Z])^2$. We could use the pdf of Z for this. The (much) easier way is:

$$E[Z^2] = E[(Y^3)^2] = \frac{1}{2}\int_{-1}^{1} y^6 dy = 1/7$$

$$E[Z] = E[Y^3] = \frac{1}{2}\int_{-1}^{1} y^3 dy = 0$$

So $Var[Z] = E[Z^2] - (E[Z])^2 = 1/7$.

Q4: [10 points] Let the pair of random variables X and Y have joint pdf

$$f_{XY}(x,y) = \begin{cases} 1/3 & 0 \le x < 1, 0 \le y < 1 \\ 2/3 & 1 \le x < 2, 1 \le y < 2 \\ 0 & \textbf{else.} \end{cases}$$

Let the pair of random variables U and V have joint pdf

$$f_{UV}(u,v) = \begin{cases} 1/2 & 0 \le u < 1, 0 \le v < 1 \\ 1/2 & 1 \le u < 2, 1 \le v < 2 \\ 0 & \textbf{else.} \end{cases}$$

[3] a) Are X and Y independent? Are U and V independent? Explain your answers.

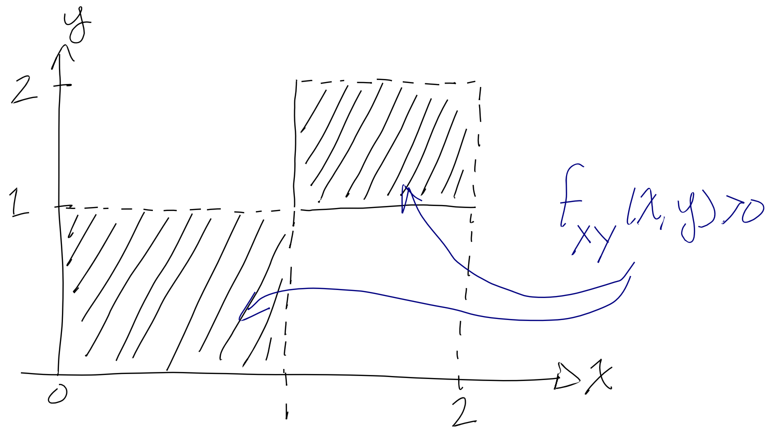
[3] b) Find the pdf $f_X(x)$ of X.

[4] c) Find the CDF of $Z = \max(X, Y)$.

Solutions:

a) No to both cases. For X and Y (the argument is the same for U and V and it is sufficient to say this), $f_{XY}(x,y) > 0$ over the region below. However, the marginal $f_X(x) > 0$ over $0 \le x < 2$ and $f_Y(y) > 0$ over $0 \le y < 2$. So $f_X(x)f_Y(y) > 0$ over the rectangle $0 \le x < 2$, $0 \le y < 2$.

Since the region where $f_X(x)f_Y(y) > 0$ is not the same as that where $f_{XY}(x,y) > 0$, X and Y are not independent.



b)

$$
\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} f_{XY}(x,y)dy \\
&= \begin{cases} \int_0^1 \frac{1}{3}dy & 0 \le x < 1 \\ \int_1^2 \frac{2}{3}dy & 1 \le x < 2 \\ 0 & \text{else} \end{cases} \\
&= \begin{cases} \frac{1}{3} & 0 \le x < 1 \\ \frac{2}{3} & 1 \le x < 2 \\ 0 & \text{else} \end{cases}
\end{aligned}
$$

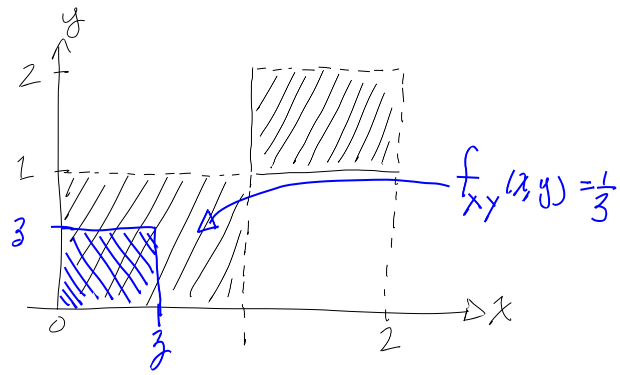
c) For $z < 0$, $F_Z(z) = 0$ and for $z \ge 2$, $F_Z(z) = 1$. So that leaves $0 \le z < 2$:

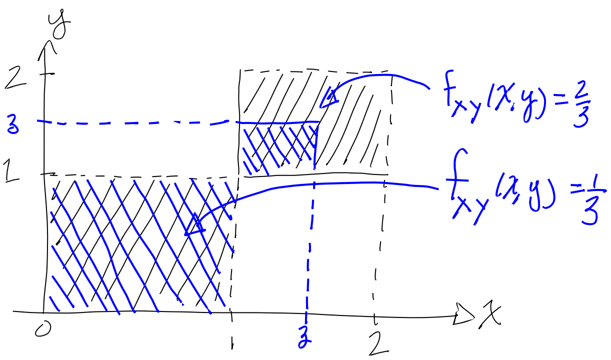
$$F_Z(z) = P[\max(X,Y) \le z].$$

There are two cases, sketched below:

Case $0 \le z \le 1$: Here, we integrate over the area shown in blue in the left fig below. The result is $\frac{1}{3}z^2$.

Case $1 \le z \le 2$: here we integrate over the area shown in blue in the right fig below. The result is $\frac{1}{3} + \frac{2}{3}(z-1)^2$.





Combining, we get

$$
F_Z(z) = \begin{cases} 0 & z < 0 \\ \frac{z^2}{3} & 0 \le z < 1 \\ \frac{1}{3} + \frac{2(z-1)^2}{3} & 1 \le z < 2 \\ 1 & 2 \le z \end{cases}
$$

Q5: [10 points] Let X and Y be independent random variables with $X \sim U(-1,1)$ and $Y \sim$ Exp(λ).

[3] a) What is $P[X = Y]$?

[3] b) If $b > 0$, what is $P[X > bY]$?

c) Let $Z = X^2/Y^2$.
[2] i) Show that if $a > 0$, then $P[Z \le a] = 1 - 2P[X > \sqrt{a}Y]$
[2] ii) Find the CDF of Z.
Hint: Use may use the results of previous parts of this problem to solve ii)

Solutions:

a)

$$\begin{aligned} P[X = Y] &= \iint\limits_{x=y} f_{XY}(x,y)\, dxdy \\ &= \int_{-\infty}^{\infty} \int_{y}^{y} f_{XY}(x,y)\, dxdy \\ &= \int_{-\infty}^{\infty} 0\, dy \\ &= 0 \end{aligned}$$

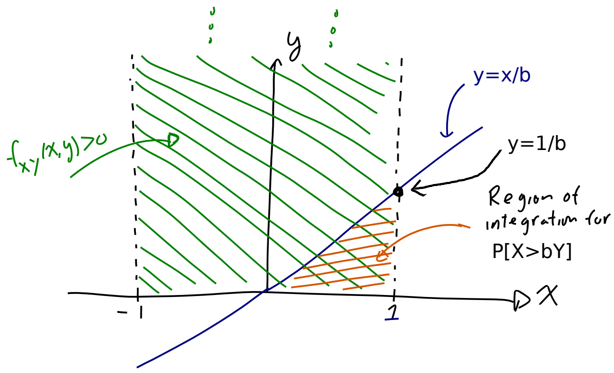
Alternatively, the region $\{x = y\}$ in the $x - y$ plane has no area, so the double integral is 0.

b) We are given

$$f_{XY}(x,y) = f_X(x) f_Y(y) = \begin{cases} \frac{\lambda}{2} e^{-\lambda y} & -1 < x < 1, y > 0 \\ 0 & \text{else} \end{cases}$$

$$P[X > bY] = \iint\limits_{x > by} f_{XY}(x,y)\, dxdy$$

The region were $f_{XY}(x,y) > 0$ is sketched below, and the region to integrate over is the shaded orange triangle region below the $y = x/b$ line.

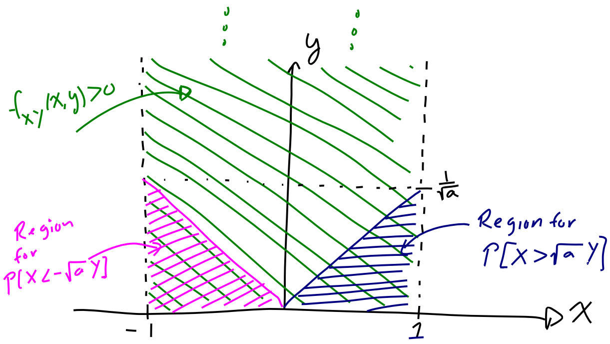


$$
\begin{aligned}
P[X > bY] &= P[Y < X/b] \\
&= \iint\limits_{y<x/b} f_{XY}(x,y)\ dxdy \\
&= \int_0^1 \int_0^{x/b} \frac{\lambda}{2} e^{-\lambda y}\ dydx \\
&= \int_0^1 \frac{1}{2} \left[-e^{-\lambda y} \right]_{y=0}^{y=x/b}\ dx \\
&= \int_0^1 \frac{1}{2} \left[1 - e^{-\lambda x/b} \right]\ dx \\
&= \frac{1}{2} - \frac{1}{2} \int_0^1 e^{-\lambda x/b}\ dx \\
&= \frac{1}{2} + \frac{b}{2\lambda} \left[e^{-\lambda x/b} \right]_{x=0}^{x=1} \\
&= \frac{1}{2} - \frac{b}{2\lambda} \left[1 - e^{-\lambda/b} \right]
\end{aligned}
$$

c) i)

$$
\begin{aligned}
P[Z \le a] &= P[X^2/Y^2 \le a] \\
&= P[X^2 \le aY^2] \\
&= P[-\sqrt{a}Y \le X \le \sqrt{a}Y] && \text{since } Y \text{ is non-negative} \\
&= 1 - P[X < -\sqrt{a}Y] - P[X > \sqrt{a}Y] \\
&=^{(a)} 1 - 2P[X > \sqrt{a}Y]
\end{aligned}
$$

where (a) follows by symmetry of the joint pdf. It is enough to claim symmetry for full credit. But to see the symmetry, the regions of integration for $P[X > \sqrt{a}Y]$ and $P[X < -\sqrt{a}Y]$ are plotted below, and that in the region where the joint pdf $f_{XY}(x,y) > 0$, the joint pdf is a function of y only.



ii) For $a \leq 0$, since Z is non-negative then $P[Z \leq a] = 0$. For $a > 0$, using the result from part i) and then part b) with $b = \sqrt{a}$:

$$\begin{aligned} P[Z \leq a] &= 1 - 2P[Y < X/\sqrt{a}] \\ &= 1 - 2\left[\frac{1}{2} - \frac{\sqrt{a}}{2\lambda}\left[1 - e^{-\lambda/\sqrt{a}}\right]\right] \\ &= \frac{\sqrt{a}}{\lambda}\left[1 - e^{-\lambda/\sqrt{a}}\right] \end{aligned}$$

Putting it all together:

$$F_Z(a) = \begin{cases} 0 & a \leq 0 \\ \frac{\sqrt{a}}{\lambda}\left[1 - e^{-\lambda/\sqrt{a}}\right] & a > 0 \end{cases}$$

Q6: [8 points] In each case below, must the conclusion always be true? [Answer yes or no for each case].

If you claim yes, it is always true, then prove why it is always true. If you claim no, give an example where the conditions are satisfied, but the conclusion is false.

[3] a) Let X be a discrete random variable, and $Y = |X|$. Then X and Y are not independent.

[3] b) If X is a continuous random variable, then $E[X^3] = (E[X])^3$.

Solutions:

a) No, this is false. Let X have pmf

$$P[X = a] = \begin{cases} 1/2 & a = 1 \\ 1/2 & a = -1 \\ 0 & \text{else} \end{cases}$$

Then Y has pmf

$$P[Y = b] = \begin{cases} 1 & b = 1 \\ 0 & \text{else} \end{cases}$$

The joint pmf is then

$$P[X = a, Y = b] = \begin{cases} 1/2 & a = 1, b = 1 \\ 1/2 & a = -1, b = 1 \\ 0 & \text{else} \end{cases}$$

and thus $P[X = a, Y = b] = P[X = a]P[Y = b]$ for every choice of a and b.

b) No, this is false. Let X be uniform on $(0, 1)$. Then $E[X] = 1/2$, but

$$E[X^3] = \int_0^1 x^3 dx = [x^4/4]_0^1 = 1/4.$$

So $(E[X])^3 = 1/8 \neq E[X^3]$.

Table of $\Phi(x)$:

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.50000	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.52790	0.53188	0.53586
0.1	0.53983	0.54380	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.62930	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.65910	0.66276	0.66640	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.70540	0.70884	0.71226	0.71566	0.71904	0.72240
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.75490
0.7	0.75804	0.76115	0.76424	0.76730	0.77035	0.77337	0.77637	0.77935	0.78230	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.86650	0.86864	0.87076	0.87286	0.87493	0.87698	0.87900	0.88100	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.90320	0.90490	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.92220	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.94520	0.94630	0.94738	0.94845	0.94950	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.96080	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.97320	0.97381	0.97441	0.97500	0.97558	0.97615	0.97670
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.98030	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.98300	0.98341	0.98382	0.98422	0.98461	0.98500	0.98537	0.98574
2.2	0.98610	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.98840	0.98870	0.98899
2.3	0.98928	0.98956	0.98983	0.99010	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.99180	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.99430	0.99446	0.99461	0.99477	0.99492	0.99506	0.99520
2.6	0.99534	0.99547	0.99560	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.99720	0.99728	0.99736
2.8	0.99744	0.99752	0.99760	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.99900
3.1	0.99903	0.99906	0.99910	0.99913	0.99916	0.99918	0.99921	0.99924	0.99926	0.99929
3.2	0.99931	0.99934	0.99936	0.99938	0.99940	0.99942	0.99944	0.99946	0.99948	0.99950
3.3	0.99952	0.99953	0.99955	0.99957	0.99958	0.99960	0.99961	0.99962	0.99964	0.99965
3.4	0.99966	0.99968	0.99969	0.99970	0.99971	0.99972	0.99973	0.99974	0.99975	0.99976